Searching for sinks of Hénon map using a multiple-precision GPU arithmetic library

Mioara Joldeș, Valentina Popescu, Warwick Tucker

Mathematical Structures of Computation
Lyon 2014

Hénon attractor

Hénon Map

$$h_{a,b}(x_1, x_2) = (1 + x_2 - ax_1^2, bx_1)$$

Hénon attractor

Hénon Map

$$h_{a,b}(x_1, x_2) = (1 + x_2 - ax_1^2, bx_1)$$

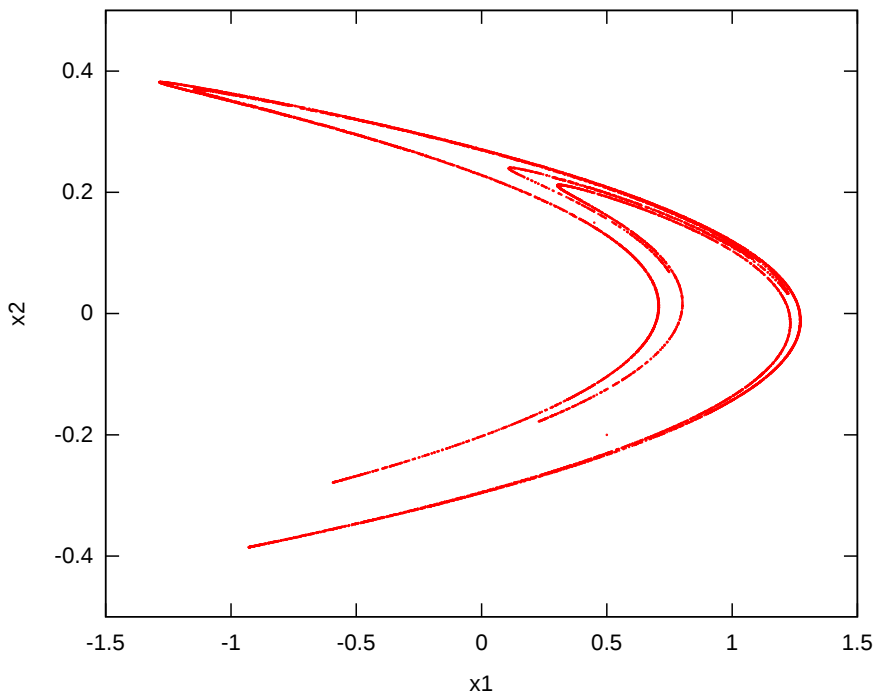
- Map iterations: $h_{a,b}^{i+1} := h_{a,b} \circ h_{a,b}^{i}, i \in \mathbb{N}^*$.

Hénon attractor

Hénon Map

$$h_{a,b}(x_1, x_2) = (1 + x_2 - ax_1^2, bx_1)$$

- Map iterations: $h_{a,b}^{i+1} := h_{a,b} \circ h_{a,b}^{i}, i \in \mathbb{N}^*$.
- For classical parameter values $a = 1.4$, $b = 0.3$ one observes the so-called Hénon attractor by iterating $h_{a,b}^{n}, n \to \infty$:

