Abstract—The problem of controlling an aerial robot connected by a passive tether or a passive rigid link to the ground is considered here. We provide a thorough characterization of this nonlinear dynamical robotic system in terms of fundamental properties such as differential flatness, controllability, and observability. We prove that the robotic system is differentially flat with respect to two possible output pairs: elevation of the link and attitude of the vehicle; elevation of the link and longitudinal stress force applied to the link (e.g., cable tension, or bar compression). We show the design of an almost globally convergent nonlinear observer of the full state that resorts only to onboard IMU measurements, namely an accelerometer and a gyroscope. We also design two almost globally convergent nonlinear controllers for the two flat output pairs. Using this controller the system is able to track any sufficiently smooth nonlinear controllers for the two flat output pairs. Using this controller, the system is able to track any sufficiently smooth input. The combination of robot dynamics, observer and controller is analytically proven to result in a stable closed-loop system. Finally, using extensive dynamical simulations, we test the robustness of the proposed method in several far-from-nominal conditions: nonlinear cross-coupling effects, parameter deviations, and noise in the measurements.

I. INTRODUCTION

Research on aerial robots has been boosted in the last decade thanks to the availability of low-cost and light-weight technology for onboard sensing and computation, which made possible for many robotics lab the use of small and cheap aerial systems (also known as UAVs, UASs, or aerial vehicles in the literature). This increase of popularity led to development of several methods for different problems such as, e.g., flight control, aerial navigation and planning, aerial sensing and mapping, and aerial transportation (see, e.g., [1] and references therein). These algorithms find application in several practical scenarios such as search and rescue, monitoring [2], and precision agriculture [3], just to mention a few. Furthermore both the fully-autonomous and the human-in-the-loop [4] scenarios have been considered in the literature.

Although the main use of aerial robots has been the one of mobile remote sensors, today their use for physically interactive tasks has become a very popular and promising topic [5], [6]. Researchers are increasingly directing their attention to problems involving, e.g., cooperative transportation of suspended loads with cables [7]–[9] and manipulation of parts to be assembled using onboard robotic arms [10].

Cables are not only used for cooperative transportation but also to tether an aerial vehicle to a ground station. In this case the cable can provide many advantages to the robotic system, such as i) improved flight stability and reliability (against wind, in dangerous maneuvers) [11], ii) physical interaction with a ground object, iii) enduring power [12], and iv) high bandwidth communication channel with the base station. Applications fields related to this kind of robotics systems are i) landing/taking-off from/on a moving platform such as a ship or a truck [13], [14], ii) heterogeneous systems composed by a ground or naval robot tethered to an aerial robot [11], iii) inspection and surveillance [15], and iv) communication reinforcing using relays [16].

Control and estimation for an aerial system that is connected to the ground is not an easy task, due to the fact that standard flight-control and estimation methods either cannot be applied straightforwardly to this case or, if applicable, provide sub-optimal performances because they do not exploit the full dynamics and capabilities available to the new interconnected system. Therefore, the only way to cope with the difficulties of the new robotic system and to exploit at best its capabilities is to design new control and estimation methods that consider the system in its whole completeness. However this is hard to accomplish, due to the nonlinear dynamics and the dynamic coupling between the aerial vehicle and the link.

Driven by the relevance of this topic, several control and estimation schemes have been presented in the robotic literature. For the case in which the aerial vehicle is an helicopter the authors of [13] present a method to land it to the deck of the ship in rough see using a cable. The controller is based on a time-scale separation technique: when far from landing the attitude is considered faster than translation and when close to landing translation is considered faster than attitude. In the context of [17], the authors of [14] present a control scheme with the goal of both improving hovering in windy conditions and land on a mobile platform. The control law, whose goal is to stabilize the flight to a constant elevation, is based on a PID scheme together with partial model inversion. A feed-forward action is used for counteracting rotational couplings. Remarkably, a viscoelastic model of the tether is employed and experiments with a real helicopter are also presented.

For the case of cable-tethered coplanar-multi-rotor system, [18] presented a controller to stabilize the elevation (angle) of the cable to a constant value using only inertial onboard sensors. Also in this case, a remarkable feature is the test on a real quadrotor aerial robot. Both controller and observer work under a quasi-static assumption. Using a conservative heuristic strategy, the cable tension is kept large.

Dynamics, Control, and Estimation for Aerial Robots Tethered by Cables or Bars

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enough during the partially-controlled transient phases in order to decrease the chances that cable becomes slack. The authors of [19] also present a controller to stabilize the elevation of a tethered multi-rotor to a constant value but at the same time they ensure the positivity of the cable tension in a theoretically proven way. The controller employs a cascaded scheme and stability is formally assessed using an ISS approach.

The state-of-the-art approaches provide a good basis for observation and control of such systems. However, they also present several fundamental drawbacks which we want to overcome with the presented work. First, the link can only be cable, while the use of rigid link, such as bar can be beneficial for, e.g., sustain part of the platform weight while hovering, thus reducing the power consumption, or allowing a full-bilateral physical interaction with a ground object. Second, the methods in the literature provide a controller for static case, i.e., able to keep a constant elevation. However in many practical cases (if not all) it is important to track a time-varying trajectory for, e.g., allowing a predictable transition between different elevations or the tracking of a moving target from the platform. Third, the methods in the literature are able, in the best case [19], to keep the link stress positive. However in many cases is important to precisely regulate the link stress to any desired value and even track a time-varying stress profile, possibly including negative stresses (pushing) if the link is a bar. This is impossible with the state-of-the-art techniques. Last, but not least, the state-of-the-art methods can, in the best case [18], observe the elevation state from onboard IMU measurements only in the case that the system is in a static equilibrium (constant elevation). Relying on a quasi-static assumption, these methods have been be employed also in the case of slowly-changing trajectories. However the larger the trajectory dynamicity, the larger the estimation errors.

The goal of this work is then to overcome all the listed shortcomings of the state-of-the-art. In order to achieve this goal we started by analyzing in details and from an exhaustive point-of-view the fundamental properties of a robotic system composed by an aerial robot and a passive link to the ground and possessing only an onboard IMU (accelerometer plus gyroscope) as sensorial device, especially in terms of intrinsic properties such as differential flatness, controllability and observability.

Generalizing the problem considered in [18], [19] we examine the case of a more generic link that can be either a cable, a strut, or a bar. For this system, we address three main control objectives, described in the following.

The first objective is to solve the tracking problem for any time-varying desired link-elevation trajectory (instead of just regulating the elevation to a constant desired value as done in the literature) while, at the same time, letting the aerial vehicle attitude track any other time-varying trajectory as well.

The second objective is to solve the elevation tracking problem while also solving the tracking problem for time-varying link-stress trajectories that can assume values both in the positive (tensions) and negative (compressions) domains. Note that this significantly differs from [19] where only the positivity of the stress is ensured.

The first control objective responds to applications requiring to place the robot at a desired altitude and with a desired orientation. For example, in the case that an end-effector or a sensor is rigidly attached to the aerial vehicle, the proposed controller allows to orient the end-effector independently from the elevation of the robot. For surveillance-task, equipping the vehicle with a camera, one could easily inspect an area or track a moving target of interest.

Whereas, in the cases where the main goal is not the orientation of the aerial vehicle but the stress along the link, the second control objective has to be solved. For example, if the link is a cable one may like to keep it always taut while at the same time avoid peaks on the tensions that can damage the cable or its attaching mechanisms. Or, in the case of a strut, a certain precise compression could be needed in order to preserve the integrity and the structure of the link while is reoriented.

The third and last objective is to close the control loop with a minimal sensorial setup. For this purpose we design an observer to retrieve the state from any dynamic case using inertial-only onboard measurements, without the needed of GPS or vision sensors. This makes the system feasible for both inside and outside environments and also in the case of very low and also zero visibility.

A. Previous Version and Summary

A preliminary conference version of this paper has been presented in [20] together with an attached technical report [21] showing some simulation results. With respect to that conference version we provide here many substantial improvements and additions (listed in the following) which let this submission fit the required evolutionary paradigm of the journal, in particular: i) we define an additional control problem (the one defined by Objective 1 in the Problem of Sec. II) an solve it with a new controller, ii) we prove the flatness of the outputs of interest, explicitly deriving the map from the output trajectory to the state and from the output trajectory to the input, iii) we have completely streamlined the description of the derivation of the observer, removing unnecessary complicated parts and justifying each design choice with a physical intuition thus easing the reader understanding, iv) we provide a completely new set of numerical results, which does not only show the results of single simulations, as in [21], but shows the overall behavior of the methods in a much more comprehensive way, v) we introduce for the first time a more general model for the system in exam and study the applicability of the proposed control methods to this more general case as well.

