Quadratic separation for feedback connection of an uncertain matrix and an implicit linear transformation ⋆

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Abstract

Topological separation is investigated in the case of an uncertain time-invariant matrix interconnected with an implicit linear transformation. A quadratic separator independent of the uncertainty is shown to prove losslessly the closed-loop well-posedness. Several applications for LTI descriptor system analysis are then given. First, some known results for stability and pole location of descriptor systems are demonstrated in a new way. Second, contributions to robust stability analysis and time-delay systems stability analysis are exposed. These prove to be new even when compared to results for usual LTI systems (not in descriptor form). All results are formulated as linear matrix inequalities (LMIs).

1 Introduction

Well-posedness of feedback systems provides a fertile framework for stability analysis of non-linear and uncertain systems. An associated fundamental concept is topological separation [26]. It states that internal signals of a multivariable feedback connection of two systems F and G are unique and bounded under external disturbances if and only if the graph of F is topologically separated from the inverse graph of G. While finding such topological separator is tricky in general, for several choices of systems F and G there exist, sometimes lossless [22], tractable techniques. Among these, major results for robust stability analysis are given in [13] and references therein. The purpose of the present paper is to extend these and show how they apply on some significant analysis problems.

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It is to be noted that the general framework of topological separation as introduced by Safonov [26] considers implicit systems. In this paper we investigate a special case of feedback connections where the first system F is a given implicit finite-dimentional linear transformation, $\mathcal{E}z = \mathcal{A}w$, and the second system G is a complex valued uncertain matrix gain, $w = \nabla z$, $\nabla \in \boldsymbol{\nabla}$. No assumption is made on the set of possible uncertainties $\boldsymbol{\nabla}$. The central result to which is dedicated section 2 proves that well-posedness of such systems is losslessly assessed by a quadratic separator (the topological separator is a quadratic functional of z and w) independent of the uncertainty ∇.

The result is a generalisation of Corollary 1 in [13] to implicit linear transformations and therefore generalizes the robust analysis results of [13] to descriptor systems. Closely related results for usual LTI systems are full-block S-procedure Theorem 1 in [27] and IQC Theorem 1 in [21]. Even closer results that consider special cases of implicit systems are: well-posedness Lemma 4 in [15] and KYP lemma for implicit systems Theorem 3 in [14]. As in all these papers, the main task is to demonstrate that there exists a quadratic separator and give (possibly conservative) LMI conditions in accordance with known data on the uncertainty set. The central contribution of the present manuscript extends previous results for the case of non-square implicit systems in a unified framework. As a by-product of the generality of the implicit transformation formulation one gets many corollaries for stability analysis of dynamic systems. Combined with redundant system modeling as in [6,7], it conducts to results that are totally new, even for usual (non singular) LTI systems.

After stating the central result in section 2, the remaining of the paper is dedicated to corollaries of the central result:

- In section 3 we illustrate how previously obtained stability analysis conditions for descriptor systems are related to topological separation. First we show that stability of $E\dot{x} = Ax$ is equivalent to well-posedness of a feedback connection of $\nabla = s^{-1}1$ and $s^{-1} \in \mathbb{C}^+$, the closed right half-plane, with an implicit linear transformation. Quadratic separation with respect to this uncertainty set proves to be related to the existence of a quadratic Lyapunov certificate. Extensions to discrete-time system stability as well as to pole location analysis are also given and compared with [19,12,11].
- In section 4 of the paper, corollaries for robust stability of descriptor systems are derived. More general than the framework by [17] and [32], the contribution considers rationally dependent models where uncertainties enter on both the A and the E matrices. To our knowledge these results are totally new. They allow to test stability of systems with structured uncertainty entering in a very general modeling naturally produced when manipulating Linear Fractional Transform (LFT) models as attested in [10,23]. But the contribution is more significant than an extension to descriptor systems of previous results. Tests with various degrees of conservatism are given that involve either constant or parameter-dependent Lyapunov functions. The sequence of recursively derived corollaries is an original contribution even when compared to similar results for usual LTI systems such

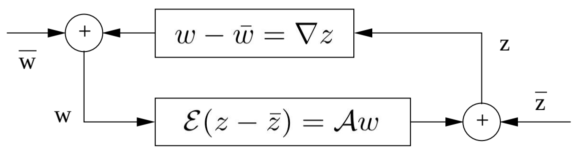

Figure 1. Feedback system

as [15,2].

- Finally, section 5 is devoted to stability analysis of descriptor time-delay systems. Both delay-independent and delay-dependent results are derived. In the former case the results are a generalization of those in [1] to descriptor systems. In the latter case, results not only encompass all existing LMI formulas as proved in [8], but extend these by proposing new tests with reduced conservatism. As for all corollaries of the other sections, the delay-dependent results are tested on a numerical example. Conservatism reduction is shown to be significant without increasing drastically the numerical burden.

Notations: $\mathbb{R}^{m\times n}$ and $\mathbb{C}^{m\times n}$ are the sets of m-by-n real and complex matrices respectively. A^T is the transpose of the matrix A and A^* is its transpose conjugate. $A^\perp$ is a full rank matrix whose columns span the null-space of A. Define as well A° as a full rank matrix whose columns span the same space as the columns of A. If $A = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \text{diag}(S, 0) \begin{bmatrix} V_1 & V_2 \end{bmatrix}^*$ is the singular value decomposition with S containing all non zero singular values, then one can choose $A^\perp = V_2$ and $A^\circ = U_1$. In addition, define $A^\circledast = A^{*\circ}$ (or $A^{T\circ}$ if real-valued) which is such that the columns of $\begin{bmatrix} A^\perp & A^\circledast \end{bmatrix}$ span $\mathbb{C}^m$ (assuming $A \in \mathbb{C}^{n\times m}$).1 and 0 are respectively the identity and the zero matrices of appropriate dimensions. For Hermitian matrices, $A > (\geq) B$ means that $A - B$ is positive (semi) definite.

2 Well-posedness condition

Consider two possibly non-square finite dimensional matrices $\mathcal{E}$ and $\mathcal{A}$. Let an uncertain matrix ∇ with appropriate dimensions that belongs to some set $\boldsymbol{\nabla}$. No assumption is made on the uncertainty set $\boldsymbol{\nabla}$ at this stage.