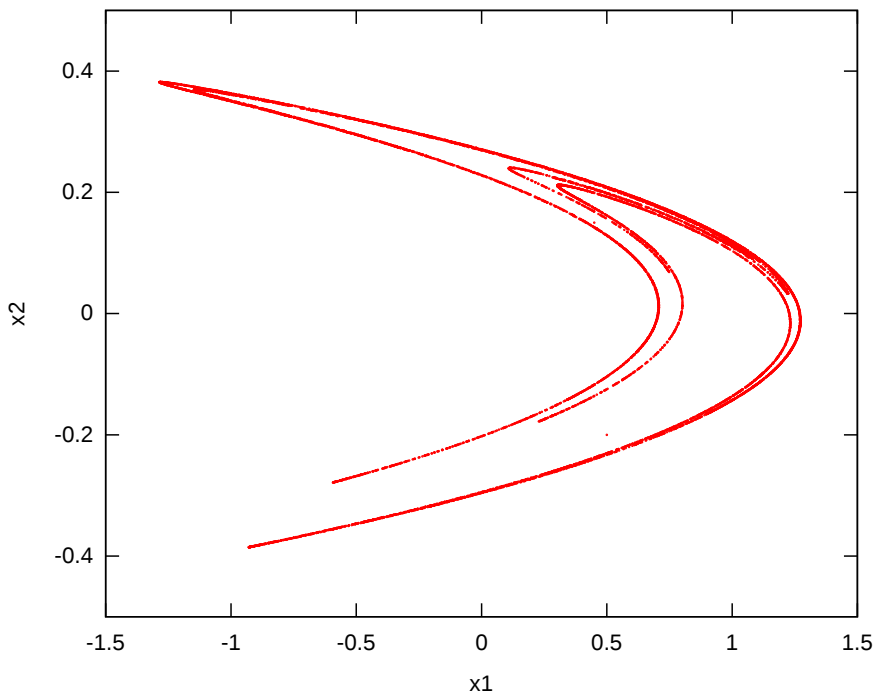


Hénon attractor

Hénon Map

$$h_{a,b}(x_1, x_2) = (1 + x_2 - ax_1^2, bx_1)$$

- Map iterations: $h_{a,b}^{i+1} := h_{a,b} \circ h_{a,b}^{i}, i \in \mathbb{N}^*$.
- For classical parameter values $a = 1.4$, $b = 0.3$ one observes the so-called Hénon attractor by iterating $h_{a,b}^{n}, n \to \infty$:



Open question: Is this a Strange Attractor ?

Some basic terminology

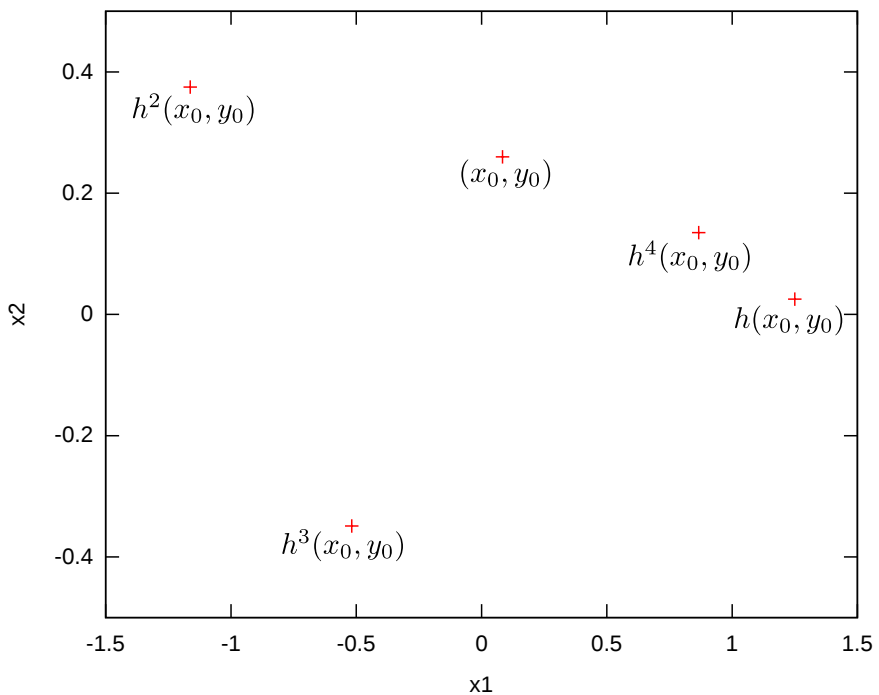
Orbit of x: $\Gamma(x) = \Gamma^+(x) \cup \Gamma^-(x)$

- Forward orbit:

$$\Gamma^+(x) := \{h^n_{a,b}(x), n \in \mathbb{N}\}$$

- Backward orbit:

$$\Gamma^-(x) := \{y : \exists n \in \mathbb{N} : h^n_{a,b}(y) = x\}$$



Some basic terminology

Orbit of x: $\Gamma(x) = \Gamma^+(x) \cup \Gamma^-(x)$

- Forward orbit:
$$\Gamma^+(x) := \{h_{a,b}^n(x), n \in \mathbb{N}\}$$
- Backward orbit:
$$\Gamma^-(x) := \{y : \exists n \in \mathbb{N} : h_{a,b}^n(y) = x\}$$

Periodic orbit $\Gamma_n(x)$ of period n if $\exists n \in \mathbb{N}^*$ s.t. $h_{a,b}^n(x) = x$.