The paper is organized as follows. In Sec. II we derive the dynamic model of the system providing a complete characterization of the stress force acting along the link and describing the addressed control problems. In the following Sec. III we prove the differential flatness of the system with respect the two set of outputs. Then, in Sec. IV we design the controllers for the tracking of the outputs of interest. The observability analysis and the derivation of the observer are presented in Sec. V. A comprehensive numerical validation of the proposed methods is shown in Sec. VI. Finally, in Sec. VII, conclusions and future developments are discussed.
internal force acting along the link, called stress. When the link is pulled the stress is called tension and \( f_L > 0 \), whereas when it is compressed the stress is called compression and \( f_L < 0 \). Depending on the type of link, i.e., cable, strut, or bar, the accepted ranges for the stresses are either only tension or only compression, or both, respectively.

In the case that the link is a tie (resp. a strut) we assume that a preliminary phase brings the link in a taut (resp. compressed) condition. For example, one can use the controller in [22] suitable for the free flight and also to exert an arbitrary 3D force in order to take off, reach the desired initial position with the cable slack and then stretch it providing the force required to achieve any initial tension.

The mechanical model of the robotic system (i.e., aerial vehicle plus link) can be then easily derived. Given the constraints of the system, the motion is restricted on a circle of radius \( l \). Thus the configuration of the system is fully described by the generalized coordinates \( \mathbf{q} = [\varphi \ \vartheta]^T \). Applying the Euler-Lagrange approach it is possible to obtain the equations of the dynamic model

\[
M(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \mathbf{Q}(\mathbf{q})\mathbf{u}
\]

where

\[
M = \begin{bmatrix} J_\varphi & 0 \\ 0 & J_\vartheta \end{bmatrix}, \quad \mathbf{g} = \begin{bmatrix} m_R g d^\perp \\ 0 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} f_R^L \\ \tau_R \end{bmatrix}
\]

and: \( \mathbf{d} = [\cos \varphi \sin \vartheta] \), \( \mathbf{d}^\perp = [-\sin \varphi \cos \vartheta] \), \( \mathbf{R}_W^B \in \mathbb{R}^{3 \times 3} \) is the rotation matrix from \( \mathcal{F}_B \) to \( \mathcal{F}_W \), \( \mathbf{e}_3 = [0 \ 0 \ 1]^T \) and \( g \approx 9.81 \) is the gravitational constant.

Defining \( \mathbf{x} = [\varphi \ \vartheta \ \dot{\vartheta}]^T = [x_1 \ x_2 \ x_3 \ x_4]^T \) as the state vector of the system the dynamic model in state space form results:

\[
\dot{\mathbf{x}} = \begin{bmatrix} x_2 \\ a_1 \cos x_1 \\ x_4 \\ 0 \\ 0 \\ a_3 \end{bmatrix} + \begin{bmatrix} 0 \\ a_2 \cos (x_1 + x_3) \\ 0 \\ 0 \\ 0 \end{bmatrix} \mathbf{u},
\]

where \( a_1 = -g/l, \ a_2 = 1/(m_R l), \ a_3 = 1/J_R \) are the constant parameters of the dynamical model. In a real scenario, computing this parameters results rather easy, indeed one can directly measure the length of the link, \( l \), and the mass of the vehicle \( m_R \), whereas, for its inertia, \( J_R \), looking at the rotational dynamic in (2), one can use a standard identification technique for linear systems.

The problems addressed in this paper are the following.

**Problem (Addressed Control Problems).** Design two control laws for the input \( \mathbf{u} \), computing the control action using only the onboard measurements \( \mathbf{a} \) and \( \mathbf{\omega} \) and achieving the following two different control objectives, respectively.

Objective 1: let \( \varphi \) and \( \vartheta \) track, independently, two desired time-varying trajectories \( \varphi^d(t) \) and \( \vartheta^d(t) \):

Objective 2: let \( \varphi \) and \( f_L \) track, again independently, two desired time-varying trajectories \( \varphi^d(t) \) and \( f_L^d(t) \), respectively.

In view of the given control objectives we define the two pairs of outputs: \( \mathbf{y}^d = [\varphi \ \dot{\varphi}]^T \) and \( \mathbf{y}^b = [\varphi \ f_L]^T \).
In order to write the outputs as a function of $x$ and $u$ we first retrieve the stress from the force balance equation

$$m_B \dot{\mathbf{p}}_B = m_B (-d_1 \dot{\varphi}^2 + d_1 \dot{\varphi}) = -f_t \mathbf{d} - f_R \mathbf{r}_B - m_B g \mathbf{z}_W. \tag{3}$$

Therefore we obtain

$$y^a = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}, \quad y^b = \begin{bmatrix} \frac{1}{a_2} x_1^2 + \frac{a_1}{a_2} \sin x_1 \\ \sin (x_1 + x_3) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u. \tag{4}$$

### III. Differential Flatness

The differential flatness is a property of certain dynamical systems for which it exists (at least) an output (with the same dimension of the input), called flat output, such that both the state and the input can be expressed as an algebraic function of the flat output and its derivatives, up to a finite order [23]. This property is commonly used to generate feasible trajectories, especially for nonholonomic and underactuated systems, considering the input limitations.

In the following we demonstrate that the system is differentially flat with respect both outputs, i.e., $x$ and $u$ can be written as function of either $y_a$ or $y_b$ and some of their derivatives.

#### A. Flatness with Respect to $y^a$

From the definition of $y^a$ we directly have that

$$y^a = \begin{bmatrix} y_1^a \\ y_2^a \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}^T.$$

Then, inverting (2) we can write the inputs as function of the output and some of its derivatives

$$u_1 = \frac{\dot{x}_2 - a_1 \cos x_1}{a_2 \cos(x_1 + x_2)} \cdot \frac{y_1^a - a_1 \cos y_1^a}{a_2 \cos(y_1^a + y_2^a)} = u_1(y_1^a, y_2^a, \dot{y}_1^a, \dot{y}_2^a) \quad \text{and} \quad u_2 = \frac{\dot{x}_4}{a_3} = u_2(y_2^a).$$

Notice that in order to compute $u_1$ as above it has to be that $y_1^a + y_2^a = \varphi + \dot{\varphi} = \pi/2 + k\pi$ where $k \in \mathbb{N}$. In the output plane, the singularity represents a set of parallel lines. This means that if initial and final outputs configurations are on the opposite side of one line, then there are no trajectories that avoid the singularity in which $u_1$ results undetermined. The same singularity will appear in the design of the controller and in Sec. IV-C we address the issue and we propose a possible solution to it.

#### B. Flatness with Respect to $y^b$

First of all we have directly

$$x_1 = \varphi = y_1^b, \quad x_2 = \varphi = y_1^b.$$  

Then consider $r = -m_B \dot{\mathbf{p}}_B(y_1^b, y_1^b, y_1^b, y_2^b) - y_2^b d(y_1^b) - m_B g \mathbf{z}_W$ that depends only on $y_1^b, y_1^b, y_1^b, y_2^b$ and some constant parameters of the system. Using (3) we get

$$u_1 \mathbf{z}_B = u_1 [-\sin x_3 \ 0 - \cos x_3]^T = r(y_1^b, y_1^b, y_1^b, y_2^b), \tag{6}$$

where $r(\cdot, \cdot, \cdot, \cdot)$ represents the function from $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$ to $\mathbb{R}^3$. If $r \neq 0$, from (6) we can retrieve $u_1$ and $x_3$ as

$$u_1 = \| r \| = u_1(y_1^b, y_1^b, y_1^b, y_2^b), \quad x_3 = \text{atan2} \left( \frac{r_1}{\| r \|}, \frac{r_2}{\| r \|} \right) = x_3(y_1^b, y_1^b, y_1^b, y_2^b). \tag{7}$$

Differentiating $x_3$ w.r.t. time we obtain $x_4$

$$x_4 = \frac{d}{dt} x_3(y_1^b, y_1^b, y_1^b, y_2^b) = x_4(y_1^b, y_1^b, y_1^b, y_1^b, y_1^b, y_2^b).$$

Further differentiating $x_4$ and using the last row of (2) we finally obtain $u_2$

$$u_2 = J_T \frac{d}{dt} x_4 = u_2(y_1^b, y_1^b, y_1^b, y_1^b, y_1^b, y_2^b, y_2^b),$$

which concludes the proof that $y^b$ is a flat output for the system in exam in the case $\| r \| \neq 0$.

1) **Case of Vanishing Thrust:** If $\| r \| = 0$ one can still obtain the state and the inputs in terms of the outputs. Differentiating both sides of (6) and substituting $u_1 = \| r \| = 0$ we obtain

$$u_1 \mathbf{z}_B = \mathbf{r}(y_1^b, y_1^b, y_1^b, y_2^b) \cdot \frac{d}{dt} y_1^b, y_2^b, y_2^b) \tag{8}$$

from which, if $\| r \| \neq 0$, we can compute $u_1$ and $x_3, x_4, u_2$ as before.

