LMI-BASED ANALYSIS OF ROBUST ADAPTIVE CONTROL FOR LINEAR SYSTEMS WITH TIME-VARYING UNCERTAINTY*

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November 4, 2008

Abstract
Passification-based adaptive control, also known as simple adaptive control, is studied with respect to its robustness to time-varying uncertainties. Results are formulated in terms of LMIs and are therefore testable in polynomial time using semi-definite programming solvers. The main result shows that the adaptive strategy allows, without measurement nor estimation of the uncertain parameters to guarantee asymptotic stability for a wide rage of these parameters. To achieve this result, the stability property is relaxed: convergence is proved to a small neighborhood of the origin, and attractive domain. It is also demonstrated that this attractor can be made as small as required the only limitation being implementation constraints.

Keywords
Adaptive control, Robustness, LMI, Passification, Attractive domains.

1 Introduction

Adaptive control schemes are conceived with the idea that online modification of control algorithms may be needed when the process dynamics are partially known and/or varying. This concept has lead to many adaptive strategies [2, 4, 8, 9, 11]. Some of these rely on parameter estimation to tune the control algorithms and in general assume time variations sufficiently slow for separation principle to be valid. Among these one can cite gain scheduling of Linear Parameter Varying (LPV) control (see [12, 18] for surveys). Another strategy takes advantage of passivity properties to perform direct tuning of controller gains by means of a differential equation. This passification-based strategy can be found as soon as 1974 for linear systems [5, 6, 1] and later extended to non-linear systems [19, 20, 10, 7]. Since it does not need any parameter estimation, it is also sometimes called simple adaptive control [11].

Although robustness to uncertainties and perturbations is an expected feature of adaptation control

*This work is supported by CNRS-RFFI research cooperation program PICS No. 4281.

which has been illustrated on numerical simulations and applications, up to recently, very few results were available for formal proof of this property. To achieve such demonstration of robustness new linear matrix inequality (LMI) based results inspired from robust control community have been produced. As a starting point these tackle the case of uncertain linear systems where uncertainties were assumed constant [17, 22, 14, 15]. With such assumptions, the adaptive control law is demonstrated to be robust to given uncertainty sets provided that a stabilizing static parameter-dependent control exists. LMI conditions for this property to hold are provided making it numerically tractable to prove robust stabilization and passification via simple adaptive control. Within the same LMI framework, extensions for disturbance rejection are given in [3, 16]. In this last paper, the adaptive strategy is proved to be at least as good in terms of L_2-gain attenuation as any virtual parameter-dependent control strategy of the type used in the demonstration.

But all these publications scarcely deal with the case of time-varying uncertainties. In [22] a section is dedicated to the question. It gives indications on how to consider the question. But no formal result with a Lyapunov certificate is provided. Adaptation schemes are by definition supposed to "adapt" to variations on the process operating conditions, but, although this issue is of major importance, there is at our knowledge no demonstration that the equilibrium point of the closed-loop system is asymptotically stable. Except for one situation: in [11] it is shown under some assumptions that if a constant static output feedback control passifies the system, then the simple adaptive control does it as well. But why do adaptive control if static control achieves the same performances?

We answer to this issue in this paper at the expense of relaxation of the stability property. Indeed adaptive control is more robust than static control, but in case of time-varying parameters, convergence of the state to the origin cannot be proved. It is replaced by convergence to a small neighborhood of the origin, an attractive domain. The reason for this is that the derivative of the Lyapunov certificate, which depends of the derivative of the parameters, may happen not to be negative near the origin. Fortunately, the attractor can be made as small as required by tuning the adaptation law. It makes asymptotic behavior convenient in practice.

The outline of the paper is as follows. First, a section is devoted to notations and problem statement. Section 3 formulates the LMI-based result of the paper and gives some comments. The two last sections are devoted to a numerical example and to conclusions.

2 Problem definition

Notations: $\mathbb{R}^{m\times n}$ is the set of m-by-n real matrices. A^T is the transpose of the matrix A. $\mathbf{1}$ and $\mathbf{0}$ are respectively the identity and the zero matrices of appropriate dimensions. For symmetric matrices, $A > (\geq) B$ if and only if $A - B$ is positive (semi) definite. For a square matrix $\langle A\rangle$ stands for the symmetric part $\langle A\rangle = A + A^T$.

The paper considers passification of uncertain LTI systems described in state-space as:

$$\dot{x} = \overbrace{\left[A_0 + \sum_{p=1}^{\bar{p}} \delta_p A_p\right]}^{A(\delta)} x + \overbrace{\left[B_0 + \sum_{p=1}^{\bar{p}} \delta_p B_p\right]}^{B(\delta)} u, \quad y = \overbrace{\left[C_0 + \sum_{p=1}^{\bar{p}} \delta_p C_p\right]}^{C(\delta)} x \; . \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control, $y \in \mathbb{R}^l$ is the measurement output. $\delta = \begin{pmatrix} \delta_1 & \dots & \delta_{\bar{p}} \end{pmatrix}$ is the vector of scalar time-varying uncertainties (which may be approximations of non-linearities) assumed to have

- bounded values $\underline{\delta}_p \leq \delta_p(t) \leq \overline{\delta}_p$,
- bounded time derivatives $\underline{\vartheta}_p \leq \dot{\delta}_p(t) \leq \overline{\vartheta}_p$.