Some basic terminology

Orbit of x: $\Gamma(x) = \Gamma^+(x) \cup \Gamma^-(x)$

- Forward orbit:
$$\Gamma^+(x) := \{h_{a,b}^n(x), n \in \mathbb{N}\}$$
- Backward orbit:
$$\Gamma^-(x) := \{y : \exists n \in \mathbb{N} : h_{a,b}^n(y) = x\}$$

Periodic orbit $\Gamma_n(x)$ of period n if $\exists n \in \mathbb{N}^*$ s.t. $h_{a,b}^n(x) = x$.

Stable orbit $\Gamma(x)$

Let $d(x, y) = ||x - y||$ and $d(y, \Gamma(x)) = \inf_{z \in \Gamma(x)} d(y, z)$.

$\Gamma(x)$ is stable if given $\varepsilon > 0, \exists \delta > 0$ s.t. $d(h_{a,b}^n(y), \Gamma(x)) < \varepsilon$, $\forall n \in \mathbb{N}^*$ and $\forall y$ s.t. $d(y, \Gamma(x)) < \delta$.

Some basic terminology

Orbit of x: $\Gamma(x) = \Gamma^+(x) \cup \Gamma^-(x)$

- Forward orbit:
$$\Gamma^+(x) := \{h^n_{a,b}(x), n \in \mathbb{N}\}$$
- Backward orbit:
$$\Gamma^-(x) := \{y : \exists n \in \mathbb{N} : h^n_{a,b}(y) = x\}$$

Periodic orbit $\Gamma_n(x)$ of period n if $\exists n \in \mathbb{N}^*$ s.t. $h^n_{a,b}(x) = x$.

Stable orbit $\Gamma(x)$

Let $d(x, y) = ||x - y||$ and $d(y, \Gamma(x)) = \inf_{z \in \Gamma(x)} d(y, z)$.

$\Gamma(x)$ is stable if given $\varepsilon > 0, \exists \delta > 0$ s.t. $d(h^n_{a,b}(y), \Gamma(x)) < \varepsilon$, $\forall n \in \mathbb{N}^*$ and $\forall y$ s.t. $d(y, \Gamma(x)) < \delta$.

Asymptotically Stable orbit $\Gamma(x)$ (sink)

$\Gamma(x)$ is asymptotically stable if it is stable and (by choosing δ smaller if necessary) $d(h^n_{a,b}(y), \Gamma(x)) \to 0$ as $n \to \infty$.

Some basic terminology II

Attractor

- describes asymptotic behaviour of typical orbits.

Some basic terminology II

Attractor

- describes asymptotic behaviour of typical orbits.

- Attracting Sets
  Let T be a compact set such that $h(T) = T$.
  T is called *attracting* if there exists an open neighborhood U of T such that $\bigcap_{i=0}^{\infty} h^i(U) = T$.

Some basic terminology II

Attractor

- describes asymptotic behaviour of typical orbits.

- Attracting Sets
  Let T be a compact set such that $h(T) = T$.
  T is called *attracting* if there exists an open neighborhood U of T such that $\bigcap_{i=0}^{\infty} h^i(U) = T$.

- Attractor: An attracting set which contains a dense orbit.

Some basic terminology II

Attractor

- describes asymptotic behaviour of typical orbits.

- Attracting Sets
  Let T be a compact set such that $h(T) = T$.
  T is called *attracting* if there exists an open neighborhood U of T such that $\bigcap_{i=0}^{\infty} h^i(U) = T$.

- Attractor: An attracting set which contains a dense orbit.
- Basin of attraction, $B(T)$: largest such U.

