A first step towards validated impulsive spacecraft rendezvous

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SEA MAC
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Simplified Tshauner-Hempel equations

We adopt the true anomaly ν of the target spacecraft as the independent variable.
Let $e \in [0, 1)$ - fixed excentricity and $\rho = 1 + e \cos \nu$.
Find a basis of solutions for:

$$\begin{cases} x'' = 2z' \\ y'' = -y \\ z'' = \frac{3z}{\rho} - 2x'. \end{cases}$$

The system can be decoupled in:

- *out of plane motion:* y - solution of the harmonic oscillator;
- *in plane motion*: (x, z) plane.
- integrating we obtain:

$$z'' + (4 - 3/\rho)z = c$$

$\rightarrow$ non-homogeneous LODE with non-polynomial coefficients
$\rightarrow$ Yamanaka-Ankerson closed form basis of solutions (2 trigonometric, 1 integral solution)

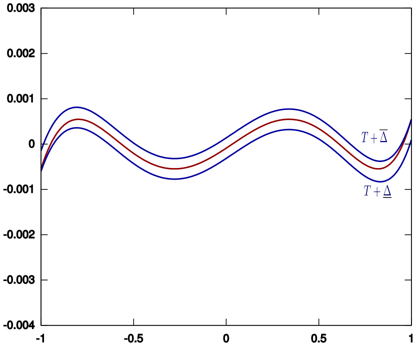
Goal: obtain validated polynomial solutions

$\rightsquigarrow$ to be used afterwards in the polynomial optimization problem

Rigorous polynomial Approximations (RPAs)

- Two steps:
  1. Obtain very good numerical polynomial approximations
  2. Validate using a fixed point method

f replaced with RPA : $(T, \boldsymbol{\Delta})$
- polynomial T of degree n
- interval $\boldsymbol{\Delta}$ s. t.
$f(x) - T(x) \in \boldsymbol{\Delta}, \forall x \in [a, b]$



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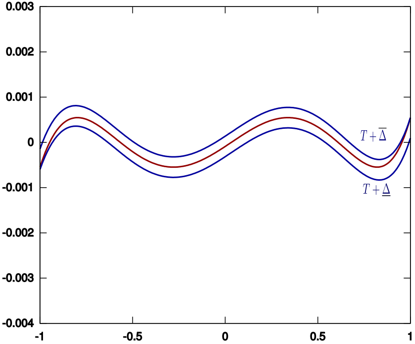

- How to deal with the resulting *"polynomial"* problem?
- We need very good numerical approximations to reduce the size of the polynomial optimization problem to be solved afterwards

Best uniform polynomial approximations

Minimax Polynomial

Let $f \in \mathcal{C}[a,b]$, $n \in \mathbb{N}$ given. Minimax polynomial $p_n^* \in \mathcal{P}_n$ solution of

$$\text{Find}\, p \in \mathcal{P}_n \text{which minimizes } \|f - p\|_\infty . \tag{1}$$

- p_n^* exists and is unique (Kirchberger, 1902)
- iterative algorithm (Remez, 1934) based on Alternation theorem
  There are at least $n+2$ points in $[a,b]$ at which $f - p_n^*$ attains its maximum absolute value (i.e $\|f - p_n^*\|_\infty$) with alternating signs.

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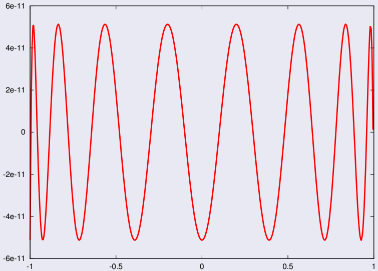
Minimax approx error: $f(\nu) = 4 - \frac{3}{1+0.052\cos\pi\nu}, \nu \in [-1,1]$, deg. 14 polynomial.