The feedback system of Figure 1 is said to be well-posed if for all uncertainties and all bounded input vectors, the internal vectors characterizing the system are unique and bounded. More specifically, consider the decomposition of z and $\bar{z}$ in the $\begin{bmatrix} \mathcal{E}^\perp & \mathcal{E}^\circledast \end{bmatrix}$ basis, *i.e.* $z = \mathcal{E}^\perp y_1 + \mathcal{E}^\circledast y$ and $\bar{z} = \mathcal{E}^\perp \bar{y}_1 + \mathcal{E}^\circledast \bar{y}$. With these notations, the feedback connected system writes

$$\begin{aligned} w - \bar{w} &= \nabla \mathcal{E}^\perp y_1 + \nabla \mathcal{E}^\circledast y \\ \mathcal{E}\mathcal{E}^\circledast (y - \bar{y}) - \mathcal{A}w &= 0 \ . \end{aligned} \tag{1}$$

As ∇ may be rank-deficient, the vector y_1 may be non-unique and unbounded, at least for some values of ∇. The vector y_1 is therefore not an internal variable of the system but rather a perturbation, possibly unbounded. The definition of well-posedness of the feedback connected system is therefore based on proving that for all uncertainties $\nabla \in \boldsymbol{\nabla}$, all vectors y_1 and all bounded inputs $\bar{w}$ and $\bar{y}$, the internal variables w and y are unique and bounded. Since only linear transformations enter the feedback system, w and z are necessarily unique if we can prove they are bounded. Well-posedness writes as

$$\exists \bar{\gamma} > 0 \ : \ \begin{array}{c} \forall(\bar{w}, \bar{z}, y_1) \\ \forall \nabla \in \boldsymbol{\nabla} \end{array} \ , \ \left\| \begin{pmatrix} w \\ y \end{pmatrix} \right\| \leq \bar{\gamma} \left\| \begin{pmatrix} \bar{w} \\ \bar{y} \end{pmatrix} \right\| . \tag{2}$$

Note that (1) implies

$$(\mathcal{E} - \mathcal{A}\nabla)\mathcal{E}^{\circledast} y = \mathcal{E}^{\circledast}\bar{y} + \mathcal{A}\nabla\mathcal{E}^{\perp} y_1 + \mathcal{A}\bar{w} .$$

Well-posedness of the system states that for all admissible $\nabla \in \boldsymbol{\nabla}$, the null space of $(\mathcal{E} - \mathcal{A}\nabla)\mathcal{E}^{\circledast}$ is empty (the matrix is non-singular if square) and one gets $((\mathcal{E} - \mathcal{A}\nabla)\mathcal{E}^{\circledast})^{\dagger}\mathcal{A}\nabla\mathcal{E}^{\perp} = 0$ since y is unique for all y_1.

Theorem 1 *The uncertain feedback system of Figure 1 is well-posed if and only if there exists a Hermitian matrix $\Theta = \Theta^*$ satisfying both conditions*

$$\begin{bmatrix} \mathcal{E}\mathcal{E}^{\circledast} & -\mathcal{A} \end{bmatrix}^{\perp *} \Theta \begin{bmatrix} \mathcal{E}\mathcal{E}^{\circledast} & -\mathcal{A} \end{bmatrix}^{\perp} > 0 \tag{3}$$

$$\begin{bmatrix} 0 & 1 \\ \nabla\mathcal{E}^{\perp} & \nabla\mathcal{E}^{\circledast} \end{bmatrix}^* \Theta \begin{bmatrix} 0 & 1 \\ \nabla\mathcal{E}^{\perp} & \nabla\mathcal{E}^{\circledast} \end{bmatrix} \leq 0 \quad , \quad \forall \nabla \in \boldsymbol{\nabla} . \tag{4}$$

If $\mathcal{E}$ and $\mathcal{A}$ are real, the equivalence still holds with Θ restricted to be real.

Before getting into the details of the proof, note that the conditions of Theorem 1 have a much simpler expression in case $\mathcal{E}$ is full column rank (which is common for many systems). In that case, $\mathcal{E}^{\circledast} = 1$ and $\mathcal{E}^{\perp}$ is an empty matrix (zero number of columns), the inequalities (3) and (4) write as

$$\begin{bmatrix} \mathcal{E} & -\mathcal{A} \end{bmatrix}^{\perp *} \Theta \begin{bmatrix} \mathcal{E} & -\mathcal{A} \end{bmatrix}^{\perp} > 0 \ , \ \begin{bmatrix} 1 \\ \nabla \end{bmatrix}^* \Theta \begin{bmatrix} 1 \\ \nabla \end{bmatrix} \leq 0$$

In addition if $\mathcal{E} = 1$ then one can choose $\begin{bmatrix} \mathcal{E} & -\mathcal{A} \end{bmatrix}^{\perp} = \begin{bmatrix} \mathcal{A}^* & 1 \end{bmatrix}^*$ which leads exactly to the well-posedness conditions of [13].

Proof of sufficiency: Assume (3) holds. It implies the existence of some positive scalar ϵ such that

$$\begin{bmatrix} \mathcal{E}\mathcal{E}^{\circledast} & -\mathcal{A} \end{bmatrix}^{\perp *} (\Theta - \epsilon 1) \begin{bmatrix} \mathcal{E}\mathcal{E}^{\circledast} & -\mathcal{A} \end{bmatrix}^{\perp} \geq 0 .$$

By definition of $\begin{bmatrix} \mathcal{E}\mathcal{E}^{\circledast} & -\mathcal{A} \end{bmatrix}^{\perp}$ and assuming (4) holds, one gets for all vectors that satisfy (1)

$$\begin{pmatrix} y-\bar{y} \\ w \end{pmatrix}^{*} (\Theta - \epsilon 1) \begin{pmatrix} y-\bar{y} \\ w \end{pmatrix} \geq 0$$
$$\begin{pmatrix} y \\ w-\bar{w} \end{pmatrix}^{*} \Theta \begin{pmatrix} y \\ w-\bar{w} \end{pmatrix} \leq 0\,.$$