Some basic terminology II

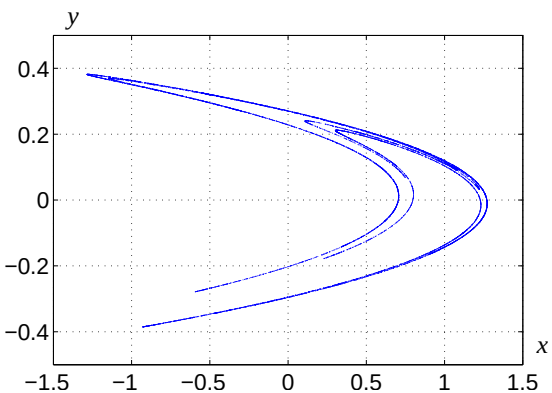
Attractor

- describes asymptotic behaviour of typical orbits.
- Attracting Sets
  Let T be a compact set such that $h(T) = T$.
  T is called *attracting* if there exists an open neighborhood U of T such that $\bigcap_{i=0}^{\infty} h^i(U) = T$.
- Attractor: An attracting set which contains a dense orbit.
- Basin of attraction, $B(T)$: largest such U.

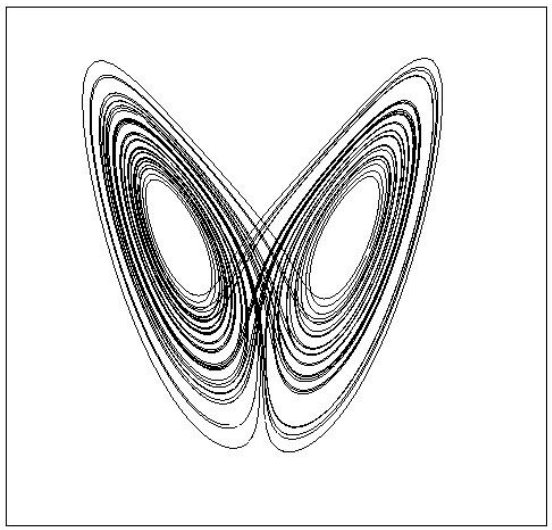
Strange Attractor

For *almost all* pairs of points $x, y \in B(T)$ there exists $k \in \mathbb{N}^*$ s.t. $h^k(x)$ and $h^k(y)$ separate by at least a constant δ_h. chaotic dynamics

Some basic terminology III - Strange Attractor



(a) Hénon Attractor

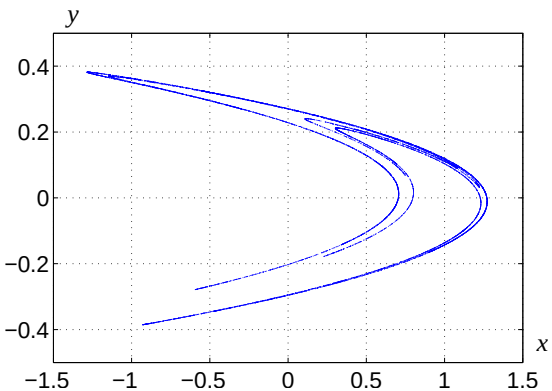


(b) Lorenz Attractor

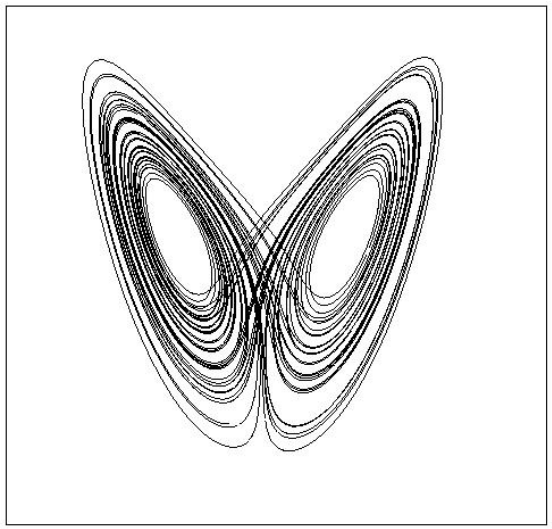
(c) Ueda Attractor

- name coined by Takens and Ruelle $\simeq 1971$

Some basic terminology III - Strange Attractor



(a) Hénon Attractor



(b) Lorenz Attractor

(c) Ueda Attractor

- name coined by Takens and Ruelle $\simeq 1971$

Quoting Ruelle...