The values $\underline{\delta}_p$, $\overline{\delta}_p$, $\underline{\vartheta}_p$ and $\overline{\vartheta}_p$ are assumed to be given. For the problem to be well-defined it is assumed that $\underline{\vartheta}_p \leq 0$ and $\overline{\vartheta}_p \geq 0$. Let $\mathbf{\Delta} = \{ \delta \ : \ \underline{\delta}_p \leq \delta_p(t) \leq \overline{\delta}_p \ , \ \underline{\vartheta}_p \leq \dot{\delta}_p(t) \leq \overline{\vartheta}_p \}$ be the set of admissible uncertainties. Define as well $\bar{\mathbf{\Delta}} = \{ (\delta, \vartheta) \ : \ \delta_p \in \{\underline{\delta}_p, \overline{\delta}_p\} \ , \ \vartheta_p \in \{\underline{\vartheta}_p, \overline{\vartheta}_p\} \}$ the set of $2^{2\bar{p}}$ vertices of the admissible values of couples $(\delta, \dot{\delta})$.

The considered control objective is to achieve some passivity property which includes asymptotic stability of the system state x. Two control strategies are adopted and compared. One is parameter-dependent static output-feedback (PDSOF) with bounded gain

$$u = \overbrace{\left[F_0 + \sum_{p=1}^{\bar{p}} \delta_p F_p \right]}^{F(\delta)} y + w \quad , \quad \text{Trace}(F^T(\delta)F(\delta)) \leq \alpha \ . \tag{2}$$

and supposes measurement or estimation of the uncertain parameters δ. The second, is simple adaptive control (SAC) with guaranteed bounded gains defined as

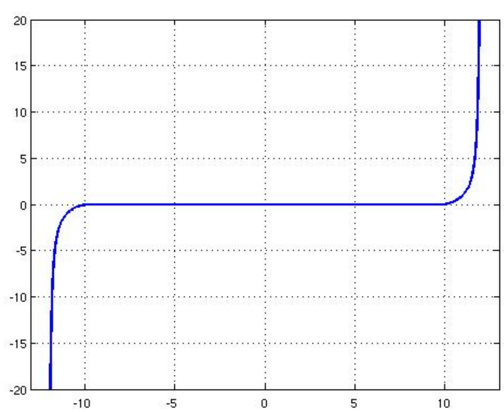
$$\begin{array}{l} u(t) = K(t)y(t) + w(t) \\ \dot{K}(t) = -Gy(t)y^T(t)\Gamma - \phi(K(t))\Gamma \end{array} \tag{3}$$

where $\phi(K) = \psi(\text{Tr}(K^T K)) \cdot K$ and ψ is a dead-zone type scalar function with logarithmic barrier behavior such that:

$$\left\{ \begin{array}{ll} \psi(k) = \mathbf{0} & \text{if} \ \ k \leq \alpha \\ \psi(k) = \frac{k-\alpha}{\alpha\beta - k} & \text{if} \ \ \alpha \leq k < \alpha\beta \end{array} \right.$$

where $\alpha > 0$ and $\beta > 1$. The function is as represented in Figure 1. Throughout the paper, closed-loop properties will be studied for any valid initial conditions such that $(x(0), K(0)) \in \mathcal{XK}$ where $\mathcal{XK} = \{(x, K) \in \mathbb{R}^n \times \mathbb{R}^{m \times l} \ : \ \text{Tr}(K^T K) < \alpha\beta\}$.

Figure 1: The dead-zone function ψ for $\alpha = 10$ and $\beta = 12/10$



The SAC law is conceived such that the term $-Gy(t)y^T(t)\Gamma$ tunes in real time the feedback gain $K(t)$ in order, hopefully, to converge towards stabilizing values. Conditions for that property to hold are given in the next section. The second term $-\phi(K(t))\Gamma$ is intended to guarantee bounded values for $K(t)$. In practice it acts on convergence of gain $K(t)$ to the bounded region $\text{Tr}(K^T K) \leq \alpha$ and the logarithmic-type barrier forces for all instants the bound $\text{Tr}(K^T K) \leq \alpha\beta$. As shown in [2, 17] such term as $-\phi(K(t))\Gamma$ also contributes to robustness with respect to external disturbances, in particular to noise on the measurements. Some properties of ϕ are now given.

Lemma 1 *If $y(t)$ is bounded for all $t \geq 0$, then $Tr(K^T(t)K(t)) < \alpha\beta$ holds for all $t \geq 0$ and all valid initial condition $(x(0), K(0)) \in \mathcal{XK}$.*

Proof: Consider the following Lyapunov function $V(K) = \mathrm{Tr}(KK^T)$. Its derivatives along the trajectories of (1) with feedback (3) writes

$$\begin{array}{rl} \dot{V}(K) & = -2\mathrm{Tr}(Gyy^T\Gamma K^T + \phi(K)\Gamma K^T) \\ & = -y^T(\Gamma K^T G + G^T K \Gamma)y - 2\frac{\mathrm{Tr}(K^TK)-\alpha}{\alpha\beta - \mathrm{Tr}(K^TK)}\mathrm{Tr}(K\Gamma K^T) \ . \end{array}$$

The last term in this formula goes to infinity as $\mathrm{Tr}(K^TK)$ goes to $\alpha\beta$. Therefore, for bounded values of y and K there exists a scalar k such that $\dot{V}(K) < 0$ for all K such that $k \leq \mathrm{Tr}(K^TK) < \alpha\beta$, *i.e.* the trajectories are decreasing and $\mathrm{Tr}(K^TK)$ cannot exceed $\alpha\beta$. ∎

Lemma 2 *For all (F, K) satisfying $Tr(F^TF) \leq \alpha$ and $Tr(K^TK) < \alpha\beta$, the inequality $Tr(\phi(K)(K - F)^T) \geq 0$ holds.*

Proof: See [16]. ∎

The control objectives are to guarantee stability of the closed-loop system and an intermediate passivity property. Passivity is defined here with respect to some input/output signals w/z where $z = Gy + D(\delta)w$ and $D(\delta)$ is some feed-through gain kept free for the design. The matrix G is assumed to be given and is the one matrix of SAC (3).