Combining both inequalities, results in a quadratic constraint on the vector $X = \begin{pmatrix} w^* & y^* \big| \bar{w}^* & \bar{y}^* \end{pmatrix}^{*}$ such as $X^* \begin{bmatrix} \epsilon 1 & T_1 \\ T_1^* & T_2 \end{bmatrix} X \leq 0$. Take any $\tilde{\epsilon}$ such that $\epsilon > \tilde{\epsilon} > 0$ and take a sufficiently large $\tilde{\gamma} > 0$ such that $\begin{bmatrix} \tilde{\epsilon} 1 & 0 \\ 0 & -\tilde{\gamma} 1 \end{bmatrix} \leq \begin{bmatrix} \epsilon 1 & T_1 \\ T_1^* & T_2 \end{bmatrix}$ to finally get $X^* \begin{bmatrix} \tilde{\epsilon} 1 & 0 \\ 0 & -\tilde{\gamma} 1 \end{bmatrix} X \leq 0$ which is the well-posedness condition (2). ■

Proof of necessity: Assume the system in Figure 1 is well-posed and equivalently that (1) is well-posed. First, note that if inequality (2) holds for $\bar{\gamma}$ it also holds for all $\gamma \geq \bar{\gamma}$. Define

$$Y^* = \begin{pmatrix} w^* & y^* \big| \bar{w}^* & \bar{y}^* \big| y_1^* \end{pmatrix}\,, \qquad \Xi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\gamma 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\,.$$
$$\mathcal{M} = \left[\begin{array}{cc|cc|c} -1 & \nabla\mathcal{E}^{\circledast} & 1 & 0 & \nabla\mathcal{E}^{\perp} \\ \mathcal{A} & -\mathcal{E}\mathcal{E}^{\circledast} & 0 & \mathcal{E}\mathcal{E}^{\circledast} & 0 \end{array}\right]\,,$$

Well-posedness of (1) implies that, for all $\gamma \geq \bar{\gamma}$ and for all $\nabla \in \boldsymbol{\nabla}$, if the equality constraint $\mathcal{M}Y = 0$ holds, then the quadratic constraint $Y^*\Xi Y \leq 0$ also holds. Due to Finsler's lemma [29], it is equivalent to

$$\mathcal{M}^{\perp^*}\Xi\mathcal{M}^{\perp} \leq 0\,. \tag{5}$$

Partition $\begin{bmatrix} \mathcal{E}\mathcal{E}^{\circledast} & -\mathcal{A} \end{bmatrix}^{\perp} = \begin{bmatrix} N_1^* & N_2^* \end{bmatrix}^{*}$ such that $\mathcal{E}\mathcal{E}^{\circledast}N_1 = \mathcal{A}N_2$. Moreover, define $R = N_1^*N_1 + N_2^*N_2$ and $Q = \gamma R - N_2^*N_2$. Since $\mathcal{E}\mathcal{E}^{\circledast}$ is full rank, N_2 is also full rank and one can choose γ sufficiently large such that $Q > 0$. Take $\mathcal{M}^{\perp *} =$

$$\left[\begin{array}{cc|cc|c} 0 & 0 & -\mathcal{E}^{\perp *}\nabla^* & 0 & 1 \\ 0 & 1 & -\mathcal{E}^{\circledast *}\nabla^* & 1 & 0 \\ N_2^* & 0 & N_2^* & -N_1^* & 0 \end{array}\right]$$, the inequality (5) then writes as

$$\begin{bmatrix} -\gamma\mathcal{E}^{\perp *}\nabla^*\nabla\mathcal{E}^{\perp} & -\gamma\mathcal{E}^{\perp *}\nabla^*\nabla\mathcal{E}^{\circledast} & \gamma\mathcal{E}^{\perp *}\nabla^* N_2 \\ -\gamma\mathcal{E}^{\circledast *}\nabla^*\nabla\mathcal{E}^{\perp} & 1-\gamma 1-\gamma\mathcal{E}^{\circledast *}\nabla^*\nabla\mathcal{E}^{\circledast} & \gamma\mathcal{E}^{\circledast *}\nabla^* N_2+\gamma N_1 \\ \gamma N_2^*\nabla\mathcal{E}^{\circledast} & \gamma N_2^*\nabla\mathcal{E}^{\circledast}+\gamma N_1^* & -Q \end{bmatrix} \leq 0 \; .$$

Applying a Schur complement argument on the block $-Q$, one gets inequality (4) where

$$\Theta = \begin{bmatrix} 1-\gamma 1+\gamma^2 N_1 Q^{-1} N_1^* & \gamma^2 N_1 Q^{-1} N_2^* \\ \gamma^2 N_2 Q^{-1} N_1^* & -\gamma 1+\gamma^2 N_2 Q^{-1} N_2^* \end{bmatrix} .$$

This matrix is real if $\mathcal{E}$ and $\mathcal{A}$ are real. Let us prove now that (3) also holds.

$$\begin{bmatrix} \mathcal{E}\mathcal{E}^{\circledast} & -\mathcal{A} \end{bmatrix}^{\perp *} \Theta \begin{bmatrix} \mathcal{E}\mathcal{E}^{\circledast} & -\mathcal{A} \end{bmatrix}^{\perp} = N_1^* N_1 - \gamma R + \gamma^2 R(\gamma R - N_2^* N_2)^{-1} R$$

Recall the matrix inversion lemma $(a+bcd)^{-1} = a^{-1} - a^{-1}b(c^{-1}+da^{-1}b)^{-1}da^{-1}$. Apply this result a first time to $(\gamma R - N_2^* N_2)^{-1}$ to get

$$\begin{bmatrix} \mathcal{E}\mathcal{E}^{\circledast} & -\mathcal{A} \end{bmatrix}^{\perp *} \Theta \begin{bmatrix} \mathcal{E}\mathcal{E}^{\circledast} & -\mathcal{A} \end{bmatrix}^{\perp} = N_1^* N_1 + N_2^*(1 - N_2(\gamma R)^{-1} N_2^*)^{-1} N_2$$

and a second time to $(1 - N_2(\gamma R)^{-1} N_2^*)^{-1}$ to conclude

$$\begin{aligned} \begin{bmatrix} \mathcal{E}\mathcal{E}^{\circledast} & -\mathcal{A} \end{bmatrix}^{\perp *} \Theta \begin{bmatrix} \mathcal{E}\mathcal{E}^{\circledast} & -\mathcal{A} \end{bmatrix}^{\perp} &= N_1^* N_1 + N_2^*(1 + N_2(\gamma R - N_2^* N_2)^{-1} N_2^*) N_2 \\ &= R + N_2^* N_2 Q^{-1} N_2^* N_2 > 0 \; . \end{aligned}$$

Both inequalities (3) and (4) hold for any γ sufficiently large to ensure $Q > 0$. ■

Remark that the heart of the proof relies on the use of Finsler's lemma. As in [3,4] this is the key tool that enables to deal with implicit linear transformation constraints.

