Certified and fast computation of supremum norms of approximation errors

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Outline

- Motivation
  - Correctly rounded elementary functions
  - Supremum norm of error functions
  - Previous approaches and difficulties
- Our approach
  - Automatic differentiation and Taylor models
  - Isolation of roots of polynomials
  - Enclosure of roots of functions
- Results & Conclusion

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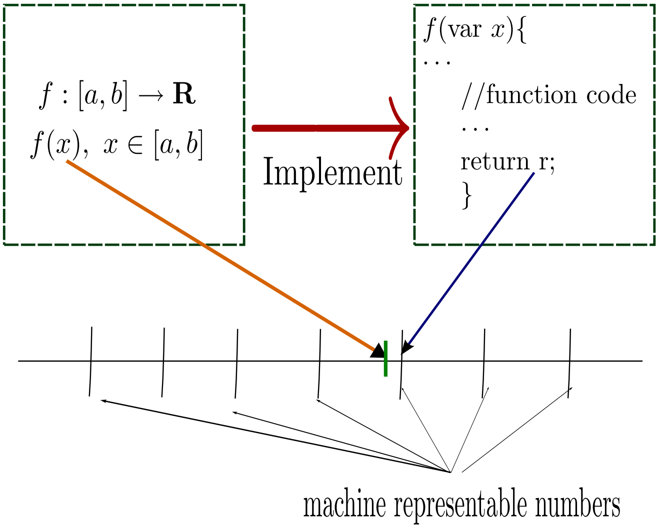
- Arenaire team develops the Correctly Rounded Libm (CRLibm)[1].

[1] http://lipforge.ens-lyon.fr/www/crlibm/

Correctly rounded functions



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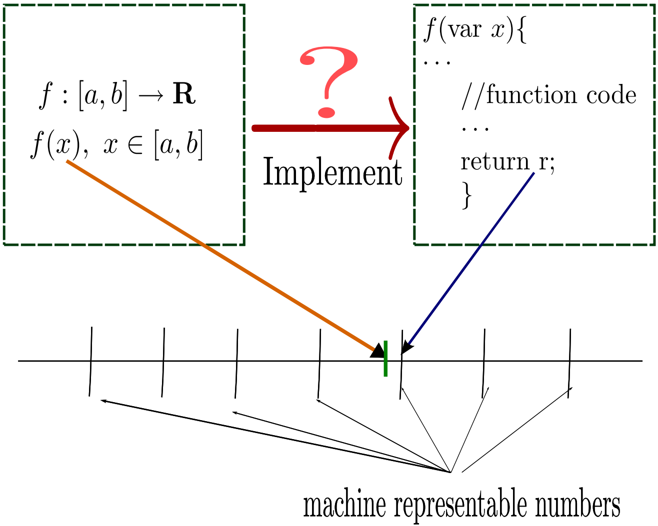
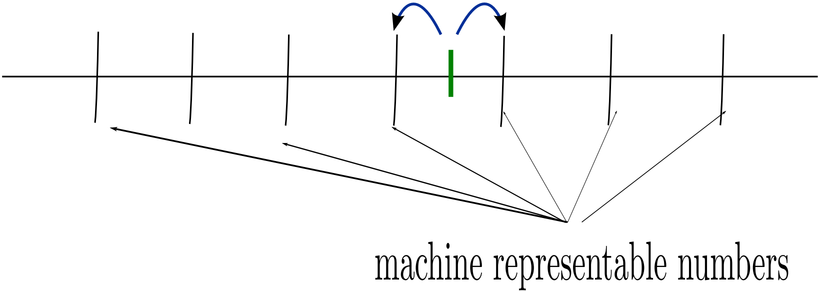