Similarly, if $\| r \| = \| r \| = \ldots = \| r^{(r-1)} \| = 0$ and $\| r \| \neq 0$, we can compute the inputs and the state differentiating (6) $r$ times and then applying the same method as before.

If all the derivatives of $u_1 = \| r \|$ are zero along the desired output trajectory then $x_3, x_4$ and $u_2$ are indefinite. In this degenerate case, the thrust is identically zero, $u_1 \equiv 0$, thus the rotational dynamics of the aerial vehicle and the one of the link are completely unrelated. The trajectory of $\varphi$ evolves only driven by the gravity force and the initial conditions of $\varphi, \dot{\varphi}$, while $\dot{\varphi}$ evolves depending only on the initial conditions of $\dot{\varphi}, \ddot{\varphi}$ and the input torque $u_2$.

Notice that in this case, considering opportune higher order of differentiation we were able to prove the flatness with respect to $y^b$ even for the singular case, i.e., $u_1 = 0$. On the other hand, for the set of flat output $y^a$, in the case in which $y_1^a + y_2^a = \pi/2 + k\pi$ is structurally impossible to retrieve the inputs from $y^a$ and some of their derivatives.

2) **Discussion on the Sign of the Thrust:** From (6) the following solution is also feasible

$$u_1 = -\| r \|, \quad x_3 = \text{atan2} \left( -\frac{r_1}{\| r \|}, \frac{r_3}{\| r \|} \right),$$

which is the ‘opposite’ of (7). Therefore, from the same output $y^b$, one can get two possible different attitude and thrust pairs, $\mathbf{z}_B = \pm \frac{r}{\| r \|}$ and $u_1 = \pm \| r \|$. These solutions represent the same force vector $f_R = -u_1 \mathbf{z}_B$, obtained with the vehicle ‘facing up’ and ‘facing down’, respectively. In order to discriminate this ambiguity, as done in [24], exploiting the system continuity, the correct attitude and thrust sign are chosen according to the previous state, taking the nearest one. Moreover, as long as the trajectory does not imply condition $\| r \| = 0$ for any $t$, the sign of $f_R$ is constant.

Although aerial vehicles that can provide also negative thrust are not very common, actually, variable-pitch propellers
can be used to generate reverse thrust as in [25]. In the more common case of aerial vehicles that allow only positive thrust, one should provide desired trajectories that are either sufficiently distant from the condition \( ||r|| = 0 \) (in order to have also some robustness against disturbances), or should plan a trajectory that avoids the change of the thrust sign when passing through the condition \( ||r|| = 0 \), i.e., planning a cusp for \( r \) in 0.

We highlighted these facts since they constitute some useful hints if one wants to develop a planning method for such a system. Developing a planning method is however not in the scope of this control/observer-oriented work, therefore we restrain ourselves from giving further details in this sense.

**IV. STATE-FEEDBACK CONTROL**

A similar concept to differential flatness is *exact feedback linearizability* [23]. Differential flatness and exact dynamic feedback linearizability are correlated, indeed they are equivalent on an open and dense set of the state space [26]. In this section we show that both \( y^a \) and \( y^b \) are also linearizing output for the system (1), and using the *static and dynamic feedback linearization* control techniques we can obtain input-output decoupling and full state linearization via static and dynamic feedback respectively. In the following two subsections we propose the design of the controllers for \( y^a \) and \( y^b \) respectively, whose global characteristics are gathered in Tab. I.

**A. Controller for \( y^a \)**

Considering the equation (4) and applying the feedback linearization technique, we see that we have to derivate twice the output in order to see the input appear, obtaining the second and fourth rows of (2):

\[
\dot{y}^a = \begin{bmatrix} \dot{y}_1^a \\ \dot{y}_2^a \end{bmatrix} = \begin{bmatrix} a_1 \cos x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} a_2 \cos (x_1 + x_3) \\ 0 \end{bmatrix} u \frac{b^a(x)}{E^a(x)} u. \tag{9}
\]

Notice that the decoupling matrix \( E^a(x) \) is invertible if and only if \( x_1 + x_3 \neq \pi/2 + k\pi \) with \( k \in \mathbb{N} \). If this condition holds, assuming \( x \) known, the control law

\[
u = \Gamma y^a(x, v^a) = E^{-1} y^a(x) [\bar{b}^a(x) + y^a], \tag{10}
\]

where \( v^a = [v_1^a v_2^a]^T \) are virtual inputs, brings the system in the form of a decoupled linear system

\[
\begin{bmatrix} y_1^{a(2)} \\ y_2^{a(2)} \end{bmatrix} = \begin{bmatrix} \bar{v}_1 \\ \bar{v}_2 \end{bmatrix} = v^a. \tag{11}
\]

Then, given a desired trajectory \( y_1^{a(2)}(t) \in C^1 \) for the link elevation \( y_1^a \) and a desired trajectory \( y_2^{a(2)}(t) \in C^1 \) for the vehicle attitude \( y_2^a \), one can achieve the tracking designing a standard state-feedback linear controller.

Moreover the total relative degree \( r^a = r_1^a + r_2^a = 2 + 2 = 4 \) is equal to the dimension of the state, thus there is no internal dynamics [27].

Notice that the control actions \( u \) required by this controller may be discontinuous due to, e.g., desired trajectories possessing a discontinuity in the second derivative (which, in fact, are not forbidden by the \( C^1 \) requirement) or simply, in the real case, due to some noise in the measurements. This has to be taken into account in case one would like, e.g., to minimize mechanical vibrations of the link. Furthermore, one as to keep in mind that discontinuous inputs cannot be performed by the physical system in exam because the acceleration of both propeller rotation and the corresponding air flow cannot be infinite. However, if one needs to enforce smoother inputs it possible to apply a dynamic compensator in order to get a sufficiently smooth control signal.

For example, if one needs at least a continuous thrust input \( u_1 \), we can assume as new control input the vector \( u' = [\tilde{u}_1 u_2]^T \), considering the velocity of the thrust intensity as new controllable input, \( \tilde{u}_1 = \dot{f}_R \). The output \( y_1^a \) has to be differentiated three times to see the input appear

\[
\begin{bmatrix} y_1^{a(3)} \\ y_1^{a(2)} \end{bmatrix} = \begin{bmatrix} -a_1(\sin x_1) x_2 - a_2 \sin (x_1 + x_3) (x_2 + x_4) u_1 \\ 0 \end{bmatrix} + E^a(x) u'.
\]

The decoupling matrix does not change, thus, if \( x_1 + x_3 \neq \pi/2 + k\pi \) with \( k \in \mathbb{N} \), we can apply the control law

\[
u' = \Gamma y^a(x, u_1) = E^{b-1} y^a \left[ -\bar{b}^a(x, u_1) + \bar{v}^a \right], \tag{12}
\]

that brings the system in the form of a decoupled linear system

\[
\begin{bmatrix} y_1^{a(3)} \\ y_2^{a(2)} \end{bmatrix} = \begin{bmatrix} \bar{v}_1 \\ \bar{v}_2 \end{bmatrix} = \bar{v}^a. \tag{13}
\]

Moreover the relative degree is still equal to the dimension of the extended state that includes also \( u_1 \), i.e., \( r^a = r^a + 1 = 5 = n + 1 \), thus the linearization is exact and without internal dynamics. This smoother form of the controller will result useful in order to close the control loop with the observer of Sec. V.

**B. Controller for \( y^b \)**

Differentially from the previous case, the static feedback with respect the output \( y^b \) is not enough in order to exactly linearize the system. Indeed, the relative decoupling matrix results always not invertible because on both elevation’s acceleration and stress the input torque does not appear. In order to delay the appearance of the inputs in the derivatives of the outputs a dynamic compensator is then needed.