Passification (making the closed-loop passive) is looked for by means of either linear PDSOF (2) or non-linear SAC (3). In both cases the closed-loop system enters the format

$$\dot{\eta} = f(\eta, \delta) + \hat{B}(\delta)w \quad , \quad y = \hat{C}(\delta)\eta \tag{4}$$

where $\eta = x$ in case of static feedback and $\eta = (x^T \ k^T)^T$ in case of adaptive feedback where k is the vector build by concatenation of all columns of K. In this last case f is a non-linear function. In the latter case $f(\eta, \delta) = A(F, \delta)x$ where $A(F, \delta) = A(\delta) + B(\delta)F(\delta)C(\delta)$.

Definition 1 *If there exists a parameter-dependent nonnegative scalar function V and a non-negative scalar function ρ such that*

$$\dot{V}(\eta(t), \delta(t)) \leq w(t)^T z(t) - \rho(\eta(t), \delta(t)) \ . \tag{5}$$

holds for all admissible uncertainties $\delta \in \mathbf{\Delta}$, all $t \geq 0$ and all trajectories starting from valid initial conditions $(x(0), K(0)) \in \mathcal{X}K$, then

- *the closed-loop system is said to be robustly passive with respect to the signals w, z,*
- *if $\rho(\eta)$ is strictly positive for all $\eta \neq 0$ the closed-loop system is said to be robustly strictly passive with respect to the signals w, z,*
- *if $\rho(\eta)$ is strictly positive for all $x \neq 0$ the closed-loop system is said to be robustly x-strictly passive with respect to the signals w, z.*

All definitions are global, *i.e.* they hold for all admissible initial conditions $(x(0), K(0)) \in \mathcal{X}K$. The adjective global is eluded in the remaining to alleviate the text. All passivity conditions imply system stability (decreasing Lyapunov function for any zero input). Additionally, strict passivity implies asymptotic stability and x-strict passivity implies that for zero inputs $w = 0$ the state x converges to zero.

As demonstrated in [16], x-strict passivity is achievable via SAC for the case of constant uncertainties ($\vartheta_{p=1\dots\bar{p}} = 0$). But, for the considered case of time-varying uncertainties, asymptotic convergence to the origin cannot be proved and objectives should be relaxed, the convergence to the origin being replaced by convergence to a small neighborhood of the origin.

Definition 2 *Given a positive definite matrix Q, the system (1) in closed-loop with SAC (3) is said to satisfy property* **P1** *with respect to the neighborhood $x^T Q x \leq 1$ and signals w, z if there exist a parameter-dependent positive definite matrix $P(\delta)$, a positive scalar $\tau > 0$, a parameter-dependent non-negative function $W(K, \delta)$, and a function $\rho(\eta)$ strictly positive for all $x \neq 0$, such that*

- *i) (5) holds with $V(\eta, \delta) = \frac{1}{2} x^T P(\delta) x + W(K, \delta)$ for all admissible uncertainties $\delta \in \mathbf{\Delta}$, all trajectories with valid initial conditions $(x(0), K(0)) \in \mathcal{X}K$ and all $t \geq 0$ as long as $x^T(t) P(\delta) x(t) > \tau$ holds,*
- *ii) $Q \leq \frac{1}{\tau} P(\delta)$ for all admissible uncertainties $\delta \in \mathbf{\Delta}$.*

Property **P1** has the same characteristics as x-strict passivity (the passivity and stability properties are global), except that asymptotic convergence is towards an ellipsoidal neighborhood of the origin defined by $x^T Q x \leq 1$. Asymptotic convergence is proved for zero disturbances $w = 0$ since for all states outside the ellipsoidal neighborhood these belong to Lyapunov equipotentials

$$\tau < \tau x^T Q x \leq x^T P(\delta) x \leq V(\eta, \delta)$$

with decreasing behavior $\dot{V}(\eta, \delta) < 0$.

3 Closed-loop passivity

Theorem 1 *If there exists two matrices H_1, H_2 a scalar $\epsilon > 0$, and $5(\bar{p}+1)$ matrices $P_{p=0\dots\bar{p}}$, $F_{p=0\dots\bar{p}}$, $D_{p=0\dots\bar{p}}$, $R_{p=0\dots\bar{p}}$, $T_{p=0\dots\bar{p}}$ such that the following conditions hold for all $(\delta, \vartheta) \in \bar{\mathbf{\Delta}}$*

$$P(\delta) > \mathbf{0} \tag{6}$$

$$\begin{bmatrix} R(\delta) & -C^T(\delta) G^T & P(\delta) \\ -GC(\delta) & \mathbf{1} & \mathbf{0} \\ P(\delta) & \mathbf{0} & \mathbf{0} \end{bmatrix} + \left\langle H_1 \begin{bmatrix} \mathbf{0} & B(\delta) & -\mathbf{1} \end{bmatrix} \right\rangle \geq \mathbf{0} \tag{7}$$

$$\begin{bmatrix} T(\delta) & F^T(\delta) \\ F(\delta) & \mathbf{1} \end{bmatrix} \geq \mathbf{0} \quad , \quad Tr(T(\delta)) \leq \alpha \tag{8}$$