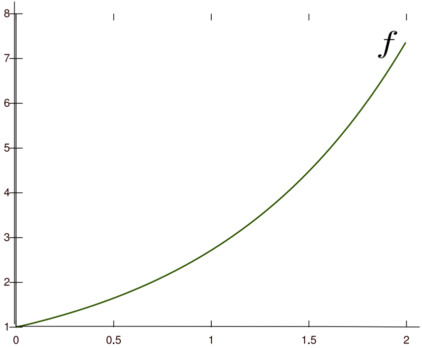
Correctly rounded functions

- Table Maker's Dilemma



- Increase working precision
- Worst cases search - V. Lefevre and J-M. Muller

Supremum Norms of Error Functions



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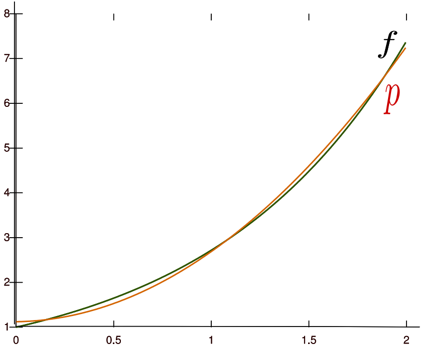

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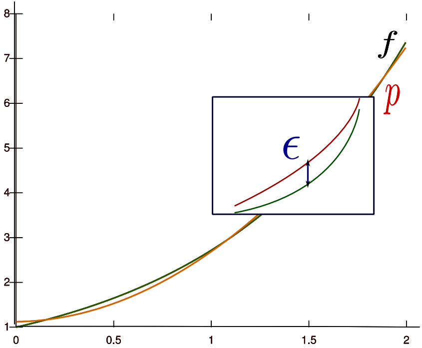

$\varepsilon(x) = f(x) - p(x),\ x \in [a, b]$ or

$\varepsilon(x) = \frac{p(x)}{f(x)} - 1,\ x \in [a, b]$

Define $\|\varepsilon\|_\infty = \sup_{x\in[a,b]}\{|\varepsilon(x)|\}$

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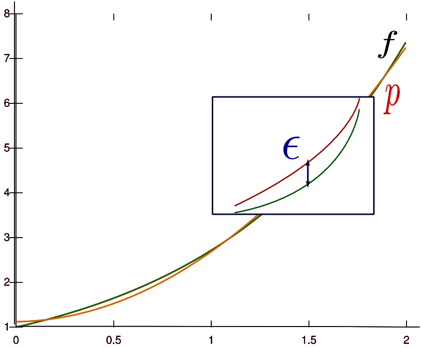

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- Compute a certified bound for the supremum norm of an error function
- "Quick and dirty" supremum norms - another class of algorithms

Supremum Norms of Error Functions

- Error $\varepsilon(x) = f(x) - p(x)$ or $\varepsilon(x) = \frac{p(x)}{f(x)} - 1, \quad x \in [a, b]$
- Define $\|\varepsilon\|_\infty = \sup_{x \in [a,\, b]}\{|\varepsilon(x)|\}$
- Purpose: Compute a certified bound for the supremum norm of an error function
- Given p and f find a narrow interval $\mathbf{r}$ such that $\|\varepsilon\|_\infty \in \mathbf{r}$.

Need for a fast and certified algorithm:

- Correctly rounded elementary functions
- For computing the minimum error between a function and thousands of polynomials with floating-point coefficients

Supremum norm of error functions

Example:

$f(x) = e^x, \; x \in [0,1]$
$p(x) = \sum_{i=0}^{5} c_i x^i$ s.t. $\|f - p\|_\infty$ is as small as possible (Remez algorithm)
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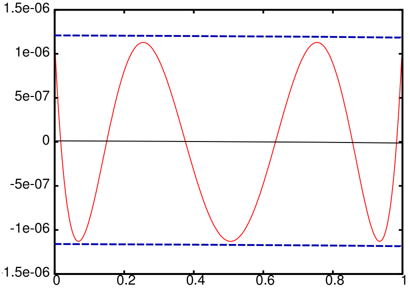