As done in [20] we consider as new input \( \bar{u} = [\bar{u}_1 \bar{u}_2]^T = [\bar{u}_1 \bar{u}_2]^T \), considering the acceleration of the thrust intensity as new controllable input, \( \bar{u}_1 = \ddot{f}_R \). Due to the dynamic compensator the actual state of the dynamical system is given by the extended vector \( \bar{x} = [\phi \, \dot{\phi} \, \theta \, \dot{\theta} \, u_1 \, \tilde{u}_1]^T \), that contains also the thrust intensity and its derivative. In this condition

\(^3\)Notice that this does not mean or imply that one has to measure or estimate \( u_1 \) or \( \bar{u}_1 \) since they are easily and safely computed as integrals of the new input \( \tilde{u}_1 \).
Then, given a desired trajectory $v$ where the form of a decoupled linear system $y$ the outputs $y_1$ and $y_2$ have to be differentiated respectively four and two times to see the new input appear

$$
\begin{bmatrix}
y_1^{(4)} \\
y_2^{(2)}
\end{bmatrix} = b^b(\tilde{x}) + \begin{bmatrix}
2a_2 \cos(x_1 + x_3) & -a_2a_1 \sin(x_1 + x_3) u_1 \\
\sin(x_1 + x_3) & a_3 \cos(x_1 + x_3) u_1
\end{bmatrix} \tilde{u},
$$

where $b^b(\tilde{x})$ depends only to the extended state and the decoupling matrix $E^b(\tilde{x})$ is always invertible except for the case $u_1 \neq 0$. Assuming $u_1 \neq 0$ and $\tilde{x}$ known, the control law

$$
\tilde{u} = \Gamma^b(\tilde{x}) = E^{b-1}(\tilde{x}) \begin{bmatrix}-b^b(\tilde{x}) + v^b\end{bmatrix},
$$

where $v^b = [v_1^b \ v_2^b]^T$ are virtual inputs, brings the system in the form of a decoupled linear system

$$
\begin{bmatrix}
y_1^{(4)} \\
y_2^{(2)}
\end{bmatrix} = \begin{bmatrix}
v_1^p \\
v_2^p
\end{bmatrix} = v^b.
$$

Then, given a desired trajectory $y_1^{bd}(t) \in C^3$ for the link elevation $y_1$ and a desired trajectory $y_2^{bd}(t) \in C^1$ for the link stress $y_2$, one can achieve the tracking by resorting to a standard state-feedback linear controller for $v^b$.

Notice that the total relative degree $r^b = r_1^b + r_2^b = 4 + 2 = 6$ is equal to the dimension of the extended state, thus there is no internal dynamics also in this case. This ensures the stability of the attitude of the vehicle while following any desired trajectories.

### C. Discussion on the Controllers

Notice that the two control laws (10,15) present a singularity in $x_1 + x_3 = \pi/2 + k\pi$ and $u_1 = 0$, respectively. If a desired trajectory passes through the singularity then the inversion of the decoupling matrix is not feasible and the controller cannot be applied. Moreover this represents an inconvenience also in the neighborhood of the singularity where small virtual inputs may cause large inputs. Nevertheless, if the singularity occurs for only an instant then the controller is still applicable in practice, e.g., approximating the control law using the damped least-square (DLS) inversion [28], $E^*$. Thus, in the neighborhood of the singularity, the inverse of the decoupling matrix is approximated with

$$
E^* = E^T(EE^T + c^2I)^{-1}.
$$

It is proven that $E^*$ is the optimal solution in order to minimize the inversion error while bounding the norm of the input, where $c \in \mathbb{R}$ is chosen to balance the two objectives. However, one has to plan trajectories that do not request the system staying in the singularity constantly.

Observe that, in order to compute both control actions (10,15), only the knowledge of the state is needed. Also the controller $\Gamma^b$ needs only the knowledge of the original state $x$. Indeed $u_1, \dot{u}_1$ are known because are internal variables of the controller, and the outputs and its derivatives can be computed from the state as done, e.g., in (5) and in the intermediate derivation steps that bring to (14).

These controllers satisfy only one of our requirements (decoupled tracking of the output) but they still need the full state information to be implemented, instead of just relying on the accelerometer and gyroscope measurements, as we want to be the case. This limitation is overcome in Sec. V.

### V. Observer and Measurement-Feedback Control

Measuring all the state using many sensors in order to compute the control action is often practically unfeasible due to, e.g., the costs and payload limitations of aerial robots. In this section we show that the standard on-board inertial sensors (i.e., an accelerometer plus a gyroscope) are sufficient to estimate the full state $x$ of the system, and we show the design of an exact nonlinear observer for that purpose. Furthermore, we show that the combination of the observer and the controller results in an almost-globally asymptotically stable closed loop system.

The on-board inertial sensor equipment is composed by a gyroscope, that measures the angular velocity, and an accelerometer that measures the specific acceleration of the aerial vehicle, i.e., the linear acceleration of the CoM, minus the gravity, expressed in the body frame, i.e.,

$$
\omega = x_4, \quad a = R^b_W(p_B + gz_W) = [a_x \ a_z]^T,
$$

where $R^b_W \in \mathbb{R}^3$ is the, previously defined, rotation matrix from $\mathcal{T}_W$ to $\mathcal{T}_B$. In the following we omit the line relative to the $y$-axis which is zero by construction.

Observing the full state of system (2) using the partial measurements (17,18) is a nontrivial nonlinear observation problem, where the nonlinearities appear both in the system dynamics and in the measurements. We show in the following how this problem can be successfully tackled.

Fist of all exploiting (17) we can define the gyroscope measurement as a new input $u_3 = \omega$ that let us reduce the system dimension from four to three. Then, substituting (3) and (2) into (18) and performing some algebraic manipulation

<table>
<thead>
<tr>
<th>Controller</th>
<th>Controlled Output</th>
<th>Feedback Linearization</th>
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<tbody>
<tr>
<td>$\Gamma^\phi$</td>
<td>$\phi, \theta$</td>
<td>Static: $u = [f_R x_4]^T$</td>
<td>$\phi + \theta = \frac{\pi}{2} + k\pi$</td>
<td>2 + 2 = 4</td>
<td>No</td>
<td>$\phi^d(\cdot) \in C^1, \theta^d(\cdot) \in C^1$</td>
<td>$f_L$</td>
<td>$x$</td>
</tr>
<tr>
<td>$\Gamma^\theta$</td>
<td>$\phi, \theta$</td>
<td>Dynamic: $u = [f_R x_4]^T$</td>
<td>$\phi + \theta = \frac{\pi}{2} + k\pi$</td>
<td>3 + 2 = 5</td>
<td>No</td>
<td>$\phi^d(\cdot) \in C^2, \theta^d(\cdot) \in C^1$</td>
<td>$f_L$</td>
<td>$x$</td>
</tr>
<tr>
<td>$\Gamma^b$</td>
<td>$\phi, f_L$</td>
<td>Dynamic: $u = [f_R x_4]^T$</td>
<td>$f_R = 0$</td>
<td>4 + 2 = 6</td>
<td>No</td>
<td>$\phi^d(\cdot) \in C^3, f_L^d(\cdot) \in C^1$</td>
<td>$\theta$</td>
<td>$x$</td>
</tr>
</tbody>
</table>
on the accelerometer measurement we obtain
\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{bmatrix}
= \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
+ \begin{bmatrix}
a_1 \cos x_1 + a_2 \cos (x_1 + x_3) u_1 \\
0 \\
u_3
\end{bmatrix}
\]
(19)
\[
a = \begin{bmatrix}
l \cos (x_1 + x_3) (x_3^2 + a_1 \sin x_1 + a_2 \sin (x_1 + x_3) u_1) \\
l \sin (x_1 + x_3) (x_3^2 + a_1 \sin x_1 + a_2 \sin (x_1 + x_3) u_1) - a_2 u_1
\end{bmatrix}.
(20)
\]

The problem is then ‘reduced’ to the observation of the state \([x_1, x_2, x_3]^T\) from the knowledge of the measurements \(a\) and the control inputs \([u_1, u_3]^T\). However this problem is still nonlinear both in the system dynamics \(x\) and in the measurement map \(a\).

In order to solve nonlinear observation problems there are mainly two classes of methods: approximate nonlinear observers and exact nonlinear observers. The first class relies on approximating the nonlinearities with linear or almost-linear maps around the current estimate\(^4\). The main disadvantage is the local approximative nature of the methods of this class. The second class of methods consists in nonlinear systems whose state is analytically proven to converge to the real state of the original system. Designing such observers is in general much more difficult since it is often hard to prove the asymptotic stability of a nonlinear dynamical system. However the observers of this class may guarantee global convergence to the real state, this is why we decided to search for an observer of the second class.

In the literature of exact nonlinear observers an important role is played by a particular class of systems known as in triangular form, that, in the case of a three-dimensional state system like (20) are written as:
\[
\dot{\tilde{x}} = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix} \tilde{x} + \begin{bmatrix}
0 \\
\phi(x, u) \\
\lambda(u)
\end{bmatrix}
(21)
\]
where \(\tilde{x} \in \mathbb{R}^3\) is the state vector, \(u \in \mathbb{R}^{m_u}\) is control input (with \(m_u \geq 0\)), \(\tilde{w} \in \mathbb{R}\) is the measurement and \(\phi : \mathbb{R}^3 \times \mathbb{R}^{m_u} \to \mathbb{R}, \lambda : \mathbb{R}^{m_u} \to \mathbb{R}^3\) are any nonlinear maps. For this category of nonlinear systems, in order to estimate the state one can use the nonlinear high gain observer (HGO)\(^{29}\) that ensures almost global convergence of the estimated state to the real one.