RPAs in 2 Steps to make use of D-finite functions

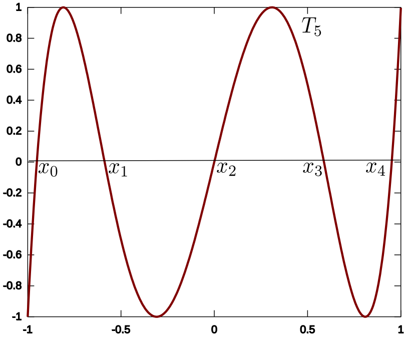
For our initial differential equation:

$$z'' + \left(4 - \frac{3}{1 + e\cos\nu}\right) z = c$$

- Approximate $4 - \frac{3}{1+e\cos\nu}$ by a minimax polynomial
- Reduce to a LODE with polynomial coefficients
- Obtain rigorous near-best uniform approximations using recent developed methods (Benoît et al. 2011).

Quick Reminder: Chebyshev Polynomials

$$T_n(cos(\theta)) = \cos(n\theta)$$



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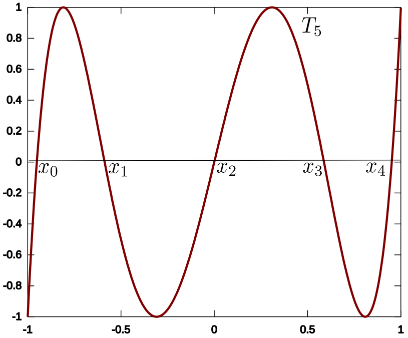

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$$x_k = \cos\left(\frac{(k+1/2)\,\pi}{n}\right), k = 0, \ldots, n-1.$$

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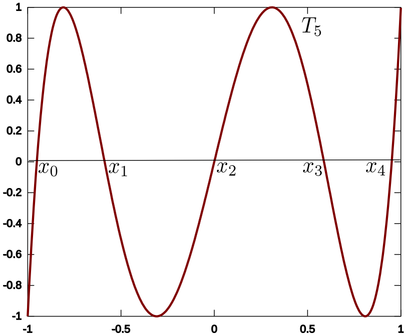

$T_{i+1} = 2xT_i - T_{i-1}, T_0(x) = 1, T_1(x) = x$

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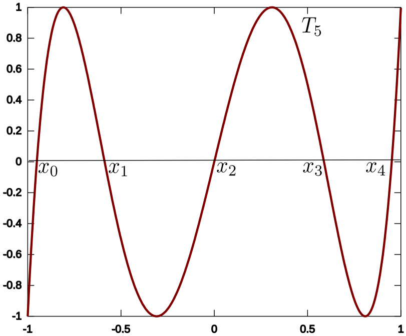

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Orthogonality:

$$\int_{-1}^{1} \frac{T_i(x)T_j(x)}{\sqrt{1-x^2}} dx = \begin{cases} 0 & \text{if } i \neq j \\ \pi & \text{if } i = 0 \\ \frac{\pi}{2} & \text{otherwise} \end{cases}$$

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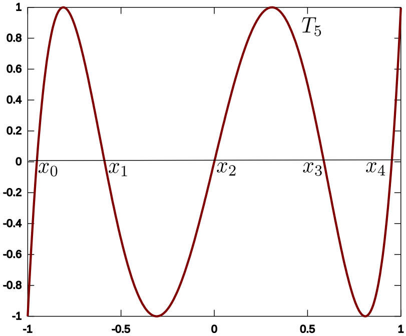

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Chebyshev Series vs Taylor Series

Two approximations of f:

- by Taylor series

$$f = \sum_{n=0}^{+\infty} c_n x^n, \; c_n = \frac{f^{(n)}(0)}{n!},$$

- or by Chebyshev series

$$f = \sum_{n=-\infty}^{+\infty} t_n T_n(x),$$

$$t_n = \frac{1}{\pi} \int_{-1}^{1} T_n(t) \frac{f(t)}{\sqrt{1-t^2}} dt.$$

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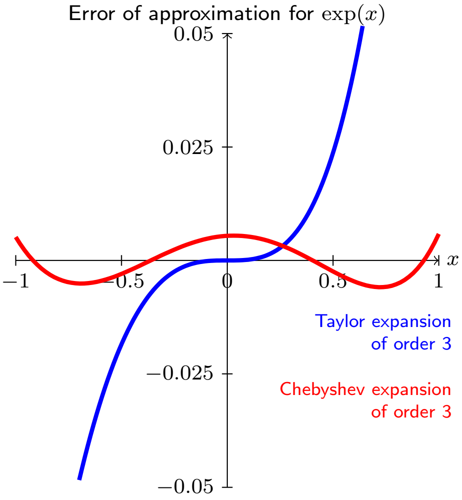

Quality of approximation of truncated Chebyshev series compared to best polynomial approximation

It is well-known that truncated Chebyshev series $\pi_d(f)$ are *near-best* uniform approximations [Chap 5.5, Mason & Handscomb 2003].