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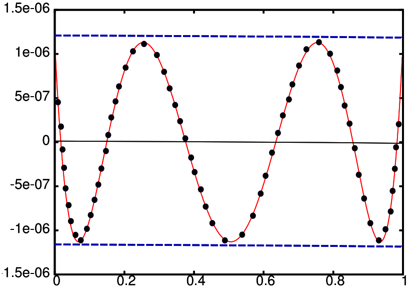

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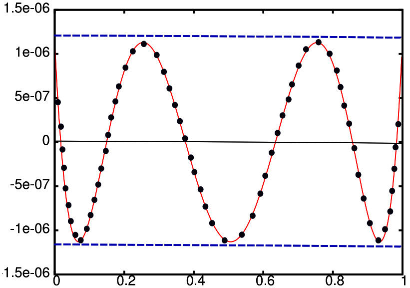

How to obtain a certified and tight bound?

Our Problem - Prone to High Dependency Phenomenon

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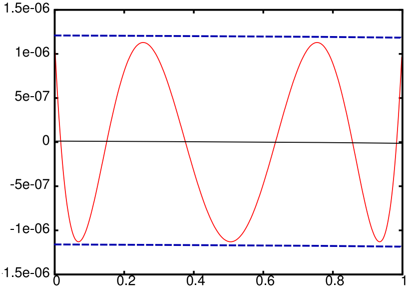

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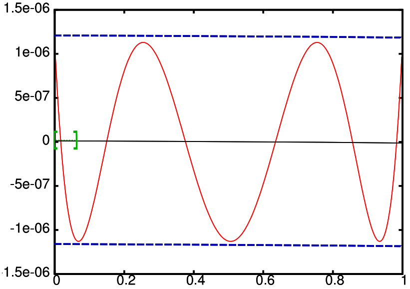

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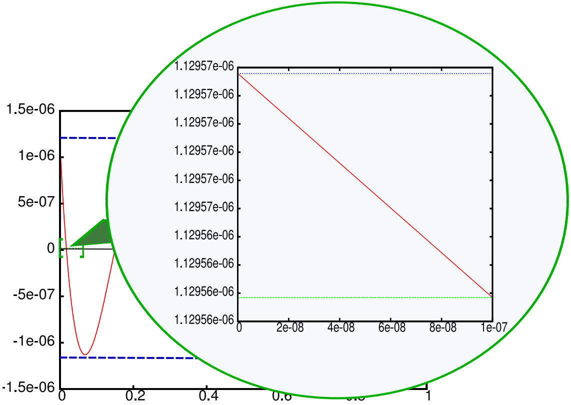

In this case, over $[0,1]$ we need 10^7 intervals!

Previous Approaches

- Floating-point techniques - not “safe“
- Existent Interval Arithmetic methods - Not sufficient
- Global optimization software (eg. Globsol) - not tailored for our specific problem
- Chevillard and Lauter’s technique: interval arithmetic, tight bounding of the zeros of the derivative of the error function, removes false singularities (like $x = 0$ for $\sin(x)/x$). High computation time for $\deg(p) > 10$.
- Techniques based on a high order Taylor expansion of the error function and a sufficiently close bounding of the remainder
- Certified polynomial approximations of analytic functions.

Our Approach

- How to obtain a certified and tight bound for $\|\varepsilon\|_\infty = \|f - p\|_\infty$ AND $\|\varepsilon\|_\infty = \|p/f - 1\|_\infty$, over given interval $[a, b]$?

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- Compute (T, Δ) s.t $f(x) - T(x) \in \Delta, \forall x \in [a, b]$.