3 Stability of descriptor systems

3.1 Continuous-time descriptor systems

A linear descriptor system characterised by the state-space equation $E\dot{x} = Ax$ fits the feedback system framework of Figure 1 if one considers

$$z = \dot{x} \ , \ w = x \ , \ \mathcal{E} = E \ , \ \mathcal{A} = A \ , \ \nabla = s^{-1}1 \ , \ s^{-1} \in \mathbb{C}^+ \ .$$

The inverse Laplace operator s^{-1} is constrained to the closed right hand-side of the complex plane and hence well-posedness proves the that there are no poles with non-negative real part. For this set of "uncertainties", a choice of quadratic separator is given in the following corollary.

Corollary 2 *The descriptor system $E\dot{x} = Ax$ is admissible, i.e. regular, stable and impulse free [19], if and only if the following LMI conditions hold*

$$E^{\circledast T} P E^{\circledast} > 0 \ , \ E^{\circledast T} P E^{\perp} = 0 \ , \ E^{\perp T} P E^{\perp} < 0 \tag{6}$$

$$\begin{bmatrix} EE^{\circledast} & -A \end{bmatrix}^{\perp T} \begin{bmatrix} 0 & E^{\circledast T} P \\ PE^{\circledast} & 0 \end{bmatrix} \begin{bmatrix} EE^{\circledast} & -A \end{bmatrix}^{\perp} < 0 \, . \tag{7}$$

Proof : One way to prove Corollary 2 is to show how this result is related to the LMI conditions of [19]

$$E^T X^T = XE \geq 0 \ , \ A^T X^T + XA < 0 \, . \tag{8}$$

For an admissible descriptor system there exist two non singular matrices V and U such that

$$VEU = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad , \quad VAU = \begin{bmatrix} A_1 & 0 \\ 0 & 1 \end{bmatrix} \, . \tag{9}$$

Based on this factorisation take

$$E^{\circledast} = U \begin{bmatrix} 1 \\ 0 \end{bmatrix} \ , \ E^{\perp} = U \begin{bmatrix} 0 \\ 1 \end{bmatrix} \ , \ \begin{bmatrix} EE^{\circledast} & -A \end{bmatrix}^{\perp} = \begin{bmatrix} A_1 \\ E^{\circledast} \end{bmatrix}$$

and $P = XV^{-1}U^{-1}$. With these notations after simple manipulations one gets that

$$E^T X^T = U^{-T} \begin{bmatrix} E^{\circledast T} P E^{\circledast} & E^{\circledast T} P E^{\perp} \\ 0 & 0 \end{bmatrix} U^{-1} \, .$$

The two first conditions of (6) are therefore equivalent to $XE = E^T X^T \geq 0$. Next, assuming $E^{\circledast T} P E^{\perp} = 0$, one gets

$$A^T X^T + XA = U^{-T} \begin{bmatrix} (7) & 0 \\ 0 & 2E^{\perp T} P E^{\perp} \end{bmatrix} U^{-1}$$

which concludes the proof. ■

Remark that conditions of Corollary 2 imply to solve a smaller LMI problem in terms of number of variables and size of the constraints. This can prove more efficient for large scale problems. On the other hand, the constraints (8) may be useful when dealing with design problems [19], it is not the case for the proposed result.

Remark also that the matrix P in Corollary 2 defines a quadratic Lyapunov function $V(x) = x^T P x$ such that $V(x) > 0$ for all x in the image of $E^{\circledast}$, while $\dot{V}(x) < 0$ for all $(\dot{x}, x)$ in the null space of $\begin{bmatrix} EE^{\circledast} & -A \end{bmatrix}$.

As an example, consider as in [20] the scalar 'switch' system $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 1 \\ \delta \end{bmatrix} x$. If $\delta \neq 0$ the system is 'turned off' and resumes to $x(t) = 0$, it is stable. Otherwise the system is unstable, $\dot{x} = x$.

Assume $\delta \neq 0$ then $E^{\circledast} = 1$ and $\begin{bmatrix} EE^{\circledast} & -A \end{bmatrix}^{\perp} = \begin{bmatrix} 1 & -1 \\ 0 & -\delta \end{bmatrix}^{\perp}$ is an empty matrix (zero number of columns). The LMI conditions of Corollary 2 are summarized by the existence of a scalar $p > 0$. Take for example $p = 1$, the stability is proved. Now what happens when $\delta = 0$? In that case $\begin{bmatrix} EE^{\circledast} & -A \end{bmatrix}^{\perp} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}^{\perp} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and the LMI conditions are $p > 0 \ , \ 2p < 0$. They cannot be fulfilled, the system is unstable.

3.2 Discrete-time descriptor systems

The discrete-time case is very much similar to the continuous-time case. The state-space representation $Ex_{k+1} = Ax_k$ is identically modelled as a feedback system in Figure 1. The unique difference is the uncertainty set $\nabla = s^{-1} 1 \quad , \quad |s^{-1}| \leq 1$. Non-conservative separators can be parameterised as

$$\Theta = \begin{bmatrix} -E^{\circledast T} P E^{\circledast} & 0 \\ 0 & P \end{bmatrix}$$

with P satisfying (6). Applying Theorem 1, condition (3) with this choice of separator is a necessary and sufficient LMI condition for the stability of the discrete-time descriptor system. The result is related to the generalised discrete Lyapunov inequality of [12] [33, lemma 1] that writes $E^T X E > A^T X A$, $E^T X E \geq 0$, by taking $P = U^{-T} V^{-T} X V^{-1} U^{-1}$ where U and V are those defined in (9).