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$$\|f - \pi_d(f)\|_\infty \leqslant \underbrace{\left(\frac{4}{\pi^2} \log d + O(1)\right)}_{\Lambda_d} \|f - p_d^*\|_\infty$$

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- $\Lambda_{10} = 2.22... \rightarrow$ we lose at most 2 bits
- $\Lambda_{30} = 2.65... \rightarrow$ we lose at most 2 bits
- $\Lambda_{100} = 3.13... \rightarrow$ we lose at most 3 bits
- $\Lambda_{500} = 3.78... \rightarrow$ we lose at most 3 bits

Uniform Approximation of D-finite Functions

Problem

Given an integer d, and a D-finite function f specified by a differential equation with polynomial coefficients and suitable boundary conditions, find the coefficients of a polynomial $p(x)$ of degree d and a "small" bound B such that $|p(x) - f(x)| < B$ for all x in $[-1, 1]$.

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Applications:

- Repeated evaluation on a line segment
  - Plot
  - Numerical integration
- Transform non-polynomial to polynomial optimization problems

Previous Work

Computation of the Chebyshev coefficients for D-finite functions

- Using a relation between coefficients Clenshaw (1957)
- Using the recurrence relation between the coefficients Fox-Parker (1968)
- The tau method of Lanczos (1938), Ortiz (1969-1993)

Validation:

- Kaucher-Miranker (1984)
- Monomial basis: Verified integration of Taylor Models (Makino & Berz, 1998)

Our Work

Given a linear differential equation with polynomial coefficients, boundary conditions and an integer d

- Compute a polynomial approximation p on $[-1, 1]$ of degree d of the solution f in the Chebyshev basis in $O(d)$ arithmetic operations.
- Compute a sharp bound B such that $|f(x) - p(x)| < B, \ x \in [-1, 1]$ in $O(d)$ arithmetic operations.

Theorem 1 (60's, BenoitJoldesMezzarobba11)

$\sum u_n T_n(x)$ is solution of a linear differential equation with polynomial coefficients iff the sequence u_n is cancelled by a linear recurrence with polynomial coefficients.

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Recurrence relation + good initial conditions $\Rightarrow$ Fast numerical computation of the coefficients

Taylor: $\exp = \sum \frac{1}{n!} x^n$
Rec: $u(n+1) = \frac{u(n)}{n+1}$

$u(0) = 1$	$1/0! = 1$
$u(1) = 1$	$1/1! = 1$
$u(2) = 0,5$	$1/2! = 0,5$
$\vdots$	$\vdots$
$u(50) \approx 3,28.10^{-65}$	$1/50! \approx 3,28.10^{-65}$

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$u(0) = 1,266$	$I_0(1) \approx 1,266$
$u(1) = 0,565$	$I_1(1) \approx 0,565$
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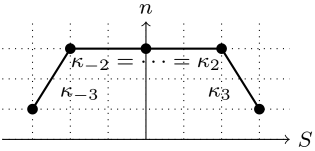
$u(50) \approx 4,450.10^{67}$	$I_{50}(1) \approx 2,934.10^{-80}$

Convergent and Divergent Solutions of the Recurrence

Study of the Chebyshev recurrence

If $u(n)$ is solution, then there exists another solution $v(n) \sim \frac{1}{u(n)}$

Newton polygon of a Chebyshev recurrence



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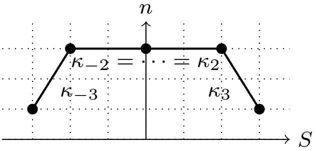

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For the recurrence $u(n+1) + 2nu(n) - u(n-1)$