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$$\|f - p\|_\infty \leq \underbrace{\|f - T\|_\infty} + \underbrace{\|T - p\|_\infty}$$

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1. Automatic Differentiation/ Taylor Models
2. Tightly bounding the polynomial difference - Roots isolation and refinement techniques

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(1) Computing (T, Δ) - Automatic differentiation (AD)

- Example:

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- Compute an enclosure of the remainder: Use AD

$$\Delta_9(x, \xi) = \underbrace{\frac{f^{(9)}(\xi)}{9!}}_{\in \frac{exp([0,1])}{9!}} \times \underbrace{(x - 1/2)^9}_{\leq (1/2)^{(-9)}}$$

(1) Computing (T, Δ) - Automatic differentiation (AD)

- Main idea:

$$\|f-p\|_\infty \leq \underbrace{\|f-T\|_\infty}_{\text{bounding a remainder}} + \underbrace{\|T-p\|_\infty}_{\text{bounding a polynomial}}$$

- Achieved so far:

$$\|\exp - p\|_\infty \leq \underbrace{\|\exp - T\|_\infty}_{\in\ 1.4630578142[1;2]\times e-8} + \underbrace{\|T-p\|_\infty}_{\text{bounding a polynomial}}$$

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Purpose: Tightly bound $\|T - p\|_\infty$ over the interval $[a, b]$

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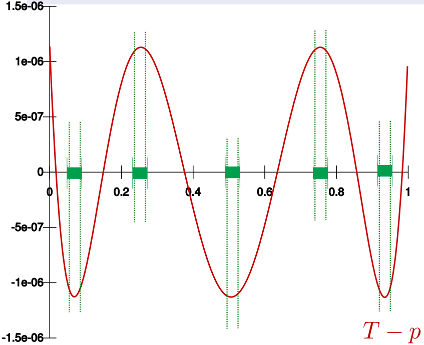

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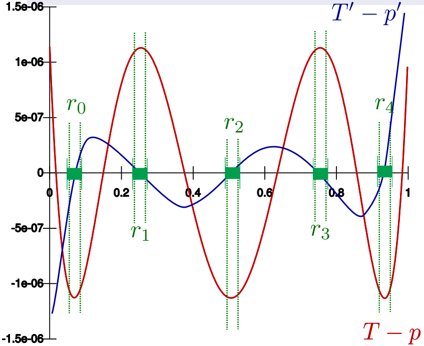

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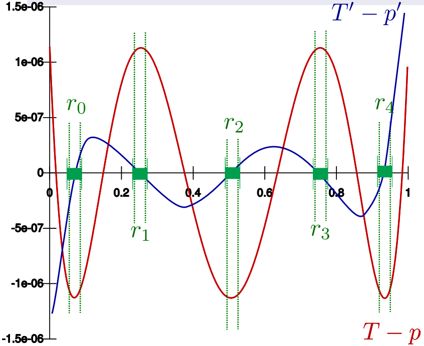

- Tightly bound the roots of the derivative
- Evaluate using interval arithmetic

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- Use dichotomy or Newton iteration process

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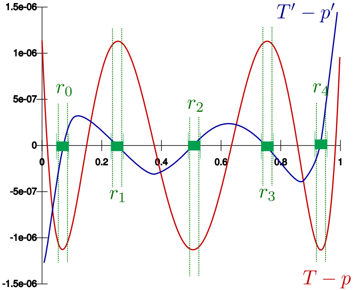

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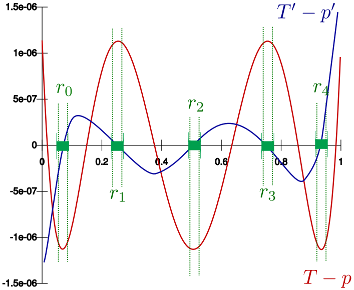

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$$\|T-p\|_\infty \in\ 1.11486943[1;2] \times e-6$$

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IA: $\|f-p\|_\infty \leq 0.4$, sampling: $\|f-p\|_\infty \simeq 1.1295e-6$.

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- Brutal application of the same principle does not work! WHY?

(1) Computing (T, Δ) - Taylor Models at a Glance

Couple of form (T, Δ), s.t. $f - T \in \Delta$.