3.3 Pole location

The procedure can be extended to pole location analysis. For example, take regions of the complex plane described by a scalar quadratic inequality:

$$\mathcal{D} = \{ \ s \in \mathbb{C} \ : \ d_1 + d_2 s + d_2^* s^* + d_3 s s^* < 0 \ \} .$$

Such regions are half-planes, interior of disks or exteriors of disks. The poles of $E\dot{x} = Ax$ (i.e. values such that $rank(Es - A)$ drops from its normal value) lie in $\mathcal{D}$ if the feedback system of Figure 1 is well-posed for all s outside the region. Pole location analysis amounts to testing well-posedness with respect to the following uncertainty set:

$$\boldsymbol{\nabla} = \left\{ \ s^{-1} 1 \ : \ d_1 s^{-1} s^{-*} + d_2 s^{-*} + d_2^* s^{-1} + d_3 \geq 0 \ \right\} .$$

Necessary and sufficient LMI condition for pole location analysis are then obtained applying Theorem 1 with the following separator:

$$\Theta = \begin{bmatrix} d_3 E^{\circledast T} P E^{\circledast} & d_2^* E^{\circledast T} P \\ d_2 P E^{\circledast} & d_1 P \end{bmatrix}$$

with P satisfying (6). For many other regions (as well as for unions of regions) separators can be chosen following the methodology in [11,14]. For intersections of regions, the procedure consists in proving pole location in each region independently.

3.4 Polynomial systems

Consider a polynomial matrix differential equation of degree d defined by $x \in \mathbb{R}^n$ and $\sum_{i=0}^{d} A_i x^{(i)} = 0$, then its stability is equivalent to well-posedness of the system of Figure 1 with

$$\mathcal{E} = \begin{bmatrix} A_d & 0 & \cdots & 0 \\ 0 & -1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & -1 \end{bmatrix} \ , \ \mathcal{A} = - \begin{bmatrix} A_{d-1} & \cdots & A_1 & A_0 \\ 1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \end{bmatrix} \ , \ \begin{array}{l} \nabla = s^{-1} 1_{dn} \\ s^{-1} \in \mathbb{C}^+ . \end{array}$$

In case A_d is full rank, stability and pole location LMI conditions obtained when applying Theorem 1 to this system are exactly the same as the one proposed in [11].

4 Robust stability of descriptor systems

Consider the following uncertain descriptor system

$$\begin{aligned}(E_A + (B\Delta - E_B)(E_D - D\Delta)^{-1}E_C)\dot{x} \\ = (A + (B\Delta - E_B)(E_D - D\Delta)^{-1}C)x\end{aligned} \tag{10}$$

where the state-space model matrices are rational functions of the uncertain parameters Δ that are assumed to belong to a set $\mathbf{\Delta}$. Note that this very general modeling naturally arises from Linear Fractional Transform (LFT) modeling as attested in [10,23]. Moreover, it can often give minimal LFT formulations which is of major interest as the numerical complexity of analysis tools grows significantly with the dimensions of the uncertain operator Δ.

Model (10) matches the framework of Figure 1 if one considers

$$\overbrace{\begin{bmatrix} E_A & E_B \\ E_C & E_D \end{bmatrix}}^{\mathcal{E}} \begin{pmatrix} \dot{x} \\ z_\Delta \end{pmatrix} = \overbrace{\begin{bmatrix} A & B \\ C & D \end{bmatrix}}^{\mathcal{A}} \begin{pmatrix} x \\ w_\Delta \end{pmatrix} \tag{11}$$

along with the set $\mathbf{\nabla} = \left\{ \begin{bmatrix} s^{-1}1_n & 0 \\ 0 & \Delta \end{bmatrix} \ : \ s^{-1} \in \mathbb{C}^+ \ , \ \Delta \in \mathbf{\Delta} \right\}$. For that type of sets, the quadratic separator can be chosen as

$$\mathcal{E}^{\circledast} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \begin{matrix} \updownarrow n \\ \updownarrow m \end{matrix} \quad , \quad \mathcal{E}^{\perp} = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \begin{matrix} \updownarrow n \\ \updownarrow m \end{matrix} \quad , \quad \Theta = \left[\begin{array}{c|cc} F_2^T\Theta_1 F_2 & -F_1^T P & F_2^T\Theta_2 \\ \hline -PF_1 & 0 & 0 \\ \Theta_2^T F_2 & 0 & \Theta_3 \end{array}\right] \tag{12}$$

where the constraints on P are as follows

$$F_1^{\circ T} P F_1^{\circ} > 0 \ , \ F_1^T P G_1 = 0 \ , \ G_1^{\circ T} P G_1^{\circ} < 0 \tag{13}$$

and the constraints on the Θ_i matrices depend on the uncertainty set $\mathbf{\Delta}$.

4.1 *Unstructured uncertainty*

Assume the uncertainties are non-structured norm-bounded: $\Delta^T\Delta \leq 1_m$. A choice of quadratic separators for $\boldsymbol{\nabla}$ is described by (12) with the constraints (13) and

$$\Theta_1 = -\tau 1 \ , \ \Theta_2 = 0 \ , \ \Theta_3 = \tau 1 \ , \ \tau > 0 \ , \ G_2 = 0 \ . \tag{14}$$

In case $\mathcal{E}$ is full rank this separator is known to be non-conservative [22]. Applying Theorem 1 with this choice of separator, an LMI condition for robust stability of the uncertain descriptor system is directly derived.

4.2 *Scalar repeated uncertainty, real-valued case*

Consider now the structured uncertainty such that $\Delta = \delta 1_m$ with δ real and norm-bounded, $|\delta| \leq \bar{\delta}$. Parametric uncertainty is assumed: δ is an unknown constant scalar. Based on mixed P-separators and vertex-separators [13], a choice of quadratic separators for $\boldsymbol{\nabla}$ is described by (12) with the constraints (13) and

$$\begin{array}{ll} F_2^T\Theta_3F_2 \geq 0 \ , & \begin{bmatrix} G_2^T\Theta_3G_2 & G_2^T\Theta_2^*F_2 \\ F_2^T\Theta_2G_2 & F_2^T(\Theta_1+\bar{\delta}\Theta_2+\bar{\delta}\Theta_2^T+\bar{\delta}^2\Theta_3)F_2 \end{bmatrix} \leq 0 \ , \\ G_2^T\Theta_3F_2 = 0 \ , & \begin{bmatrix} G_2^T\Theta_3G_2 & G_2^T\Theta_2^*F_2 \\ F_2^T\Theta_2G_2 & F_2^T(\Theta_1-\bar{\delta}\Theta_2-\bar{\delta}\Theta_2^T+\bar{\delta}^2\Theta_3)F_2 \end{bmatrix} \leq 0 \ . \end{array} \tag{15}$$

Applying Theorem 1 with this choice of separator, we get an LMI condition for robust stability of the descriptor system with scalar, repeated, real-valued, bounded uncertainty.