Although system (19–20) does not resemble at first view to a system in triangular form (21), we shall demonstrate in the following that it can be put in that form using a few appropriate nonlinear transformations.

A. State/Output Transformations and HGO Design

In the following we prove that there exist a change of coordinates from the original state \(x\) to a new state \(z = [z_1, z_2, z_3]^T\) and from the original measurements \(a\) to a new measurement \(w\) such that the system (19–20) appears in triangular form. While doing so we highlight also the intuitions that led us to discover this particular change of coordinates.

First of all since the term \(x_1 + x_3\) occurs frequently in (19–20) a simplifying choice is to assume \(z_1 = x_1 + x_3\). With this choice we have \(\dot{z}_1 = \dot{x}_1 + \dot{x}_3 = x_2 + u_3\) and therefore it is natural to choose \(z_2 = x_2\) to obtain \(\dot{z}_1 = \dot{z}_2 + u_3\), thus matching with the first row of the sought triangular form (21).

Now, if we compare the second and third rows of (19), with the corresponding rows of the sought triangular form (21) we see that: i) in the triangular form (21) the state-dependent nonlinearity \(\phi\) appears only in the last row of the dynamics, but, on the other hand ii) in (19) the nonlinearity appears already in the second row. Therefore, in order to push this ‘undesired’ nonlinearity down from the second to the third row we can define \(z_3 = \dot{z}_2 = a_1 \cos x_1 + a_2 \cos (x_1 + x_3) u_1\).

Summarizing, we propose the following change of variables
\[
\begin{align*}
z_1 &= x_1 + x_3 \\
z_2 &= x_2 \\
z_3 &= \dot{z}_2 = a_1 \cos x_1 + a_2 \cos (x_1 + x_3) u_1,
\end{align*}
\]
(22)
that transforms the system (19–20) in the following form
\[
\dot{z} = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix} z + \begin{bmatrix}
0 \\
\sigma'(z_1, \sin x_1, u_1, u_3) + \begin{bmatrix}
u_3 \\
1 \\
0
\end{bmatrix} \\
\lambda(u_3)
\end{bmatrix} + \begin{bmatrix}
\dot{u}_1 \\
\dot{u}_2 \\
\dot{u}_3
\end{bmatrix},
(23)
\]
where the sole state-dependent nonlinearity \(\sigma' = a_2 \sin x_1 + a_2 \cos \sin z_1 u_1 - a_2 \sin z_1 (z_2 + u_3) u_1\) is now appearing in the third row, as desired. Notice that we have, on purpose, left the term \(\sin x_1\) untransformed both in (23) and (24). In the following we show why this choice is convenient instead of directly computing \(\sin x_1\) from (22).

To reach the form of (21) it remains to extract a direct measurement of \(z_1\) from the accelerometer reading. From (20), defining
\[
\eta = \sqrt{a_1^2 + (a_2 + 2u_2)^2} = \pm f_1 \left(\frac{z_3^2 + a_1 \sin x_1 + a_2 \sin z_1 u_1}{\eta}\right),
(25)
\]
we can obtain a direct measure of \(z_1\) writing
\[
w = \arctan \left( \frac{\pm a_1}{\eta}, \frac{a_2 + 2u_1}{\eta} \right) = z_1 + k \pi,
(26)
\]
where \(k \in \{0, 1\}\). Notice that the transformation is possible only if \(\eta \neq 0\). In particular, from equations (5) and (25) it results that \(\eta = \pm \frac{1}{m_2} \left(\frac{z_3^2 + a_1 \sin x_1 + a_2 \sin z_1 u_1}{\eta}\right) = \pm \frac{m_2}{f_1}\), thus the transformation (26) requires non zero stress along the link, \(f_1 \neq 0\). This correspondence highlights, intuitively, that the condition \(\eta \neq 0\) is not just related to our particular transformation choice but is a structural observability requirement. Indeed, if the stress along the link is zero, the aerial vehicle and the link become two independent systems\(^5\) and the onboard inertial sensor is not enough to estimate the entire state. In the design of the observer we need to consider this singularity.

\(^1\)In fact, due to the assumption that the mass and rotational inertia of the link are negligible, the stress force represents the only coupling force between the two sub-systems.
especially for the cases where the desired stress passes from a desired tension to a desired compression.

Now we can exploit the accelerometer in order to replace \( \sin x_1 \) into (23) obtaining the sought triangular form (21). In particular from (25) we can write

\[
\sin x_1 = \frac{1}{a_1} \left( \pm \frac{1}{\eta} - z_2^2 - a_2 \sin z_1 u_1 \right). \tag{27}
\]

This fact allows us to write \( \sigma' = \sigma(z, \mu) \), i.e., in terms of only the new state \( z \) and the known quantities \( \mu = [u_1 u_1 u_3 \pm \eta]^T \).

Observe that, the sign to be put in front of \( \eta \) is ambiguous. It is convenient to recast this ambiguity putting always a positive sign and considering two possible values: \( \eta_+ = +\eta \) and \( \eta_- = -\eta \). Corresponding to these quantities we then get two different pairs of model and measure, \((\sigma(z, \mu_+), w_+)\) and \((\sigma(z, \mu_-), w_-)\) corresponding to the use of \( \eta_+ \) and \( \eta_- \), respectively. At each time instant only one choice for \( \eta \) is correct, i.e., \( \eta = f_1/m_1 \), to which corresponds the correct measure \( w = z_1 \) and the correct model. It is not possible, however, to discriminate the correct choice instantly from the measurements only. In Sec. V-D we propose a discriminating solution based on the prediction error.

Although less intuitive, the computation of \( \sin x_1 \) into (23) exploiting the accelerometer readings, allows to concentrate the ambiguity only on the sign of \( \eta \) obtaining two possible pairs of model-measure. Whereas with the more canonical technique, i.e., inverting the state transformation, we would have four different pairs of model-measure. Indeed from (22) we would get

\[
\sin x_1 = \pm \sqrt{1 - \left( \frac{z_3 - a_2 \cos z_1 u_1}{a_1} \right)^2}
\]

that presents another ambiguity on the sign obtaining two possible models. Considering then the measure ambiguity as well, we would have four possible combinations of model-measure.

For the design of the observer let us assume that we know which is the correct choice between \( \eta_+ \) and \( \eta_- \). In Sec. V-D we shall propose a method to obtain this knowledge. Under this assumption, the described transformation of the state and the measurements have finally transformed the original system (19–20) in an equivalent one in a triangular form

\[
\begin{align*}
\dot{z} &= A z + B \sigma(z, \mu) + \lambda(\mu) \\
w &= [1 \ 0 \ 0 \ C] z,
\end{align*}
\]

for which is possible to use the following high gain observer [29]

\[
\dot{\hat{z}} = A \hat{z} + B \sigma(\hat{z}, \mu) + \lambda(\mu) + H(w - C \hat{z}), \tag{28}
\]

where \( H = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix}^T \), \( \epsilon \in \mathbb{R}_{>0} \), and \( \alpha_i \in \mathbb{R}_{>0} \) are set such that the roots of \( s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3 \) have negative real part\(^6\).

\(^6\)The difference \( w - C \xi \) stands here for the unique angle \( \beta \in (-\pi, \pi] \) such that \( \beta + C \xi = w + k2\pi \) for a certain \( k \in \mathbb{Z} \).

\[\Box\]

B. Observation of Original State \( \hat{x} \)

From the estimation of \( z \), in order to obtain the estimation of the original state \( x \), we note that the state transformation (22) is not directly invertible. One can notice that the only knowledge of \( z \) is not enough to retrieve \( x_1 \), indeed from (22) one can extract only \( \cos x_1 \). Nevertheless, exploiting also the accelerometer measurements (27), we can write

\[
\begin{bmatrix}
\cos x_1 \\
\sin x_1 
\end{bmatrix} =
\begin{bmatrix}
\frac{1}{a_1}(z_3 - a_2 \cos z_1 u_1) \\
\frac{1}{a_1}(z_2 - z_3 - a_2 \sin z_1 u_1)
\end{bmatrix} = h(z, u_1, \eta).
\]

The state \( x_1 \) can then be computed as the phase of the unit vector \( h(z, u_1, \eta) \) denoted by \( \angle h(z, u_1, \eta) \). Thus the state estimation of the original state is given by

\[
\hat{x} = T(\hat{z}, u_1, \eta) = \begin{bmatrix}
\angle h(z, u_1, \eta) \\
\hat{z}_1 - \angle h(z, u_1, \eta) \\
\hat{z}_3 - \angle h(z, u_1, \eta)
\end{bmatrix}.
\tag{29}
\]

The full observer chain is then depicted in Fig. 2.

C. Closed-loop System Stability with State Observation

We show now how the estimated state can be then used to compute the control action, closing the control loop with only on-board sensors, preserving the stability of the global system.