- Propagation of error bounds combined with AD
- Start with trivial Taylor Model: $(1, [0, 0])$.
- Easily define arithmetic operations: $+, -, *$. Eg. Multiplication:
  $(T_1, \Delta_1) * (T_2, \Delta_2) =$
  $((T_1 T_2)_{0..n}, \Delta_1 B(T_2) + \Delta_2 B(T_1) + B((T_1 T_2)_{n..2n}))$
- Note: The bound $B(T)$ computed is propagated in the remainder. It influences the quality of the remainder!
- Usually the bounds on the occuring polynomials are rough!
- Tight bounds imply a computational effort at each operation.

Relative Error Issues

Example:

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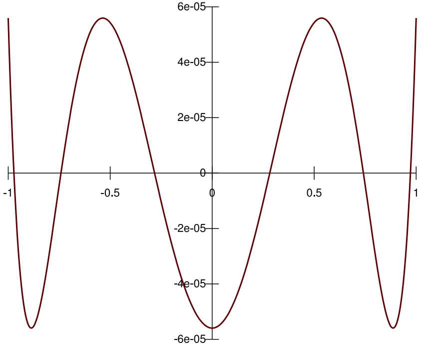

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Compute the Taylor Model for $\varepsilon(x)$, order 12.

TM for $f(x)$, remainder: $[-1.351322e-10; 1.351322e-10]$
TM for $\frac{1}{f(x)}$, remainder: $[-3.1634304e-2; 0.258216]$

Remainder bounds are unsatisfactory in our case.

Our Approach - Relative Error

- Compute: $\|\varepsilon\|_\infty = \|p/f - 1\|_\infty$ over I
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- $\varepsilon' = (p'f - pf')/f^2$. Consider $\tau = p'f - pf'$.
- Solve $\tau(x) = 0$
- Find small intervals that enclose each root of τ.

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- Brutal application of the same principle does not work:
  - Tight bounds for the polynomials interfering in Taylor Models have to be computed
  - Some compositions (inverse function, logarithm) generate intervals that are too wide in our case
- Idea: Tightly enclose all the extrema of ε: they are among the zeros of the first derivative.
- $\varepsilon' = (p'f - pf')/f^2$. Consider $\tau = p'f - pf'$.
- Solve $\tau(x) = 0$
- Find small intervals that enclose each root of τ.
- Evaluate ε on these intervals.

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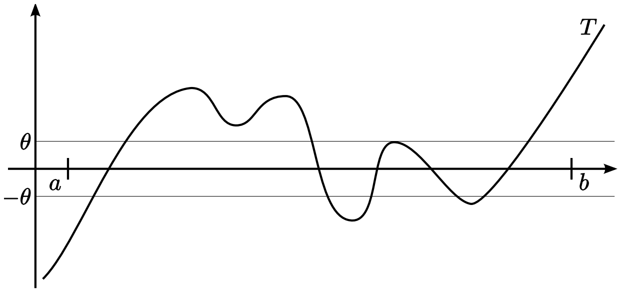

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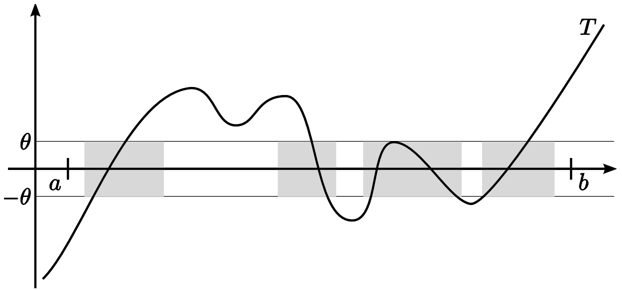