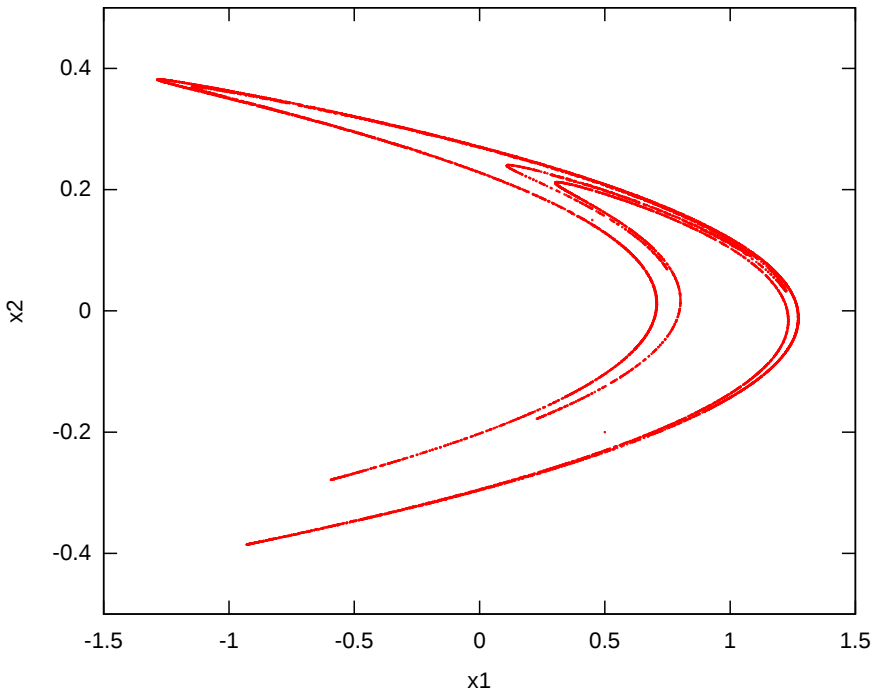
The Mathematical Intelligencer 2, 126 (1980)

I asked Floris Takens if he had created this remarkably successful expression. Here is his answer: "Did you ever ask God whether he created this damned Universe? ... I don't remember anything ... I often create without remembering it..." The creation of strange attractors thus seems to be surrounded by clouds and thunder. Anyway, the name is beautiful, and well suited to these astonishing objects, of which we understand so little.

Open Question: Is Hénon Attractor a Strange Attractor?

Hénon Map

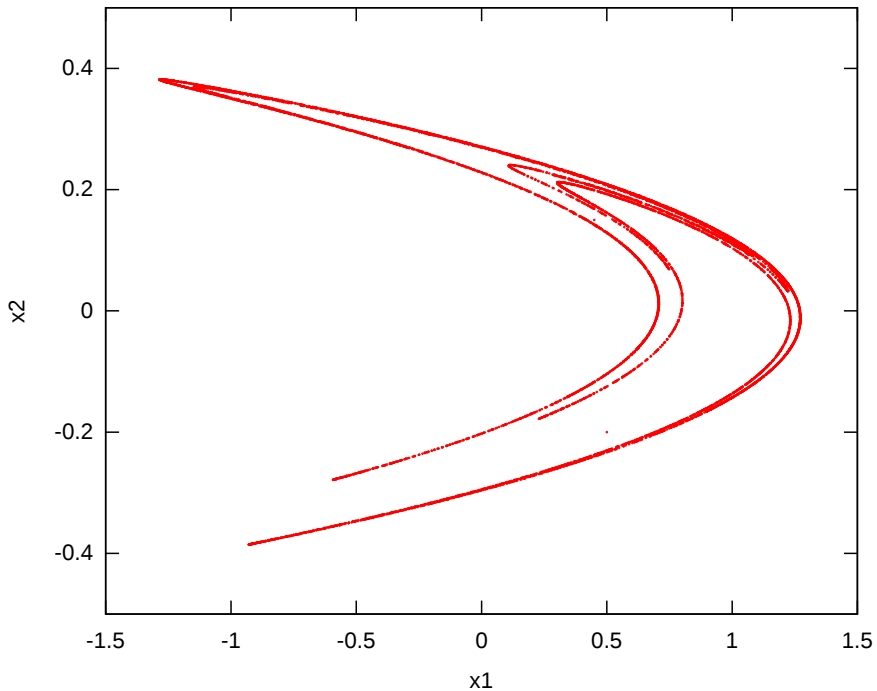
$$h_{a,b}(x_1, x_2) = (1 + x_2 - ax_1^2, bx_1)$$



Open Question: Is Hénon Attractor a Strange Attractor?