First of all notice that if controller \( \Gamma^u \) in (10) is used to control \( y^d \) then it is not possible to close the loop with the observer (28). In fact, the observer requires the knowledge of \( \dot{u}_1 \), while the static controller \( \Gamma^u \) directly computes the thrust \( u_1 \) without passing first through its derivative, and, even worse, \( u_1 \) can be even discontinuous and so not differentiable. Nevertheless, one can use the controller \( \Gamma^d_1 \) in (12), thus obtaining a differentiable \( u_1 \) and the knowledge of \( \dot{u}_1 \) that becomes an available internal state of the controller. Summarizing, \( \Gamma^d_1 \) can be used with the observer while \( \Gamma^u \) can be used only with a full knowledge of the state coming directly from the measurements.

That having said, let us now analyze the closed-loop system composed by the plant (2) and the output feedback controller (12) computed on the basis of the estimated state \( \hat{x} \). For (29) we can write

\[
u' = \Gamma^d_1(\hat{x}) = \Gamma^d_1(\hat{z}, u_1, \eta)
\tag{30}
\]

i.e., the control law depends directly on the estimation of the HGO, the measures and the state of the dynamic compensator. Since the controller is almost globally exponentially convergent, except the case \( \varphi + \vartheta = \pi/2 + k\pi \), then there exists \( \epsilon^* > 0 \) such that, for every \( 0 < \epsilon < \epsilon^* \), the closed loop system with controller (30) and observer (28) is exponentially convergent for every state except the cases for which \( \varphi + \vartheta = \pi/2 + k\pi \) and with zero stress [29].

![Fig. 2: Observer.](image-url)
Similarly for the output feedback controller (15) computed using the estimated state \( \hat{x} \) there exists \( \varepsilon^* > 0 \) such that, for every \( 0 < \varepsilon < \varepsilon^* \), the closed loop system with controller
\[
\hat{u} = \Gamma^\delta(\hat{x}, u_I, u_I) = \Gamma^\delta(\hat{z}, u_I, u_I, \eta)
\]
and observer (28) is exponentially convergent for every state except the cases with zero thrust and zero stress.

Although the convergence of the observer is almost globally, an initialization phase of the estimation can be useful in order to minimize the transient. In a quasi static condition, one can use the method proposed in [18] in order to get a first approximated estimation of the state.

### D. Disambiguation of \( \eta \)

As explained before, the transformation of the measurements presents an ambiguity on the sign of \( \eta \), that can be considered positive, \( \eta_+ \), or negative, \( \eta_- \). We show in this section how the correct choice can be easily made. Refer to Fig. 3 for a graphical representation.

For each of the two possible choices let us implement an HGO equal to (28)
\[
\begin{align*}
\dot{\hat{z}}_+ &= A\hat{z}_+ + B\sigma(\hat{z}_+, \mu_+) + \lambda(\mu_+) + H(w_+ - C\hat{z}_+) \\
\dot{\hat{z}}_- &= A\hat{z}_- + B\sigma(\hat{z}_-, \mu_-) + \lambda(\mu_-) + H(w_- - C\hat{z}_-)
\end{align*}
\]

obtaining two different estimations of the state \((\hat{z}_+, \hat{z}_-)\), and therefore two different estimations of the original state \((\hat{x}_+, \hat{x}_-)\) of which only one is correct. In order to select the correct state we propose a discrimination method based on the comparison of the measurement prediction errors. At the first observer we assign a prediction error \( \tilde{e}_+ \), smoothed with an exponential discount factor:
\[
\hat{e}_+ = \lambda(\|a - \tilde{a}_+\| - \tilde{e}_+),
\]
where \( \lambda \in \mathbb{R}_{>0} \) sets the discount rate, and \( \tilde{a}_+ \) is defined as
\[
\tilde{a}_+ = \left[ \frac{l \cos(\hat{x}_1, \hat{z}_1, \hat{z}_3) + a_1 \sin(\hat{x}_1, \hat{z}_1, \hat{z}_3) + a_2 \sin(\hat{x}_1, \hat{z}_1, \hat{z}_3) \sin(\hat{z}_1, \hat{z}_1, \hat{z}_3) u_I}{l \sin(\hat{x}_1, \hat{z}_1, \hat{z}_3) \sin(\hat{z}_1, \hat{z}_1, \hat{z}_3) \sin(\hat{z}_1, \hat{z}_1, \hat{z}_3) \sin(\hat{z}_1, \hat{z}_1, \hat{z}_3) u_I - a_2 u_I} \right] (31)
\]

Then we select the estimation provided by the observer implementation which produces the smallest prediction error
\[
\hat{x} = \begin{cases} 
\hat{x}_+ & \text{if } \hat{e}_+ \leq \tilde{e}_- \\
\hat{x}_- & \text{if otherwise.}
\end{cases}
\]

In Fig. 3 the full observer with discrimination chain is represented.

Notice that the disambiguation of the two observers is not done directly using \( \hat{z} \) because it is not possible to write the predicted measure \( \tilde{a} \) as function of \( \hat{z} \) without introducing another ambiguity. Indeed, as we saw in Sec. V-A, trying to replace \( \tilde{a} \) with \( \hat{z} \) into (31) inverting the state transformation (22), we introduce an ambiguity on the sign of \( \sin(x_1) \). Whereas the problem does not hold if we apply this discrimination technique on the original state \((\hat{x}_+, \hat{x}_-)\).

Notice that if the initial elevation rate is zero and the desired elevation rate is zero as well (i.e., \( x_2 = 0 \) and \( y^d = 0 \)), since \( w_\pm = z_1 + k\pi \) after a transient the two observers converge to
\[
\dot{\hat{x}}_{1\pm} = z_1 + k\pi \quad \dot{\hat{x}}_{2\pm} = 0 \quad \dot{\hat{x}}_{3\pm} = 0,
\]
and, using (29), we obtain
\[
\dot{\hat{x}}_{1\pm} = x_1 + k\pi \quad \dot{\hat{x}}_{2\pm} = 0 \quad \dot{\hat{x}}_{3\pm} = x_3 \quad \dot{\hat{x}}_{4\pm} = x_4.
\]

Under this condition, from equation (31) it results that the prediction errors of the two observers converge both to zero thus making ineffective the proposed discrimination strategy. Nevertheless, in practice this is not a problem. Indeed, the controller based on the wrong estimation will implement a control law different form the only one that keeps \( x_2 = 0 \) causing \( x_2 \neq 0 \) and thus, in turn, the predictions errors to be different from each other.

Finally, notice that the ambiguity issue discussed in this section is present only in the initial phases. Whenever the good observer is selected with sufficient certainty, one can switch off the other. For this purpose one can set a confidence threshold on the tracking error of the desired output. If an observer reaches the confidence threshold then this is identified as the correct one and the other one is switch off.

### E. Discussion on the Proposed Observer

1) **Applicability:** If the stress is zero then \( \eta \) is also zero and \( w \) cannot be determined. In the very special case that the stress has to be constantly equal to zero, the proposed observer is not applicable\(^7\) (while the controller would still work). Nevertheless, if the desired stress is passing through zero for a sufficiently short (ideally zero) time interval, then the proposed observer can be still used in practice by updating the filter without the correction term in that time instants, i.e., imposing
\[
\dot{\hat{x}} = A\hat{x} + B\sigma(\hat{x}, \mu) + \lambda(\mu) \quad \text{if } w = 0.
\]

In this way the error dynamics becomes non strictly stable for a short moment but the dynamics returns asymptotically stable as soon as the stress returns to a non-zero value.

2) **Robustness:** In order to deal with the drawbacks of the HGO, as peaking phenomenon and noise sensitivity, many common practical solutions has been presented in the literature, see e.g., [29], [30]. For example, to overcome the peaking phenomenon, it is sufficient to saturate the estimated state on a bounded region that defines the operative regions of the state for the system in exam. In the presence of measurement noise, the use a switched-gain approach can guarantee a quick convergence to the real state during the first phase while reducing the noise effects at steady state [30].