$$L(\delta) + \left\langle H_2 \begin{bmatrix} A(\delta) & B(\delta) & -\mathbf{1} & \mathbf{0} \\ C(\delta) & \mathbf{0} & \mathbf{0} & -\mathbf{1} \end{bmatrix} \right\rangle \leq \mathbf{0} \tag{9}$$

where $L(\delta) =$

$$\begin{bmatrix} R(\delta) + \tilde{P}(\vartheta) + \epsilon P(\delta) & -C^T(\delta) G^T & P(\delta) & \mathbf{0} \\ -GC(\delta) & -\langle D(\delta) \rangle & \mathbf{0} & \mathbf{0} \\ P(\delta) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \alpha\beta\mathbf{1} + \langle G^T F(\delta) \rangle \end{bmatrix}$$

and

$$\begin{array}{ll} P(\delta) = P_0 + \sum_{p=1}^{\bar{p}} \delta_p P_p & \tilde{P}(\vartheta) = \sum_{p=1}^{\bar{p}} \vartheta_p P_p \\ F(\delta) = F_0 + \sum_{p=1}^{\bar{p}} \delta_p F_p & D(\delta) = D_0 + \sum_{p=1}^{\bar{p}} \delta_p D_p \\ R(\delta) = R_0 + \sum_{p=1}^{\bar{p}} \delta_p R_p & T(\delta) = T_0 + \sum_{p=1}^{\bar{p}} \delta_p T_p \end{array}$$

then define $z = Gy + D(\delta) w$

- *the PDSOF (2) robustly x-strictly passifies the system with respect to the signals w, z,*
- *there exists a matrix Q such that the SAC (3) satisfies property* **P1** *with respect to the neighborhood $x^T Q x \leq 1$ and signals w, z.*

An admissible matrix Q can be computed applying the following 2-step algorithm

1- *Choose a scalar τ subject to the following LMI constraints (the variables are the matrices $\underline{S}_p$, $\overline{S}_p$ and the scalars $\underline{\epsilon}_p$, $\overline{\epsilon}_p$)*

$$\tau \geq \frac{1}{\epsilon} \sum_{p=1}^{\bar{p}} \max(\underline{\vartheta}_p \, Tr(\underline{S}_p), \overline{\vartheta}_p \, Tr(\overline{S}_p)) \tag{10}$$

where for all $p = 1 \dots \bar{p}$ and all $\delta \in \bar{\mathbf{\Delta}}$:

$$\begin{array}{l} \underline{S}_p \leq \mathbf{0} \ , \ \left[\begin{array}{cc} \underline{S}_p + \left\langle F_p \Gamma^{-1} F^T(\delta) \right\rangle + \underline{\epsilon}_p \alpha \beta \mathbf{1} & F_p \Gamma^{-1} \\ \Gamma^{-1} F_p^T & -\underline{\epsilon}_p \mathbf{1} \end{array} \right] \leq \mathbf{0} \\ \overline{S}_p \geq \mathbf{0} \ , \ \left[\begin{array}{cc} \overline{S}_p + \left\langle F_p \Gamma^{-1} F^T(\delta) \right\rangle - \overline{\epsilon}_p \alpha \beta \mathbf{1} & F_p \Gamma^{-1} \\ \Gamma^{-1} F_p^T & \overline{\epsilon}_p \mathbf{1} \end{array} \right] \geq \mathbf{0} \end{array} \tag{11}$$

2- *Admissible values of Q are such that the LMI constraint $\tau Q \leq P(\delta)$ holds for all vertices $\delta \in \bar{\mathbf{\Delta}}$.*

Proof: Note that the LMIs (6), (7), (8) and (9) are all linear with respect to (δ, ϑ). Therefore if the LMIs hold for all vertices $(\delta, \vartheta) \in \bar{\mathbf{\Delta}}$ then they also hold for all the elements of their convex hull $\delta \in \mathbf{\Delta}$. Inequality (6) therefore implies that the parameter-dependent function $V_1(x, \delta) = \frac{1}{2} x^T P(\delta) x$ is positive definite. Let the matrices

$$M_1(\delta) = \left[\begin{array}{ccc} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & B^T(\delta) \end{array} \right] \ , \ M_2(\delta) = \left[\begin{array}{cccc} \mathbf{1} & \mathbf{0} & A^T(\delta) & C^T(\delta) \\ \mathbf{0} & \mathbf{1} & B^T(\delta) & \mathbf{0} \end{array} \right] .$$

Pre and post multiply (7) by $M_1(\delta)$ and its transpose respectively. The operation eliminates the slack variable H_1 and implies that for all $\delta \in \mathbf{\Delta}$:

$$\left[\begin{array}{cc} R(\delta) & P(\delta) B(\delta) - C^T(\delta) G^T \\ B^T(\delta) P(\delta) - G C(\delta) & \mathbf{1} \end{array} \right] \geq \mathbf{0}. \tag{12}$$

Pre and post multiply (9) by $M_2(\delta)$ and its transpose respectively. The operation eliminates the slack variable H_2 and implies that for all $\delta \in \mathbf{\Delta}$:

$$\left[\begin{array}{cc} R(\delta) + \tilde{P}(\dot{\delta}) + \epsilon P(\delta) + \hat{L}(\delta) & P(\delta) B(\delta) - C^T(\delta) G^T \\ B^T(\delta) P(\delta) - G C(\delta) & -\left\langle D(\delta) \right\rangle \end{array} \right] \leq \mathbf{0} \tag{13}$$

where $\hat{L}(\delta) = \left\langle P(\delta) A(\delta) \right\rangle + C^T(\delta) \left(\alpha \beta \mathbf{1} + \left\langle G^T F(\delta) \right\rangle \right) C(\delta)$.