Consider the following simple example

$$\begin{bmatrix} E_A & E_B \\ E_C & E_D \end{bmatrix} = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{array}\right] \ , \ \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \left[\begin{array}{cc|cc} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ \hline 1 & 0 & 0.1 & 0 \\ 0 & 1 & 0 & 0.1 \end{array}\right] .$$

This uncertain descriptor system also writes as

$$\begin{bmatrix} 1 & 2\delta_a \\ 2\delta_a & 1 \end{bmatrix} \dot{x} = (\delta_a - 1)x \ \ , \ \ \delta_a = \frac{\delta}{1-0.1\delta} \ .$$

It is quite simple to see that for $|\delta| < \frac{1}{2.1}$, $E(\delta)$ is non singular and $E(\delta)^{-1}A(\delta)$ is stable. For $\delta = \frac{1}{2.1}$ $E(\delta)$ is no longer invertible but the system is driven by

$$\dot{x}_1 + \dot{x}_2 = -\frac{1}{4}(x_1 + x_2) \quad , \quad x_1 - x_2 = 0 \; ,$$

it is asymptotically stable. For any value of $\delta > \frac{1}{2.1}$, $E(\delta)$ is non singular but $E(\delta)^{-1}A(\delta)$ is unstable.

All this analytical analysis can be done here because the example is simple. For a real problem it would be more involved. But in all cases the LMI results can be tested efficiently for example using YALMIP [18]. For the given example, the LMIs are feasible for the limit bound $\bar{\delta} = \frac{1}{2.1}$. For any larger value of $\bar{\delta}$ the LMIs are infeasible. Results prove to be non-conservative for this example. This may not be the case for all systems and there is a need for less conservative methods. One way to reduce the conservatism is to work on the separator constraints. For example, (15) is only sufficient for the separation with respect to the scalar interval uncertainty. The conservatism of this constraint may be reduced following the result of [28, Theorem 7.1] which, based on a generalization of Pólya's theorem, gives new LMI conditions for a polynomial matrix function to be positive. This result has some asymptotic properties in terms of conservatism, nevertheless it may not by itself reduce to zero the overall conservatism and another approach based on parameter-dependent Lyapunov functions may be needed. The two approaches are we believe complementary and should be combined for systems with structured uncertainties.

4.3 Scalar repeated uncertainty, real-valued case - parameter-dependent Lyapunov functions

The parametric uncertainty δ is constant, therefore it is always possible to write redundant equations such as:

$$\begin{bmatrix} E_A & E_B \\ E_C & E_D \end{bmatrix} \begin{pmatrix} x^{(i+1)} \\ z_\Delta^{(i)} \end{pmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{pmatrix} x^{(i)} \\ w_\Delta^{(i)} \end{pmatrix} \quad , \quad \begin{array}{c} x^{(i)} = s^{-1} x^{(i+1)} \\ w_\Delta^{(i)} = \delta z_\Delta^{(i)} \end{array}$$

where $\bullet^{(i)}$ is the i-th derivative of $\bullet$. Introducing derivatives up to order r, system (10) equivalently writes as

$$\begin{array}{c} z^T = \begin{pmatrix} \dot{x}^T & \dots & {x^{(r)}}^T & z_\Delta^T & \dots & {z_\Delta^{(r-1)}}^T \end{pmatrix} \\ w^T = \begin{pmatrix} x^T & \dots & {x^{(r-1)}}^T & w_\Delta^T & \dots & {w_\Delta^{(r-1)}}^T \end{pmatrix} \end{array}$$

$$\overbrace{\begin{bmatrix} \begin{bmatrix} 1_{(r-1)n} & 0_{(r-1)n\times n} \end{bmatrix} & 0 \\ 1_r \otimes E_A & 1_r \otimes E_B \\ 1_r \otimes E_C & 1_r \otimes E_D \end{bmatrix}}^{\mathcal{E}} z = \overbrace{\begin{bmatrix} \begin{bmatrix} 0_{(r-1)n\times n} & 1_{(r-1)n} \end{bmatrix} & 0 \\ 1_r \otimes A & 1_r \otimes B \\ 1_r \otimes C & 1_r \otimes D \end{bmatrix}}^{\mathcal{A}} w \quad (16)$$

with a feedback operator $\nabla = \begin{bmatrix} s^{-1}1_{rn} & 0 \\ 0 & \delta 1_{rm} \end{bmatrix}$. Based on this expanded model description an other corollary to Theorem 1 is:

Corollary 3 *Choose an integer order r. For $\mathcal{E}$ and $\mathcal{A}$ defined in (16), if there exist a matrix $P \in \mathbb{R}^{rn}$ and matrices $\Theta_{i=1,2,3} \in \mathbb{R}^{rm}$ that satisfy (3), (12), (13) and (15), then the uncertain system (10) is stable for all $\Delta = \delta 1$, $\delta \in \mathbb{R}$, $|\delta| \leq \bar{\delta}$.*

As the order r grows, the LMIs of Corollary 3 grow in numerical complexity. More precisely, the number of decision variables is given by $rn(rn+1)/2+rm(2rm+1)$ and the number of rows of the LMI constraints grow by a factor $2r$. The growth of the numerical complexity goes along with reducing conservatism. It is illustrated on the following example.

First take the example from [2] and consider the last case of one real-valued uncertainty that enters the considered framework when taking:

$$\begin{bmatrix} E_A & E_B \\ E_C & E_D \end{bmatrix} = 1_6 \ , \quad \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \left[\begin{array}{ccc|ccc} -12+\alpha & -7 & 7 & 1 & 3 & 0 \\ -11 & -13+\alpha & -5 & -2 & -1 & 2 \\ -2 & 9 & -8+\alpha & -1 & 3 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right] .$$

The aim is to evaluate a robustness margin for real valued uncertainties bounded by $\bar{\delta} = 1$, that is to maximize α under the constraint for the system to be robustly stable. For $r = 1$ the LMIs of Corollary 3 are feasible up to $\alpha_1 = 3.249$. For $r = 2$ and $r = 3$ the results are identical and give $\alpha_{2,3} = 5.4176$. Note that for $\alpha = 5.4177$ and $\delta = 1$ the system is unstable which indicates that the maximal bound is reached at a precision of 10^{-4}. These results can be compared to those of [2]. That paper builds following a comparable technique of successive LMI-based relaxations of the robust stability problem. Applied to the example, the three first estimates are respectively 3.24, 5.39 and 5.41. The dimensions of these LMI problems are comparable to those of Corollary 3.