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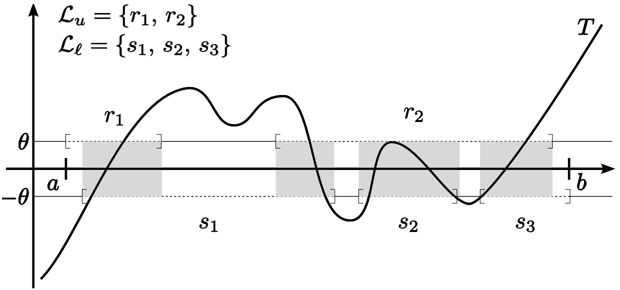

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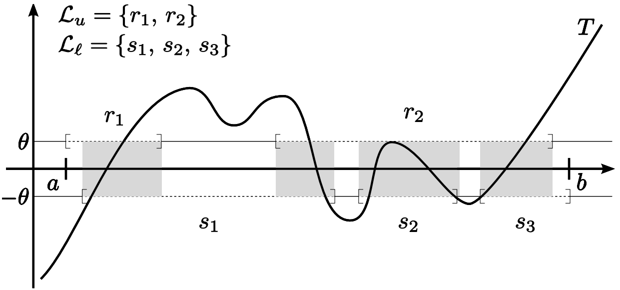
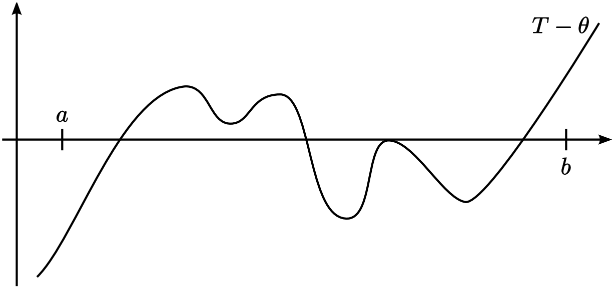


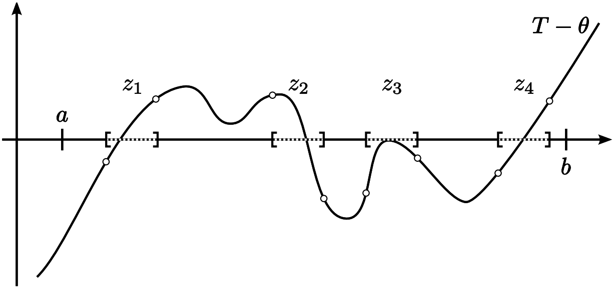
Finding enclosures of roots of a function

- Compute a list of intervals where $T - \theta \leq 0$, T is a polynomial



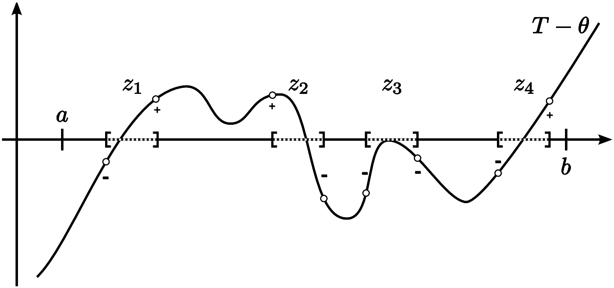
Finding enclosures of roots of a function

- Compute a list of intervals where $T - \theta \leq 0$, T is a polynomial
- Compute enclosures of the roots of $T - \theta$



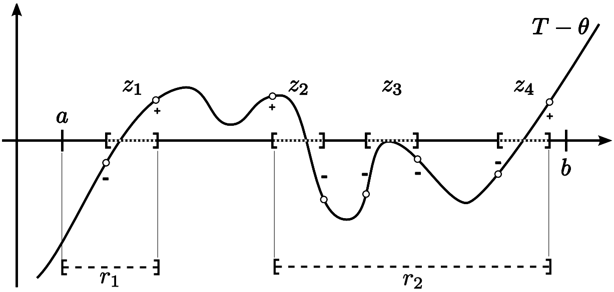
Finding enclosures of roots of a function

- Compute a list of intervals where $T - \theta \leq 0$, T is a polynomial
- Compute enclosures of the roots of $T - \theta$
- Compute sign changes