\(^7\)The static observer in [18] requires a nonzero stress as well to work.
VI. NUMERICAL VALIDATION

In this section we present a comprehensive validation of the method with the main purpose of testing it under difficult non-ideal conditions, thus showing both its strengths and possible limits when applied on a real system. We simulate a standard aerial vehicle whose nominal parameters are \( m_R = 1\, [\text{kg}] \), \( J_R = 0.25\, [\text{kg} \cdot \text{m}^2] \) and a link of length \( l = 2\, [\text{m}] \). Concerning the controller we set the gains of the linear outer control loop, \( k_1 \) and \( k_2 \), such that the error dynamics of \( y^1 \) and \( y^2 \) has poles in \((-1, -1.5, -2, -2.5)\) and \((-1, -1.5)\) respectively. For the gains of the observer we set \( \epsilon = 0.1 \) and \((\alpha_1, \alpha_2, \alpha_3) \) such that the root of \( s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3 \) are \((-6, -4.5, -3)\). Those values guarantee the stability of the closed loop system and a sufficiently rapid convergence of the observer and controller. For the controller \( \Gamma^o \), the desired trajectory is a smooth step, continuous up to the third order for \( \varphi \) and up to the second order for \( \vartheta \), from the initial values \( \varphi_0^d = 10^\circ, \vartheta_0^d = 30^\circ \) to the final values \( \varphi_f^d = 50^\circ, \vartheta_f^d = 5^\circ \), respectively. For the controller \( \Gamma^b \), the desired trajectory is a smooth step, continuous up to the fourth order for \( \varphi \) and up to the second order for \( f_L \), from the initial values \( \varphi_0^d = 45^\circ, f_L_0^d = 3\, [\text{N}] \), to the final values \( \varphi_f^d = 135^\circ, f_L_f^d = 5\, [\text{N}] \), respectively. Smooth step-like trajectories, as it will be clear later, have the benefit of clearly showing the performances of the controllers under three important conditions: the initial transient, the tracking of a fast time-varying signal, and the steady state.

To obtain a complete validation, in the following we show the results about the stability and robustness of the proposed method under different non ideal conditions, such as: a) nonzero initial tracking and estimation errors, b) parametric variations, c) generic CoM position and non-negligible link mass, and d) noisy sensor measurements.

In this paper we decide to not provide videos or stroboscopic animations of the controlled system since one would not gain too much information from such illustrations. The reader that is interested in watching animations to get a rough idea of how the system evolves under a subset of the control actions presented in this manuscript is encouraged to have a look to the technical report [21] and to the video attached to [20], which can be also seen here http://homepages.laas.fr/afranchi/robotics/sites/default/files/2015a-TogFra.mp4.

Furthermore in these simulations we also do not show the capability of the controller \( \Gamma^b \) to track both positive and negative stresses since this has been already clearly demonstrated.
in [20].

A. Validation for Nonzero Initial Tracking/Estimation Errors

In order to show the asymptotic convergence performances of both the controller and the observer we initialize the control system with nonzero initial tracking and estimation errors. From Fig. 4 one can see that after the convergence of the observer, that takes less than one second, the controller $\Gamma^c$ is able to steer the outputs along the desired trajectories with zero error. A similar behavior is obtained for the controller $\Gamma^d$, we do not report the plots here for the sake of brevity. We then performed many other similar simulations with different initial errors and we observed always the same asymptotically convergent behavior, as expected from the almost-global nature of the proposed observer and control laws.

B. Parametric Variations

In a real scenario we can not know exactly each parameter of the system. The purpose of this section is to investigate the robustness of the proposed method in this scenario. In particular we introduce discrepancies between the real model and the nominal one used in the controller/observer.

We notice that in principle one could try to design an adaptive control law that is able to compensate for parametric uncertainties. However, this is clearly a tough objective, because the system is nonlinear and the available measurements are only the (nonlinear) accelerometer and the gyroscope readings. Therefore this goal is left as future work. Instead, we concentrate in this section on assessing the ranges of parameter variations that causes a degradation of the performance that remains within an acceptable bound. By doing so, we shall see in fact that the proposed control scheme possesses a remarkable robustness even without the presence of an adaptive design.

Considering $l_0$, $m_R$, and $J_R$ the real parameter value and $l_0 m_R$ and $J_R$ the nominal one, we set $l = (1 + \Delta l)l_0$, $(1 + \Delta m_R)m_R$ and $(1 + \Delta J_R)J_R$, where $\Delta m_R$, $\Delta l$ and $\Delta J_R$ denote the relative parametric variations.

Fig. 5 shows the performances of controller $\Gamma^c$, with a relative variation of 0.05 for each parameter (the performances of controller $\Gamma^d$ are not shown because very similar to the one of Fig. 5). We can notice that due to the uncertainty of the model we have nonzero errors in the tracking and in the estimation of the state. Nevertheless the system remains stable and the tracking and estimation errors result bounded. However, this is just the result of a particular case which does not let us really understand how the system behaves for any parameter variation. Therefore, for obtaining a comprehensive analysis we tested the behavior for several different parametric variation combinations. The results are plotted in Fig. 6 and Fig. 7, where we show the mean tracking error, $\bar{e}_{\text{track}}$, and the
corresponding standard deviation $\sigma_{\text{track}}$, defined as

$$e_{\text{track}}(t) = \left\| y_1^f(t) - y_1(t) \right\| + \left\| y_2^f(t) - y_2(t) \right\|$$

$$\bar{e}_{\text{track}} = \frac{1}{t_f - t_0} \int_{t_0}^{t_f} e_{\text{track}}(t) dt$$

$$\sigma_{\text{track}} = \sqrt{\frac{1}{t_f - t_0} \int_{t_0}^{t_f} (e_{\text{track}}(t) - \bar{e}_{\text{track}})^2 dt},$$

where $t_0$ and $t_f$ are the initial and final time, respectively. In the plots the solid line corresponds at the mean tracking error plus minus its standard deviation.

The effect of an unknown parameter could also change with respect to the trajectory and in particular with respect to the velocity and acceleration at which the trajectory is followed. Consequently, we analyze the error behavior in three phases: in the Phase 1 (transient) the desired trajectory is constant and the analysis is more focused on the convergence of the observer; Phase 2 constitutes the dynamic part where the desired trajectory quickly goes from the initial value to the final one; the Phase 3, the last, corresponds to the steady state condition where the desired trajectory is again constant. We show the tracking error for each of the three phases to better understand if a parameter variation affects more the transient, the dynamic phase, or the static one. We denote with $\bar{e}_{\text{track}1}$, $\bar{e}_{\text{track}2}$ and $\bar{e}_{\text{track}3}$, the tracking errors in those three phases, respectively.

From Fig. 6 and Fig. 7 one can notice that, as expected, the performances get worse increasing the parametric variation. Furthermore, the same variation has more effect if the trajectory is more “aggressive” and it is followed with higher speed. This is due to the fact that with higher speed and acceleration the inertial and Coriolis/centripetal terms become larger, and thus also the error in the feedback linearization increases that in turn implies a worse tracking of the desired trajectories.

Comparing the performances between the controllers in Fig. 6 and Fig. 7 we can see that the controller $\Gamma_d$ results to be more robust then the controller $\Gamma_b$ in term of mean tracking error. This is due to the fact that for the controller $\Gamma_d$, the dynamics of one of the controlled outputs, namely $\dot{\theta}$, is not influenced by the parameters such as mass and length of the link. This means that any variation on these parameters does not generate a worse tracking of $\dot{\theta}$, which results to a lower mean tracking error.

One can also notice that the mean tracking error is not in general symmetric with respect the sign of the relative parametric variation. For example for the controller $\Gamma_d$ is better to overestimate the mass and the length of the link rather than underestimating them, while for controller $\Gamma_b$ it results to be the opposite, even if these consideration are more relevant for the dynamic phase. Indeed, during the steady state phase the behavior is almost symmetrical.

Another facts that appears clear from the plots is that the variation that most influences the performances is the one on the mass of the aerial vehicle. Fortunately, in reality this parameter can be easily measured with high precision using, e.g., a scale.

C. General Model with Generic CoM Position and Non-negligible Link Mass

The controller developed in this paper assumes that the system can be represented with the model given in Fig. 1, where the CoM of the aerial vehicle coincides with the attachment point of the link to the vehicle and the link has a negligible mass. Figure 8 represents instead a more general model for which the assumptions done in Sec. II are not fulfilled. Taking into account the definitions made in Sec. II we then define a body frame, $\mathcal{F}_L$, attached to the link, with axes $\{x_L, y_L, z_L\}$ and origin $O_L$ coinciding with the center of mass (CoM) of the link. The position of $O_L$, defined in $\mathcal{F}_W$, is denoted with $p_L = [x_L, y_L, z_L]^T$. As for $\mathcal{F}_W$ we have that $y_L \equiv y_B \equiv y_W$ and $y_L \equiv 0$. For the validation we model the link as a rigid body of mass $m_L \in \mathbb{R}_{>0}$ and inertia $J_L \in \mathbb{R}_{>0}$. Considering the inertia of the link as the inertia of an infinitesimal thin rigid tie with uniform distributed mass, we have also that $J_L = m_L l^2/12$.

Assuming links with high stiffness, the deformations and the elongations results negligible with respect the length of the cable itself, in the range of stresses of our concern. Therefore the link length is fixed. The link is connected at one end to a fixed point coinciding with $O_W$ and at the other end to a point rigidly attached to the aerial vehicle whose constant position in $\mathcal{F}_W$ is denoted with $r_{BL} = [r_o, 0, r_l]^T$. If $\|r_{BL}\| = 0$ then the link is directly attached to the CoM of the aerial vehicle.