Two independent solutions are $I_n(1) \sim \frac{1}{(2n)!}$ and $K_n(1) \sim (2n)!$

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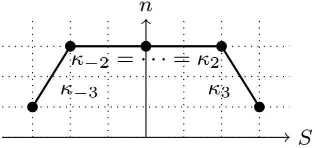

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Miller's algorithm

To compute the first N coefficients of the most convergent solution of a recurrence relation of order 2

- Initialize $u(N) = 0$ and $u(N-1) = 1$ and compute the first coefficients using the recurrence backwards
- Normalize u with the initial condition of the recurrence

Example 2

$y(x) = e^x = \sum_{n=-\infty}^{\infty} c_n \, T_n(x)$ $c_{n+1} + 2n \, c_n - c_{n-1} = 0$

$$\begin{array}{l} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{50} \\ u_{51} \\ u_{52} \end{array} \qquad \begin{array}{l} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{50} \\ c_{51} \\ c_{52} \end{array}$$

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$$\begin{aligned}
u_0 & \\
u_1 & \\
u_2 & \approx -4,72 \cdot 10^{80} \\
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$$S = \sum_{n=-50}^{50} u_n \, T_n(0) \approx -3,48 \cdot 10^{81}$$

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$c_n := u_n / S$

$$\begin{aligned} c_0 &\approx 1,27 \\ c_1 &\approx -5,65 \cdot 10^{-1} \\ c_2 &\approx 1,36 \cdot 10^{-1} \\ &\vdots \\ c_{50} &\approx 2,93 \cdot 10^{-80} \\ c_{51} &\approx 2,88 \cdot 10^{-82} \\ c_{52} &\approx 0 \end{aligned}$$

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Algorithm for Computing the Coefficients

Algorithm 1

Input: *a differential equation of order r with boundary conditions*
Output: *a polynomial approximation of degree N of the solution*

- *compute the Chebyshev recurrence of order $2s \geq 2r$*
- *for i from 1 to s*
  - *using the recurrence relation backwards, compute the first N coefficients of the sequence $u^{[i]}$ starting with the initial conditions*

$$\left(u^{[i]}(N+2s), \cdots, u^{[i]}(N+i), \cdots, u^{[i]}(N+1)\right) = (0, \cdots, 1, \cdots, 0)$$

- *combine the s sequences $u^{[i]}$ according to the r boundary conditions and the $s-r$ symmetry relations*

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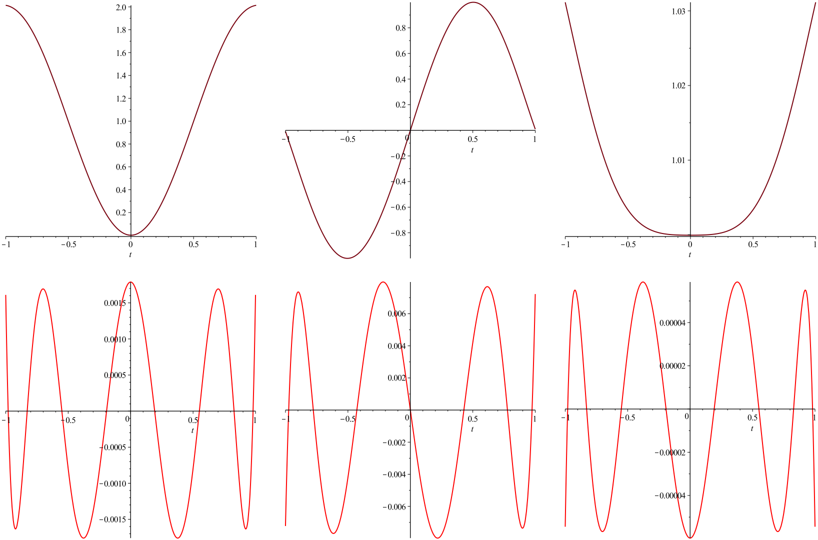
Theorem

This algorithm runs in $O(N)$ arithmetic operations

Quality of polynomial approximations in our case

Sol. Basis

Error



degree = 4 | degree = 4 | degree = 4

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- Compute a sharp bound B such that $|f(x) - p(x)| < B,\ x \in [-1, 1]$ in $O(d)$ arithmetic operations.