Hénon Map

$$h_{a,b}(x_1, x_2) = (1 + x_2 - ax_1^2, bx_1)$$

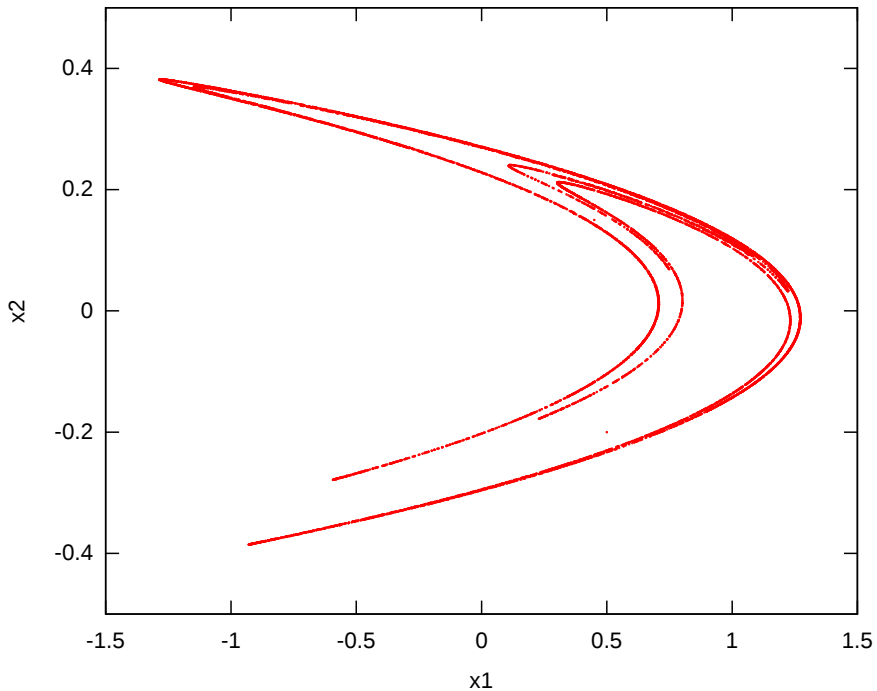


- Chaotic map: aperiodic trajectories (generally believed)

Open Question: Is Hénon Attractor a Strange Attractor?