Based on the fact that (12), (8) and (13) hold for all $\delta \in \mathbf{\Delta}$, we shall now prove that the PDSOF x-strictly passifies the system.

Pre and post multiply inequality (12) by $\left[\begin{array}{cc} \mathbf{1} & -C^T(\delta) F^T(\delta) \end{array} \right]$ and by its transpose respectively to get:

$$\begin{array}{l} C^T(\delta) \left\langle G^T F(\delta) \right\rangle C(\delta) + R(\delta) \\ \qquad \geq \left\langle P(\delta) B(\delta) F(\delta) C(\delta) \right\rangle - C^T(\delta) F^T(\delta) F(\delta) C(\delta) \ . \end{array} \tag{14}$$

A Schur complement argument on (8) gives that $\mathrm{Tr}(F^T(\delta) F(\delta)) \leq \mathrm{Tr}(T(\delta)) \leq \alpha$. It implies that $F^T(\delta) F(\delta) \leq \alpha \mathbf{1}$. As $\beta > 1$ one gets $F^T(\delta) F(\delta) - \alpha \beta \mathbf{1} \leq \mathbf{0}$. Combining this last inequality and inequality (14) with (13) implies that

$$\left[\begin{array}{cc} \left\langle P(\delta) A(F, \delta) \right\rangle + P(\dot{\delta}) + \epsilon P(\delta) & P(\delta) B(\delta) - C^T(\delta) G^T \\ B^T(\delta) P(\delta) - G C(\delta) & -\left\langle D(\delta) \right\rangle \end{array} \right] \leq \mathbf{0} \tag{15}$$

where $A(F,\delta) = A(\delta) + B(\delta)F(\delta)C(\delta)$ stands for the dynamics matrix of the closed-loop system. Pre and post multiply the matrix inequality (15) by $\left(\begin{array}{cc} x^T & w^T \end{array} \right)$ and its transpose respectively, to get

$$2x^T P(\delta)(A(F,\delta)x + B(\delta)w) + x^T \tilde{P}(\dot{\delta})x \leq 2z^T w - \epsilon x^T P(\delta)x \tag{16}$$

and hence $\dot{V}_1(x,\delta) \leq z^T w - \frac{\epsilon}{2} x^T P(\delta)x$. The function V_1 proves that the PDSOF x-strictly passifies the system with respect to the signals w, z.

Let us now prove the passifying property **P1** of SAC. The storage function to be used is

$$V(\eta,\delta) = V_1(x,\delta) + \frac{1}{2}\mathrm{Tr}\left((K - F(\delta))\Gamma^{-1}(K - F(\delta))^T\right)$$

where $\Gamma > \mathbf{0}$ is a positive definite matrix. The derivative of V along the trajectories of the closed-loop system with SAC is

$$\begin{array}{c} \dot{V}(\eta,\delta) = x^T P(\delta)(A(\delta)x + B(\delta)Ky + B(\delta)w) + \frac{1}{2} x^T \tilde{P}(\dot{\delta})x \\ +\mathrm{Tr}\left((\dot{K} - \tilde{F}(\dot{\delta}))\Gamma^{-1}(K - F(\delta))^T\right) \end{array}$$

where $\tilde{F}(\dot{\delta}) = \sum_{p=1}^{\bar{p}} \dot{\delta}_p F_p$. Pre and post multiply the matrix inequality (13) by $\left(\begin{array}{cc} x^T & w^T \end{array} \right)$ and its transpose respectively, to get

$$\begin{array}{ll} 2x^T P(\delta)(A(\delta)x + B(\delta)w) + x^T \tilde{P}(\dot{\delta})x & \\ +x^T R(\delta)x + \alpha\beta y^T y + 2y^T G^T F(\delta)y & \leq 2z^T w - \epsilon x^T P(\delta)x \end{array} \tag{17}$$

and therefore

$$\begin{array}{rl} \dot{V}(\eta,\delta) \leq & z^T w - \frac{\epsilon}{2} x^T P(\delta)x + x^T P(\delta)B(\delta)Ky - \frac{1}{2} x^T R(\delta)x \\ & -\frac{1}{2}\alpha\beta y^T y - y^T G^T F(\delta)y \\ & +\mathrm{Tr}\left((\dot{K} - \tilde{F}(\dot{\delta}))\Gamma^{-1}(K - F(\delta))^T\right) \end{array}$$

Pre and post multiply inequality (12) by $\left[\begin{array}{cc} x^T & -y^T K^T \end{array} \right]$ and by its transpose respectively to get

$$2x^T P(\delta)B(\delta)Ky - x^T R(\delta)x \leq 2y^T K^T Gy + y^T K^T Ky \tag{18}$$

and therefore

$$\begin{array}{rl} \dot{V}(\eta,\delta) \leq & z^T w - \frac{\epsilon}{2} x^T P(\delta)x + \frac{1}{2} y^T (K^T K - \alpha\beta\mathbf{1})y + y^T G^T (K - F(\delta))y \\ & +\mathrm{Tr}\left((\dot{K} - \tilde{F}(\dot{\delta}))\Gamma^{-1}(K - F(\delta))^T\right) \end{array}$$