Validation

Fixed Point Theorem Applied to a Differential Equation: f is solution of

$$y'(x) - a(x)y(x) = 0, \text{ with } y(0) = y_0,$$

if and only if f is a fixed point of τ defined by

$$\tau(y)(t) = y_0 + \int_0^t a(x)y(x)dx.$$

Validation

Fixed Point Theorem Applied to a Differential Equation: f is solution of

$$y'(x) - a(x)y(x) = 0, \text{ with } y(0) = y_0,$$

if and only if f is a fixed point of τ defined by

$$\tau(y)(t) = y_0 + \int_0^t a(x)y(x)dx.$$

For all functions $a(x)$, if $\frac{\|a\|_\infty^j}{j!} < 1$ then $\forall i \geqslant j$, τ^i is a contraction map.

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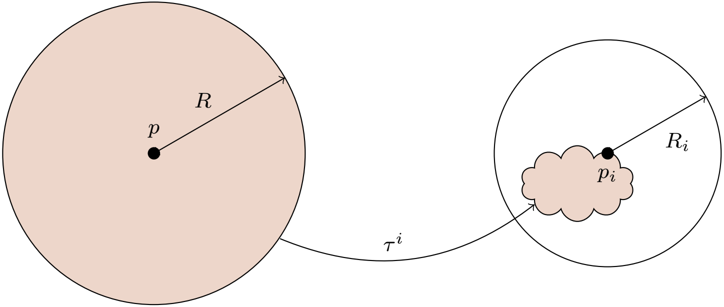

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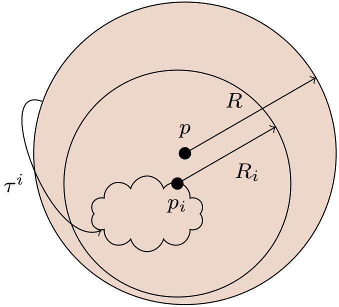

$$f \in B(p,R) \implies f = \tau^i(f) \in B(p_i,R_i)$$

$$\|f-p\|_\infty \leq \|f-p_i\|_\infty + \|p-p_i\|_\infty \leq R_i + \|p-p_i\|_\infty \leq R$$

ATV Mission Example

Excentricity e	0.0052
ν_0	0
ν_f	π
$X_0 = [x_0, z_0, x'_0, z'_0]$	$[-30156, 5026, 7146, 0]$
$X_f = [x_f, z_f, x'_f, z'_f]$	$[-994.8, 0, 0.003, 0]$
N_{max}	4

Polynomial Optimization problem

Minimize a given consumption objective $\sum_{i=1}^{N} |\Delta V_i|$ under dynamic constraints:

$$
\min_{\Delta V_i, \nu_i} \quad \sum_{i=1}^{N} |\Delta V_i|
$$

$$
\text{w.r.t.} \quad \begin{cases} X(\nu) = \sum_i \tilde{P}(\nu)\Delta V_i & \text{(Dynamics)} \\ X(\nu_1) = X_1,\ X(\nu_f) = X_f & \text{(RdV cond.)} \\ -\overline{\Delta v}_i \leq \Delta V_i \leq \overline{\Delta v}_i & \text{(Saturation)} \\ \Gamma(X, \nu) \leq 0 & \text{(Path const.)} \end{cases}
$$

Tests with Gloptipoly, SparsePOP

⇝ With Gloptipoly we manage to solve for 2 impulses (both SDPT3 and SeDuMi).
⇝ With SparsePOP:

- 2 impulses, deg. 4, SeDuMi:

Relax order	Timings(s)
3	35.72
4	78.44

- 3 impulses, deg. 4, SeDuMi:

Relax order	Timings(s)
3	64.64
4	-

- works up to 3 impulses and polynomial of degree 4
- lacks certification of Gloptipoly

ATV mission solution:

ν_i	$[0.2, 3.1415]$
$\Delta V x_i$	$[1228.7, 24.98]$
$\Delta V z_i$	$[3084.7, 4499.6]$
Cost	8838.4