Hénon Map

$$h_{a,b}(x_1, x_2) = (1 + x_2 - ax_1^2, bx_1)$$

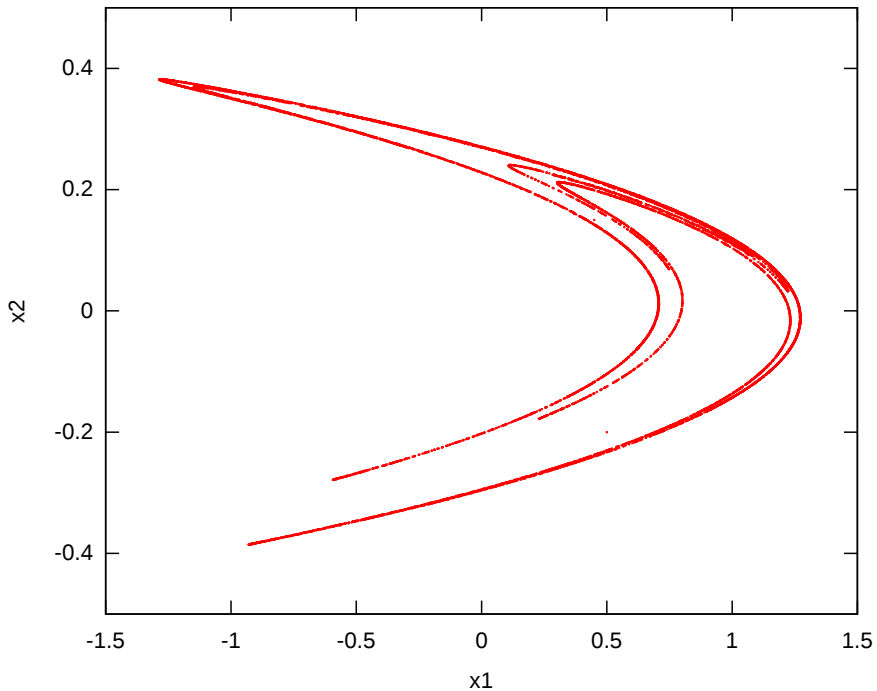


- Chaotic map: aperiodic trajectories (generally believed)
- there is a set of parameters (near $b = 0$) with positive Lebesgue measure for which the Hénon map has a strange attractor. [Benedicks & Carleson,'91].

Open Question: Is Hénon Attractor a Strange Attractor?

Hénon Map

$$h_{a,b}(x_1, x_2) = (1 + x_2 - ax_1^2, bx_1)$$

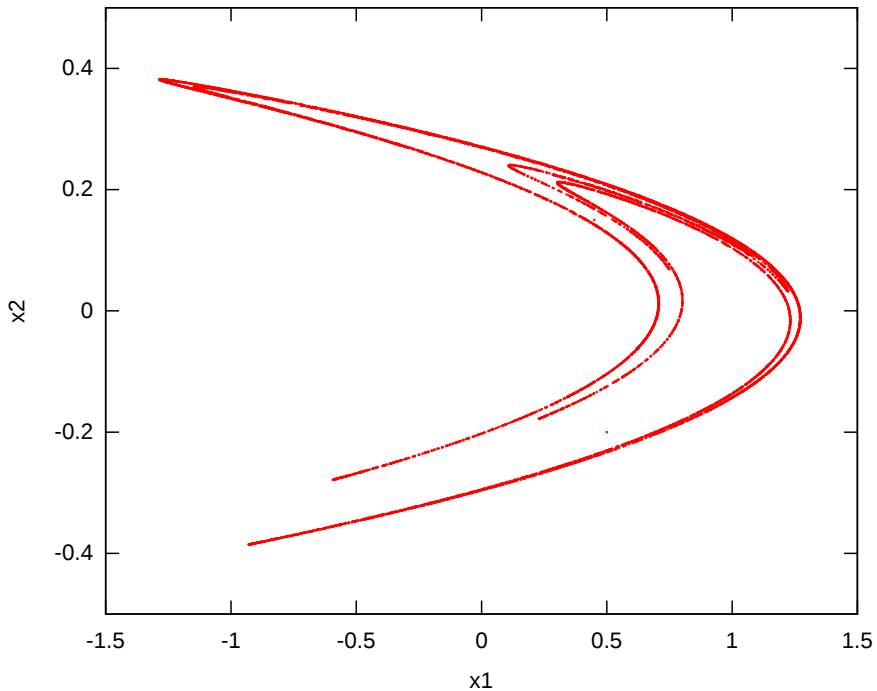


- Chaotic map: aperiodic trajectories (generally believed)
- there is a set of parameters (near $b = 0$) with positive Lebesgue measure for which the Hénon map has a strange attractor. [Benedicks & Carleson,'91].
- the parameters space is believed to be densely filled with regions, where the attractor is periodic.

Open Question: Is Hénon Attractor a Strange Attractor?

Hénon Map

$$h_{a,b}(x_1, x_2) = (1 + x_2 - ax_1^2, bx_1)$$



- Chaotic map: aperiodic trajectories (generally believed)
- there is a set of parameters (near $b = 0$) with positive Lebesgue measure for which the Hénon map has a strange attractor. [Benedicks & Carleson,'91].
- the parameters space is believed to be densely filled with regions, where the attractor is periodic.