Finding enclosures of roots of a function

- Compute a list of intervals where $T - \theta \leq 0$, T is a polynomial
- Compute enclosures of the roots of $T - \theta$
- Compute sign changes
- Find suitable intervals



Our example - Relative error

Example:

$f(x) = \cos(x)$ over $[-1, 1]$, $p(x) = \sum_{i=0}^{5} c_i x^i$,
$\varepsilon(x) = p(x)/f(x) - 1$

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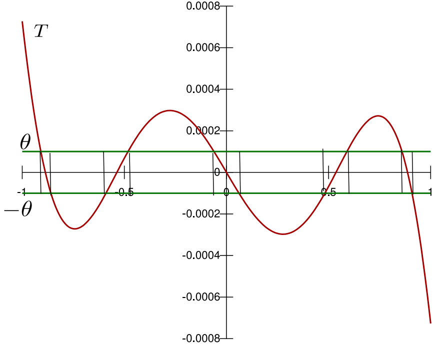
Compute Taylor Model for $\tau = p'f - pf'$, order 12.

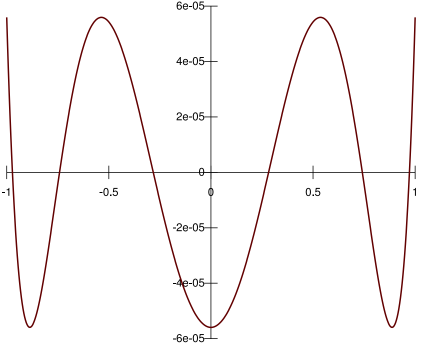
Remainder: $[-1.67352e - 7; 1.67352e - 7]$

Our example - Relative error



$r_0 \in -0.886[9;8]$, $r_1 \in -0.536[6;4]$,
$r_2 \in [-0.000001; 0.000001]$, $r_3 \in 0.536[4;6]$,
$r_4 \in 0.886[8;9]$

Our example - Relative error



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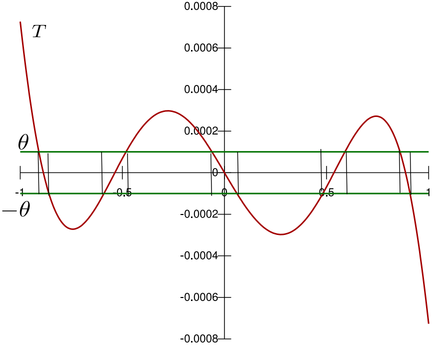
$\|\varepsilon\|_\infty \in 5.59[3/4] \times e - 5.$

Our Approach - Summary

- Purpose: fast and safely compute the supremum norm $\|f - p\|_\infty$ and $\|p/f - 1\|_\infty$ over an interval $[a, b]$
- Enclose all the zeros of the first derivative of the error
- Evaluate the approximation error on these small intervals only, using IA
- Use a Taylor Model based approach to overcome the dependency
- Enclose the zeros of a function using our algorithm

Our example - Relative error - How small should the remainder be?

$\tau(x) - T(x) \in [-\theta, \theta], \forall x \in I$
$\tau(x) = 0 \implies T(x) \in [-\theta, \theta]$



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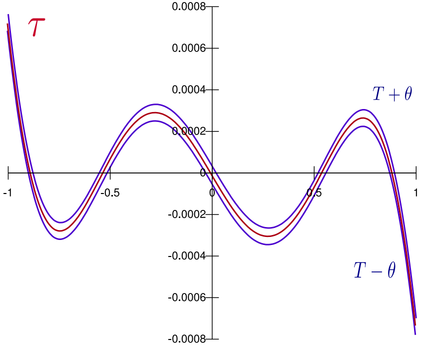