Note the following result based on the fact that $\mathrm{Tr}(M_1 M_2) = \mathrm{Tr}(M_2 M_1)$:

$$y^T (K - F(\delta))^T Gy = \mathrm{Tr}(y^T (K - F(\delta))^T Gy) = \mathrm{Tr}(Gyy^T (K - F(\delta))^T) \ .$$

Therefore, replacing $\dot{K}$ by its value, one obtains

$$y^T (K - F(\delta))^T Gy + \mathrm{Tr}\left(\dot{K}\Gamma^{-1}(K - F(\delta))^T\right) = -\mathrm{Tr}\left(\phi(K)(K - F(\delta))^T\right)$$

which is negative due to Lemma 2. Moreover, Lemma 1 guarantees that $\mathrm{Tr}(K^T K) \leq \alpha\beta$ and hence $K^T K - \alpha\beta\mathbf{1} \leq \mathbf{0}$. The derivative of the storage function function along the closed-loop trajectories is therefore, for all $t \geq 0$, bounded by:

$$\dot{V}(\eta,\delta) \leq z^T w - \frac{\epsilon}{2} x^T P(\delta)x - \mathrm{Tr}\left(\tilde{F}(\dot{\delta})\Gamma^{-1}(K - F(\delta))^T\right) \ .$$

Condition (5) therefore holds as long as

$$\begin{array}{rl} x^T P(\delta)x > & \frac{2}{\epsilon}\mathrm{Tr}\left(\tilde{F}(\dot{\delta})\Gamma^{-1}(K - F(\delta))^T\right) \\ & = \frac{2}{\epsilon}\sum_{p=1}^{\bar{p}} \dot{\delta}_p \mathrm{Tr}\left(F_p \Gamma^{-1}(K - F(\delta))^T\right) \ . \end{array}$$

We shall now prove that the 2-step procedure for finding admissible Q matrices is valid.

First note that the LMIs are affine with respect to δ, therefore if these hold for all vertices $\delta \in \bar{\mathbf{\Delta}}$ they also hold for their convex hull $\delta \in \mathbf{\Delta}$. Pre and post multiply the right-hand side inequalities in (11) by $\left[\begin{array}{cc} \mathbf{1} & -K \end{array}\right]$ and its transpose respectively to get

$$\begin{array}{c} \underline{S}_p + \left\langle F_p \Gamma^{-1} (F(\delta) - K)^T \right\rangle \leq -\underline{\epsilon}_p (\alpha\beta \mathbf{1} - KK^T) \\ \overline{S}_p + \left\langle F_p \Gamma^{-1} (F(\delta) - K)^T \right\rangle \geq \overline{\epsilon}_p (\alpha\beta \mathbf{1} - KK^T) \end{array}$$

Lemma 1 guarantees that $\mathrm{Tr}(KK^T) \leq \alpha\beta$ and hence $\alpha\beta\mathbf{1} - KK^T \geq \mathbf{0}$. The LMIs impose that $\underline{\epsilon}_p$ and $\overline{\epsilon}_p$ are positive (bottom-right blocks) therefore one gets that

$$\begin{array}{c} \mathrm{Tr}(\underline{S}_p) \leq 0 \ , \ \mathrm{Tr}(\underline{S}_p) \leq 2\mathrm{Tr}\left(F_p \Gamma^{-1} (K - F(\delta))^T\right) \\ \mathrm{Tr}(\overline{S}_p) \geq 0 \ , \ \mathrm{Tr}(\overline{S}_p) \geq 2\mathrm{Tr}\left(F_p \Gamma^{-1} (K - F(\delta))^T\right) \end{array}$$

which implies that condition (5) does hold for all $x^T P(\delta) x > \tau$ with τ solution of (10). ■

Having stated this global result several remarks follow for the usage of the result.

First, consider the constraints (6,7,8,9).

Remark 1 *If constraints (6,7,8,9) are feasible for a given* ϵ*, then the inequalities also hold for any* $\hat{\epsilon} \in]0 \ \epsilon]$*. Moreover, for a given* $\epsilon > 0$ *the constraints (6,7,8,9) are LMIs. Therefore, to find a feasible solution to constraints (6,7,8,9) one can select a small enough* ϵ *and solve the LMI problem (for example using the tools [13, 21]).*

Second, let us state the sub-cases for which the attractive ellipsoidal domain reduces to the origin. These are the cases when $\tau = 0$ is admissible.

Remark 2 *The closed-loop with SAC is such that the state* x *converges asymptotically to the origin in either following cases*

- *the uncertainties are constant (this case corresponds to results in [16]),*
- *there exists a static output-feedback, unique for all parameters (*$F_{p=1\dots\bar{p}} = \mathbf{0}$*), solution of the inequalities (6,7,8,9).*

This second sub-case is of weak importance. Indeed in such case the solution to the LMIs is a unique static-output feedback $F(\delta) = F_0$ *that has the same* x*-strict passivity properties as SAC. The former being much easier to implement than SAC, it should be preferred.*

Third, let us discuss methodologies for selecting appropriate matrices Q, the goal being to obtain the smallest ellipsoidal approximations of the attractor as possible.