The mechanical model of the more general robotic system can be then derived writing the dynamics as the one in (1) plus a disturbance due to the non idealities

$$M(q)\ddot{q} + g(q) + \delta(q, \dot{q}, q, u) = Q(q)u,$$

where

$$\delta(q, \dot{q}, q, u) = \bar{M}(q)\dot{q} + \bar{c}(q, \dot{q}, u) + \bar{g}(q) - \bar{Q}(q)u,$$

$$\bar{M} = \begin{bmatrix} J_\phi & J_{\phi\theta} \\ J_{\phi\theta} & J_\theta \end{bmatrix}, \quad \bar{c} = \begin{bmatrix} \dot{\phi}^2 \\ \dot{\theta}^2 \end{bmatrix},$$

$$\bar{g} = \begin{bmatrix} m_l g d_1 \cdot e_3 \\ -m_R gR_B^w r_{BL} \cdot e_3 \end{bmatrix}, \quad \bar{Q} = \begin{bmatrix} 0 & 0 \\ -r_{BL} \cdot e_3 & 0 \end{bmatrix},$$

in which $R_B^w = \partial R_B^w / \partial \phi$, $e_3 = [0, 0, 1]^T$, $J_\theta = m_l l^2/3$, $J_\phi = m_l ||r_{BL}||^2$, $J_{\phi\theta} = J_{\theta\phi} = -m_R \bar{R}_B^w r_{BL} \cdot d$, and $\bar{c} = \bar{m}_R \bar{R}_B^w r_{BL} \cdot d$.

Fig. 9 shows the performances of the proposed method $\Gamma_b$ for a plausible case in which the link consists of a cable of
mass $m_L = 0.01 m_R$ and inertia (during taut condition) $J_L = m_L l_2^2 / 12$. The link is attached to the robot in the position $r_{BL} = [0.03 0.03]^T [\text{m}]$ with respect $\mathcal{R}_B$. One can notice that the controlled system is stable but the error does not converge exactly to zero. Indeed, due to the nonzero $|r_{BL}|$, the stress along the link generates an extra torque on the aerial vehicle that is not compensated and so a constant steady state error appears.

In order to understand how each parameter of the more general model affects the tracking performances, as done in Sec. A-B, Fig. 10 shows the mean tracking error and its standard deviation for different parameter values and in the three Phases described before. In particular in Fig. 10c the mass of the link is taken as $m_L = \Delta m_L m_R$. We noticed that the negative effects due to a nonzero offset $r_{BL}$ reduce or increase if the rotational inertia is increased or reduced, respectively. Indeed, looking at the rotational dynamics in the case of nonzero offset:

$$\dot{\vartheta}_R = \frac{\tau_R}{J_R} - f_L d \cdot \frac{r_{BL}}{J_R},$$

one can notice that the effect of the stress on the angular acceleration decreases if the inertia increases. Intuitively, a bigger inertia would mean a bigger mass or a bigger dimension of the vehicle. In the second case, the bigger the vehicle the more the effect of the offset becomes negligible. For this reason in Fig. 10 we plot the mean tracking error with respect $r_{BLx}/J_R$ and $r_{BLz}/J_R$, thus normalizing this effect.

We do not report the plots relative to the controller $\Gamma'$ since it resulted to be much more sensitive to link mass and to the offset than the controller $\Gamma^b$. This is due to the fact that one of the output, namely the attitude of the aerial vehicle $\vartheta$, is directly influenced by the offset as it is shown in (32). Furthermore, even with small offset the tracking error is such that the actual trajectory passes through the singularity of the controller $\Gamma'$ (see Sec. IV-A) causing an unstable behavior.

On the other hand, the controller $\Gamma^b$ turned to be much more robust to these sort of structural model variations. From the plots of Fig. 10 we can see that the parameters that are most affecting the performances are the entries of $r_{BL}$, i.e., $r_{BLx}$ and $r_{BLz}$. One can notice that it is more advisable to attach the link such that one is sure that $r_{BLz} \leq 0$, especially if ‘aggressive’ motions are required. The effect of the displacement along $x_R$ is instead almost symmetric. The small asymmetry is due to the particular trajectory passing from the first to the second quadrant. The mean tracking error increases instead almost linearly with respect to the mass of the link. Nevertheless, even with $m_L$ equal to the 20% of $m_R$ the closed loop system remains still perfectly stable.

D. Noise on the Measurements

In this last section we investigate the robustness of the proposed method in the presence of noisy measurements,
of elevation angle and link stress. To close the control loop we designed a nonlinear observer able to retrieve the state from any dynamic condition using an inertial-only on-board sensor. This showed that inertial only measurements are enough to estimate the full state of the system. An exhaustive numerical validation of the proposed method was presented considering some possible non-ideal conditions such as parametric variation, more general model and noisy measurements.

As future development we are going to implement the method on a real platform in order to test the method under real experiments. Another possible improvement would be to exploit the flatness property for an initial planning phase to generate the desired trajectory respecting the limits of the robotic system and to avoid the singularity of the controllers. Automatic identification of the system parameters and control adaptation are other possible directions of development.

REFERENCES

APPENDIX A
ADDITIONAL SIMULATIONS

Within this section we shall provide some more detailed results obtained by simulations in order to prove the validity of the proposed method.

Regarding the parameters of the system we used the same value presented in Sec. VI. Analogously for the desired trajectories. To have a more complete validation, in the following we show the results of the convergence and robustness of the proposed method under different non ideal conditions, such as: a) initial tracking and estimation error, b) parametric variations, and c) noisy sensor measurements.

A. Initial errors

In this section we want to show the closed loop stability of the system in dynamic condition even with some initial tracking and estimation error. In Fig. 12 one can see that after the convergence of the observer, that takes less than one second, the controller $\Gamma^d$ exponentially steers the outputs along the desired trajectories. Regarding the controller $\Gamma^b$, the plots are shown in Sec. VI.

B. Parametric variations

The purpose of the next sections is to investigate the robustness of the proposed methods. In particular we consider some parameter variation between the real model and controller/observer. Indeed in a real scenario we can not know exactly each parameter of the system, thus the controller and observer would be based on some nominal parameter value different from the real one.

Fig. 13 shows the performances of the controller $\Gamma^d$ with a relative variation of 0.05 for each parameter. Regarding the controller $\Gamma^b$, the plots are shown in Sec. VI.

With a relative parameter variation, $\Delta m_i, \Delta I$ and $\Delta I_b$ equal to 0.05, we can notice that due to the uncertainty of the model we have some nonzero errors in the tracking and in the estimation of the state. Nevertheless the system remains stable and the tracking and estimation errors result bounded. In principle an integrator term in the controller could overcome the constant steady state error, thought the control action is computed from an estimation of actual output based on the input and on the estimation of the state. The last one does not converge to the real one since the observer has a steady state error and so also the estimation of the output.

As we already said in Sec. VI, the add of an integrator term in the controller does not implies an improvement of the performances since the observer output has a bias due to the estimation error. On the contrary, in the case the
feedback control could be done based on the real outputs, then the integrator term would solve the steady state error due to the parameter variations. Indeed, in Fig. 14, where the performances are showed in the case that the real output is assumed measured and an integrator term is added on the controller, one can observe that the mean tracking error reduces a lot, especially in the steady state phase.

C. Noise on the Measurements

In this last section we investigate the robustness of the proposed method with the presence of noise in the measurements. We considered the accelerometer and the gyroscope with white noise component of variance 0.1[m/s²] and 0.01[rad/s], respectively, that correspond to typical values of real sensors.

From Fig. 15, for the controller Γ², we can observe that the estimated state shows some noise but the corresponding error remains always limited. Due to the noisy component on the estimated state the control action presents some oscillations that imply a non exact tracking of the desired trajectory. Nevertheless the tracking error remains small and always bounded. To achieve these results we had to reduce the gains of both controller and observer. Indeed, high gain values increase the convergence speed but also the negative effects of the noisy measurements. In general the two controllers Γ² and Γ³ do not show particular different behaviors.

Fig. 13: Controller Γ²: possible real parametric variation with a relative variation of 0.05 (i.e., 5%) for each parameter.

Fig. 14: Parametric variation - Controller Γ³ with real output and an integrator term. The subscript 1, 2, and 3 corresponds to the three different trajectory times: 1) 7[s], 2) 5[s] and 3) 3[s], respectively. Over the displayed range of parametric variation the performances are very bad or the system results to be even unstable.
(a) Controller Results.

(b) Observer Results.

Fig. 15: Controller $\Gamma^d$: Noisy measurements.