Degree: 12

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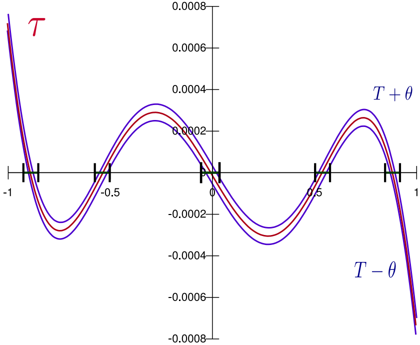

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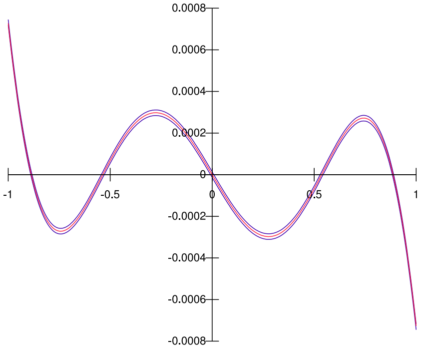

Degree: 14

Our example - Relative error - How small should the remainder be?

$$\tau(x) - T(x) \in [-\theta, \theta], \forall x \in I$$
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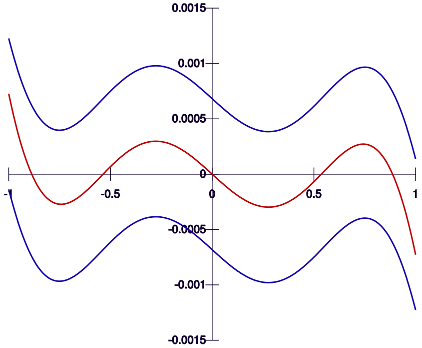

Degree: 8

Our example - Relative error - How small should the remainder be?

- Heuristics
- Start with a value close to the approx. max value of τ
- In our example: $\theta_i = 6.44e - 4$

θ	degree T	$\|\varepsilon\|_\infty$
θ_i	8	$[5.59e - 5; 1.84]$
$\theta_i/2$	9	$[5.59e - 5; 1e - 3]$
$\theta_i/10$	10	$[5e - 5; 8e - 5]$
$\theta_i/10^2$	11	$0.5[5; 7]e - 4$
$\theta_i/10^3$	12	$0.559[3; 4]e - 4$
$\theta_i/10^4$	13	$0.55935[2; 4]e - 4$
$\theta_i/10^5$	14	$0.5593528[2; 4]e - 4$

Results

Experiments were made on an Intel Pentium D 3.00GHz with a 2GB RAM.

f	$[a,\, b]$	d_p[1]	m[2]	acc[3]	time[4]
$\exp(x) - 1$	$[-0.25,\, 0.25]$	5	r	37.6	412
$\log_2(1 + x)$	$[-2^{-9},\, 2^{-9}]$	7	r	83.3	2, 186
$\cos(x)$	$[-0.5,\, 0.25]$	15	r	19.5	2, 235
$\exp(x)$	$[-0.125,\, 0.125]$	25	r	42.3	7, 753
$\sin(x)$	$[-0.5,\, 0.5]$	9	a	21.5	520
$\exp(\cos(x)^2 + 1)$	$[1,\, 2]$	15	r	25.5	10, 984
$\tan(x)$	$[0.25,\, 0.5]$	10	r	26.0	1, 072
$x^{2.5}$	$[1,\, 2]$	7	r	15.5	1, 362

[1] Degree of p

[2] Error mode considered: a=absolute, r=relative

[3] Accuracy

[4] Timings in ms

Conclusion

- Safe and fast algorithm for bounding the supremum norm of the error functions
- Combination and reusal of various techniques (TM, polynomial roots isolation, interval arith)
- Absolute and Relative errors handled
- Faster and more accurate than other current approaches
- Future works:
  - Formal proof (AD, isolation of roots, multiple precision interval arithmetic are needed in the proof checker)
  - Generalization of the algorithm for multivariate functions.

Questions

Thank you for your attention!

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Results

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