In case of usual LTI systems ($E_A = 1$, $E_B = 0$, $E_C = 0$, $E_D = 1$), closely related results were obtained by [15]. For example, our Corollary 3 corresponds to the problem solved in their Theorem 5. For an exact comparison, their result corresponds to the following expanded model description

$$\overbrace{\left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ -C & 0 & 0 & 1 \end{array}\right]}^{\mathcal{E}} \left(\begin{array}{c} \dot{x} \\ \dot{z}_\Delta \\ \hline z_\Delta \\ \dot{z}_\Delta \end{array}\right) = \overbrace{\left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ A & 0 & B & 0 \\ 0 & 0 & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & D \end{array}\right]}^{\mathcal{A}} \left(\begin{array}{c} x \\ z_\Delta \\ \hline w_\Delta \\ \dot{w}_\Delta \end{array}\right)$$

with a feedback operator $\nabla = \begin{bmatrix} s^{-1}1_{n+m} & 0 \\ 0 & \delta 1_{2m} \end{bmatrix}$. Both this result and that of Corollary 3 read as analysis conditions where stability is proved with a parameter-dependent Lyapunov function $V_\Delta(x) = x^T P(\Delta)x$. But the choice of $P(\Delta)$ is different. In Corollary 3 stability of $\dot{x} = A(\Delta)x = (A + B\Delta(1 - D\Delta)^{-1}C)x$ is assessed by

$$P(\Delta) = \begin{bmatrix} A(\Delta) \\ 1 \end{bmatrix}^T P \begin{bmatrix} A(\Delta) \\ 1 \end{bmatrix}$$

while in [15] the parameter-dependent Lyapunov matrix is such that

$$P(\Delta) = \begin{bmatrix} \Delta(1 - D\Delta)^{-1}C \\ 1 \end{bmatrix}^T P \begin{bmatrix} \Delta(1 - D\Delta)^{-1}C \\ 1 \end{bmatrix} .$$

Note that this last type of Lyapunov matrix includes the former. Moreover, on the given example conditions of [15] are feasible up to $\alpha = 5.4176$.

4.4 Scalar repeated uncertainty, complex-valued case

Consider the uncertainty with identical diagonal structure $\Delta = \delta 1_m$ with δ complex-valued and norm-bounded, $|\delta| \leq \bar{\delta}$. Based on mixed P-separator and D-scalling [13], a choice of quadratic separators for $\boldsymbol{\nabla}$ is described by (12) with the constraints (13) and

$$\Theta_1 = -Q \ , \ \Theta_2 = 0 \ , \ \Theta_3 = Q \ , \ F_2^{\circ T} Q F_2^\circ > 0 \ , \ F_2^T Q G_2 = 0 \ , \ G_2^{\circ T} Q G_2^\circ < 0 \ . \tag{17}$$

Based on this description of the separator one gets the following corollary.

Corollary 4 *Choose an integer order r. For $\mathcal{E}$ and $\mathcal{A}$ defined in (16), if there exist a matrix $P \in \mathbb{R}^{rn}$ and a matrix $Q \in \mathbb{R}^{rm}$ that satisfy (3), (12), (13) and (17), then the uncertain system (10) is stable for all $\Delta = \delta 1$, $\delta \in \mathbb{C}$, $|\delta| \leq \bar{\delta}$.*

As an illustration take again the example from [2] with $|\delta| \leq 1$, complex valued. For $r = 1$ and $r = 2$, Corollary 4 gives respectively 3.249 and 4.147 as maximal admissible bounds on the robustness margin α. Increasing $r \geq 3$ does not bring any improvement. For comparison the same bounds 3.24 and 4.14 where found in [2]. This result is in fact expected because, for the case when $E = 1$ (usual LTI systems) with a single scalar complex parameter, the result of Corollary 4 is identical to that in [2]. This reference moreover proves that the conservatism of the method vanishes asymptotically as r grows to infinity.

5 Stability of descriptor time-delay systems

Consider the following time-delay descriptor system

$$E\dot{x}(t) + E_h\dot{x}(t-h) = Ax(t) + A_h x(t-h)\ . \tag{18}$$

Define $\eta(t) = Ex(t) + E_h x(t-h)$. The model matches the framework of Figure 1 if one considers

$$\overbrace{\begin{bmatrix} 1 & -A \\ 0 & E \end{bmatrix}}^{\mathcal{E}} \begin{pmatrix} \dot{\eta}(t) \\ x(t) \end{pmatrix} = \overbrace{\begin{bmatrix} 0 & A_h \\ 1 & -E_h \end{bmatrix}}^{\mathcal{A}} \begin{pmatrix} \eta(t) \\ x(t-h) \end{pmatrix} \tag{19}$$

along with the set: $\boldsymbol{\nabla} = \left\{ \begin{bmatrix} s^{-1}1_n & 0 \\ 0 & e^{-hs}1_n \end{bmatrix} \ : \ s^{-1} \in \mathbb{C}^+ \right\}$.

5.1 Delay-independent case

Delay-independent stability is achieved if the system is stable for all delays $h \in [0\ +\infty[$. In that case, the delay operator e^{-hs} is equivalent to a complex norm-bounded uncertainty $e^{-hs} = \delta$, $|\delta| \leq 1$. Therefore this case is a subcase of the previously considered problem of robust stability analysis with scalar repeated complex valued uncertainty, and hence all previous results apply. Note that for $E = 1$, the expanded versions based on Corollary 4 are exact reformulations of results in [1]. Moreover, that last reference proves that as r grows, the conservatism of the method vanishes.