Remark 3 *Linear characterizations of the "size" of the ellipsoidal outer approximation of the attractor are: the length of the largest semi-axis or the mean value of all semi-axes lengths.*

- *In order to sub-optimally reduce the largest semi-axis length, one can minimize* λ_Q *at step 2 of the 2-step procedure with the additional constraint* $\lambda_Q \mathbf{1} \leq Q$*. Notice that the result being dependent of the values of the* P_p *matrices obtained when solving (6,7,8,9), the result is inevitably sub-optimal. To reduce the conservatism one can add the following constraint* $\lambda_P \mathbf{1} \leq P(\delta)$ *for all* $(\delta, \vartheta) \in \bar{\mathbf{\Delta}}$ *when solving (6,7,8,9) and maximize* γ_P*.*

- *In order to sub-optimally reduce the mean value of the semi-axes lengths, one can minimize Tr(Q) at step 2 of the 2-step procedure. Again the result being dependent of the previously obtained values of the P_p matrices, the result is inevitably sub-optimal. To reduce the conservatism one can add the following constraint $\lambda_P \leq Tr(P(\delta))$ for all $(\delta, \vartheta) \in \bar{\boldsymbol{\Delta}}$ when solving (6,7,8,9) and maximize γ_P.*

Finally, consider the choice of matrix Γ.

Remark 4 *The size of the attractive domain (as described in the previous remark) is increased if τ is increased. One should therefore minimize τ while solving the LMI constraints (10,11) at step 1- of the 2-step procedure. The LMIs to solve at that step are linear with respect to Γ^{-1}. One can therefore also minimize τ keeping Γ^{-1} a free variable. But the constraints are such that $\min(\tau)$ goes to zero for $\Gamma = \gamma \mathbf{1}$ when γ goes to infinity. This type of situation should be avoided because would induce extremely large derivatives of the gain $K(t)$ of SAC ($\dot{K}$ is proportional to Γ) and would be hardly implementable in practice. Therefore, we suggest to keep Γ^{-1} free while minimizing τ with an additional constraint that $\underline{\gamma}^{-1}\mathbf{1} \geq \Gamma^{-1} \geq \overline{\gamma}^{-1}\mathbf{1}$ for some chosen positive $\underline{\gamma}$, $\overline{\gamma}$. The choice of $\underline{\gamma}$, $\overline{\gamma}$ is done according to implementation constraints. $\underline{\gamma}$ should not be too small for the adaptation algorithm to be effective (if $\Gamma = \mathbf{0}$ there is no adaptation). $\overline{\gamma}$ should not be too large to avoid too rapid variations of the control gain K.*

To summarize: using LMI optimization tools (for example [13, 21]), results of Theorem 1 can be tested automatically for systems modeled as (1) assuming given values of G, $\underline{\gamma}$, $\overline{\gamma}$ and ϵ. Choices of ϵ, $\underline{\gamma}$ and $\overline{\gamma}$ are discussed in remarks 1 and 4. Choosing the matrix G is a major issue for which partial answers have been provided in [16, 15].

4 Numerical example

The numerical example of [16] is considered.

$$\left[\begin{array}{cc} A(\Delta) & B(\Delta) \end{array}\right] = \left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 12-7.5\delta_1 & -0.6+0.7\delta_1 & 5-4.5\delta_1 & 0 \\ 0 & 0 & 0 & -20+\delta_2 & 20-\delta_2 \end{array}\right]$$

$$C(\Delta) = \left[\begin{array}{cccc} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1+0.1\delta_2 \end{array}\right], \quad G = \left[\begin{array}{ccc} 400 & 300 & 200 \end{array}\right]$$

The second parameter is assumed bounded such that $\delta_2 \in [\ 0 \ \ 2.5\ ]$ and constant ($\underline{\vartheta}_2 = \overline{\vartheta}_2 = 0$). The first parameter is assumed bounded in $\delta_1 \in [\ -1 \ \ \overline{\delta}_1\ ]$ and we aim at computing the maximal allowable $\overline{\delta}_1$ for various cases. All computations are done in Matlab. LMIs are coded using YALMIP [13] and SeDuMi solver [21] is used. All tests are made for $\alpha = 10$, $\beta = 1.2$, $\underline{\gamma} = 100$ and $\overline{\gamma} = 0.5$. The first two parameters are assumed to be chosen based on implementation considerations (limited amplification capacities, see [16] for a discussion about these parameters). γ_0 is assumed to be chosen based on considerations with respect to limitations on the derivative of the control gain (see Remark 4).

Case 1. Assume the derivatives of δ_1 are bounded such that $\overline{\vartheta}_1 = -\underline{\vartheta}_1 = 10$. The LMIs are solved choosing $\epsilon = 0.01$ (see Remark 1), optimizing the mean value of the semi-axes of the ellipsoid (see Remark 3). Iterating on $\overline{\delta}_1$ we get that the LMIs (6,7,8,9) are feasible for $\overline{\delta}_1 = 0.686$ and unfeasible for $\overline{\delta}_1 = 0.687$. One can thus conclude that for all time-varying uncertainties such that $\delta_1 \in [\ -1 \ \ 0.686\ ]$ with derivatives bounded in $[\ -10 \ \ 10\ ]$ and whatever constant $\delta_2 \in [\ 0 \ \ 2.5]$ the system is asymptotically stable, in the

Table 1: Tests for case 1

$\overline{\delta}_1$	-0.5	0	0.5	0.6	0.655	0.686
Tr(Q)	3.158	2.413	0.976	0.757	0.635	0.457

sense that the state converges to some neighborhood of the origin. If δ_1 stops varying ($\dot{\delta}_1$ goes to zero) then the state converges to the origin (see Remark 2).

In Table 1, results concerning the obtained values of Tr(Q) are given for various values of $\overline{\delta}_1$. These results indicate that as the uncertain domain increases coming close to include possibly destabilizing values, the outer approximation of the attractor grows in size.