5.2 Delay-dependent case

The delay-dependent case amounts to proving asymptotic stability for all bounded delays $0 \leq h \leq \bar{h}$. To do so, papers such as [34,8] introduce the following bounded operator $\delta_h = s^{-1}(1 - e^{-hs})$ that operates on the system signals as

$$\delta_h[\dot{\eta}(t)] = \eta(t) - \eta(t-h) \; .$$

This operator is bounded such that:

$$|\delta_h| \leq \bar{h} \;\; , \;\; \forall \;\; s \in \mathbb{C}^+ \;\; , \;\; 0 \leq h \leq \bar{h} \; . \tag{20}$$

Bounding in this way the operator amounts to an approximation that may be reduced if considering fractions of the delay h:

$$|\delta_{h/r}| \leq \frac{\bar{h}}{r} \;\; , \;\; \forall \;\; s \in \mathbb{C}^+ \;\; , \;\; 0 \leq h \leq \bar{h} \; . \tag{21}$$

As the fractioning integer r goes to infinity the bounded approximation $|\delta_{h/r}|$ tends to zero . Based on these considerations, for a given r, we introduce the signals $x(t - \frac{ih}{r})$ where $i \in \{0 \ldots r+1\}$ and the augmented system signals:

$$v(t) = \begin{pmatrix} x(t-\frac{h}{r}) \\ x(t-\frac{2h}{r}) \\ \vdots \\ x(t-h) \end{pmatrix} \;\; , \;\; z(t) = \begin{pmatrix} \dot{\eta}(t) \\ x(t) \\ v(t) \\ \dot{\eta}(t) \end{pmatrix} \;\; , \;\; w(t) = \begin{pmatrix} \eta(t) \\ v(t) \\ x(t - \frac{(r+1)h}{r}) \\ \eta(t) - \eta(t-\frac{h}{r}) \end{pmatrix} \; .$$

These signals make system (18) match the framework of Figure 1 with:

$$\overbrace{\begin{bmatrix} 1 & -A & 0 & 0 \\ 0 & E & 0 & 0 \\ 0 & 0 & 1_{rn} & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}^{\mathcal{E}} z = \overbrace{\begin{bmatrix} 0 & \begin{bmatrix} 0 \cdots 0 & A_h \end{bmatrix} & 0 & 0 \\ 1 & \begin{bmatrix} 0 \cdots 0 & -E_h \end{bmatrix} & 0 & 0 \\ 0 & 1_{rn} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & \begin{bmatrix} E & 0 \cdots 0 \end{bmatrix} & E_h & 1 \end{bmatrix}}^{\mathcal{A}} w \tag{22}$$

and an uncertainty set defined as:

$$\boldsymbol{\nabla} = \left\{ \begin{bmatrix} s^{-1} 1_n & 0 & 0 \\ 0 & e^{-\frac{hs}{r}} 1_{(r+1)n} & 0 \\ 0 & 0 & \delta_{h/r} 1_n \end{bmatrix} \; : \; s^{-1} \in \mathbb{C}^+ \; , \; 0 \leq h \leq \bar{h} \right\} .$$

Define the following row partitioning

$$\mathcal{E}^{\circledast} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} \begin{matrix} \updownarrow n \\ \updownarrow (r+1)n \\ \updownarrow n \end{matrix} \ , \ \mathcal{E}^{\perp} = \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix} \begin{matrix} \updownarrow n \\ \updownarrow (r+1)n \\ \updownarrow n \end{matrix} \ . \tag{23}$$

Based on mixed P-separator, D-scallings [13], a choice of quadratic separators for ∇ is described by

$$\Theta = \left[\begin{array}{c|ccc} -F_2^T Q F_2 - \frac{\bar{h}^2}{r^2} F_3^T R F_3 & -F_1^T P & 0 & 0 \\ \hline -PF_1 & 0 & 0 & 0 \\ 0 & 0 & Q & 0 \\ 0 & 0 & 0 & R \end{array}\right] \tag{24}$$

and constrained by

$$\begin{array}{c} F_1^{\circ T} P F_1^{\circ} > 0 \ , \ F_2^{\circ T} Q F_2^{\circ} > 0 \ , \ F_3^{\circ T} R F_3^{\circ} > 0 \ , \\ F_1^T P G_1 = 0 \ , \ F_2^T Q G_2 = 0 \ , \ F_3^T R G_3 = 0 \ , \\ G_1^{\circ T} P G_1^{\circ} < 0 \ , \ G_2^{\circ T} Q G_2^{\circ} < 0 \ , \ G_3^{\circ T} R G_3^{\circ} < 0 \ . \end{array} \tag{25}$$

Corollary 5 *Choose an integer fractioning r. For $\mathcal{E}$ and $\mathcal{A}$ defined in (22), if there exist a matrix $P \in \mathbb{R}^n$, a matrix $Q \in \mathbb{R}^{(r+1)n}$ and a matrix $R \in \mathbb{R}^n$ that satisfy (3), (24) and (25), then the time-delay system (18) is stable for all $0 \leq h \leq \bar{h}$.*

Here again, as the fractioning r grows, the LMIs of Corollary 5 grow in numerical complexity and this goes along with reducing conservatism. It is illustrated on the following example.

To illustrate this result consider the time delay system defined by

$$E = 1 \ , \ E_h = 0 \ , \ A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} \ , \ A_h = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} \tag{26}$$

This example has for long been used to illustrate delay-dependent results. The left hand-side of Table 1 summarizes these previously published results by giving the reference, the obtained bound and the number of variables involved in the corresponding LMIs. The right hand-side gives the results of Corollary 5 for various fractionings.

On the considered example the proposed method shows to be more effective, except compared to that of [35]. In this last reference, the improvement is essentially due to a Padé approximation of e^{-hs} which goes further than the basic operator $s^{-1}(1-$

Table 1
Maximum allowable delay

Methods	h_{max}	nb vars	Corollary 5	h_{max}	nb vars
[16]	0.8571	9	r=1	4.4721	9
[25]	0.99	11	r=2	5.71	16
[24]	4.3588	16	r=3	5.96	27
[9]	4.4721	9/18	r=4	6.05	42
[31]	4.4721	17	r=9	6.149	177
[30]	4.4721	38	r=15	6.164	471
[35]	6.150	81	r=25	6.169	1281
Theoretical bound	6.172	∞	r=30	6.171	1836

e^{-hs}). Combining the Padé approximation and the proposed fractioning scheme is a promising approach currently under investigation.

6 Conclusion

A novel quadratic separation framework for feedback connected systems with implicit linear transformation is described. Directly related robust stability analysis results are derived for descriptor and time-delay systems. Only two special cases of uncertainties were considered but extensions can be obtained for more complex and time-varying structured uncertainties following results of [15,5]. Results for delay-dependent systems are given for the case of systems with one single delay and without uncertainties but extensions are trivial for such more involved problems. The procedure amounts to combinations of constraints on the quadratic separator. Prospective work will be dedicated to analysing the relative conservatism of several independent results when stability is proved with parameter-dependent Lyapunov certificates.

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