Case 2. δ_1 is assumed constant and the same experimental conditions are considered as in case 1 ($\epsilon = 0.01$). The LMIs (6,7,8,9) are of course a little less restrictive than those for the time-varying case. The maximal admissible bound found is $\overline{\delta}_1 = 0.732$. Note that this result is different from that in [16] due to a modified value of β (in [16] one gets $\overline{\delta}_1 = 0.7$ for $\beta = 2$).

Case 3. Assume same experimental conditions as in case 1 but restricting $F_{p=1\ldots\overline{p}} = \mathbf{0}$, *i.e.* testing the existence of a static output-feedback for the system. The maximal admissible bound found is $\overline{\delta}_1 = 0.655$ thus illustrating the advantage of adaptive control compared to static output-feedback. The static output feedback, without measurement of the parameters, is limited in use to the interval $\delta_1 \in [\ -1 \ \ 0.655\ ]$, while the adaptive control stabilizes the system for the interval $\delta_1 \in [\ -1 \ \ 0.686\ ]$. Note that in Table 1 for $\overline{\delta}_1 = 0.655$ we had found Tr(Q) $= 0.635$ indicating a non reduced to the origin outer approximation of the attractor. This approximation in now proved conservative since case 3 proves convergence to the origin as long as $\delta_1 \in [\ -1 \ \ 0.655\ ]$ (see Remark 2).

Numerical simulations. Numerical simulations are conducted with δ_1 decomposed as two sinusoids $\delta_1(t) = 0.75\sin(0.125t + 3\pi/2) + 0.1\sin(49t + 3\pi/2) - 0.15$. The value of Γ that is tested is the one obtained when solving case 1 for $\overline{\delta}_1 = 0.686$:

$$\Gamma = \begin{bmatrix} 0.0279 & -0.0135 & 0.0785 \\ -0.0135 & 0.0207 & -0.0512 \\ 0.0785 & -0.0512 & 0.4810 \end{bmatrix}.$$

The maximal value of δ_1 is greater than the proved admissible values ($\delta_1 \in [\ -1 \ \ 0.7\ ]$) and the derivatives are within the $[\ -10 \ \ 10\ ]$ interval. Starting with zero initial conditions an impulse signal is send on the input of the system every 20 seconds. Time histories of x, K and Tr(K^TK) are plotted in Figures 2, 3 and 4 respectively. Convergence to the origin of the system state is confirmed as well as the fact that the control gains remain bounded (Tr(K^TK) $< \alpha\beta = 12$). Big, rapidly vanishing oscillations after the initial impulse input are due to first convergence of control gain parameters to stabilizing values. These do not happen any more for the following impulses. Notice that stability is not violated even if δ_1 goes beyond the proved stabilizing values. This illustrated the conservatism of the results (see as well [16]for maximal admissible values in case of constant uncertainties).

5 Conclusions

Extensions of previous results for robust adaptive control have been obtained for time-varying uncertainty case. To obtain the results, global asymptotic stability of the equilibrium point has been relaxed to global asymptotic stability of an attractor neighborhood. LMI formulas are provided for computing outer approximations of this attractor. Simulations show that these approximations are conservative: the adatptive control even better in practice. Extensions of these results can be to consider non-linear systems where non-linear parts are described as embedded in some domain, called the uncertainty set.

Results are all based on the assumption of a given G matrix. The current results allow to analyze the stability of the closed-loop with the adaptive control dependent of that G matrix (and several other coefficients α, β, γ_0 that should be chosen based on implementation constraints). The design of matrix G is a central issue for future work.

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Figure 2: Time history of $x(t)$

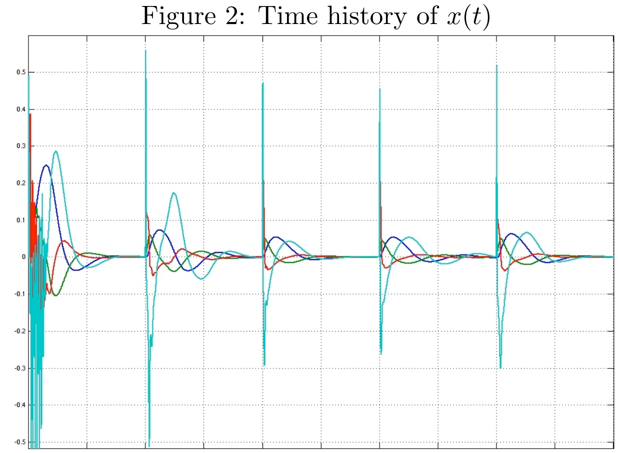

Figure 3: Time history of $K(t)$

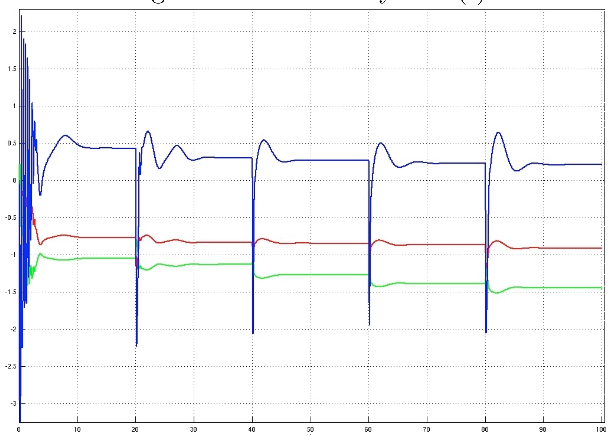

Figure 4: Time history of $\mathrm{tr}(K^T(t)K(t))$

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