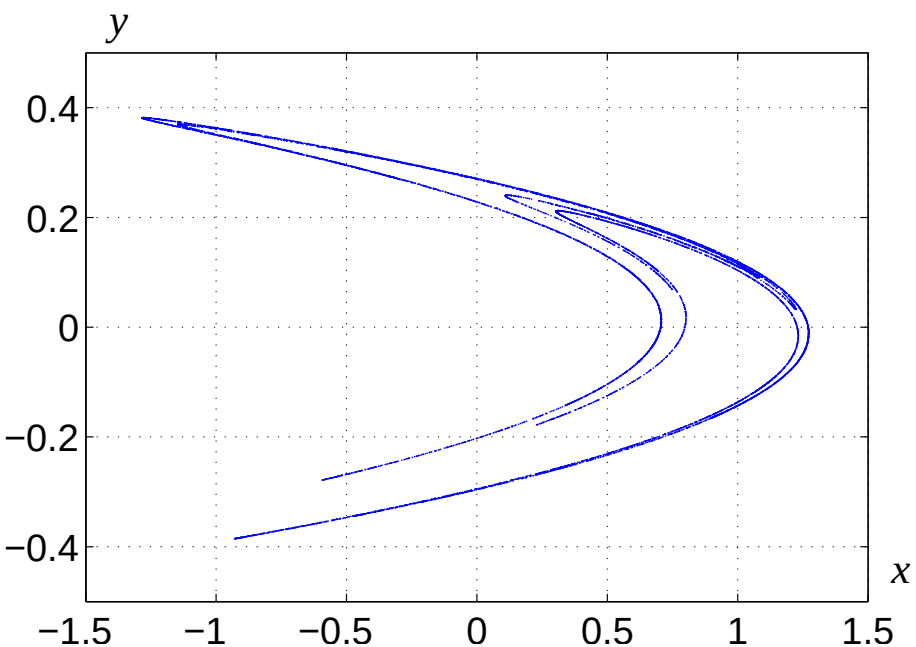
Goal: Given (a, b), prove existence of sinks (stable periodic orbits).

Example of sink in Hénon attractor [Galias & Tucker 2013]

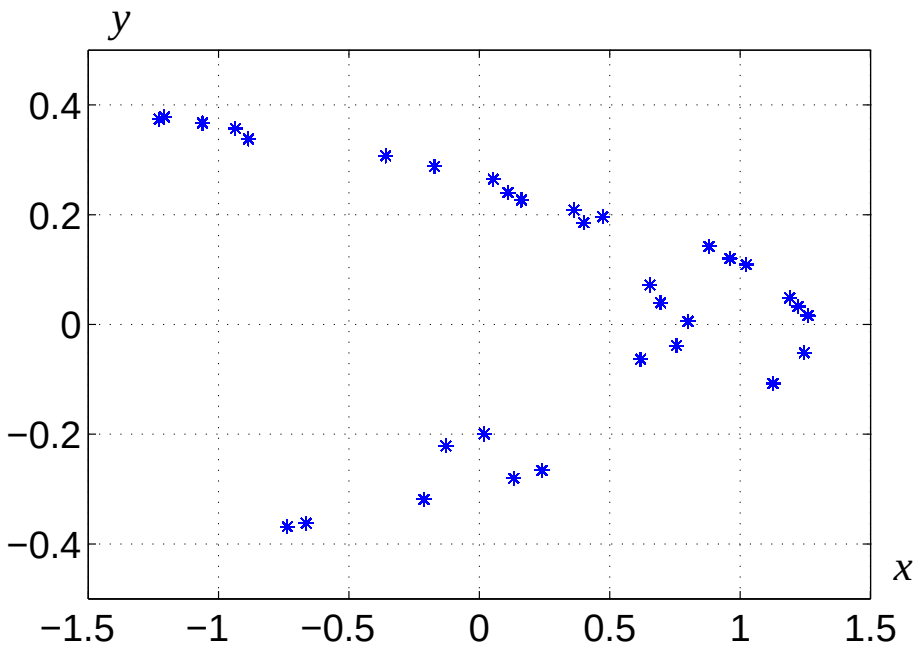
Hénon Map

$$h_{a,b}(x_1, x_2) = (1 + x_2 - ax_1^2, bx_1)$$

Let $a = 1.399999486944$, $b = 0.3$. Trajectory composed of 10000 points:



(a) $5 \cdot 10^9$ iterations skipped



(b) $6 \cdot 10^9$ iterations skipped.

Hénon attractor

Hénon Map

$$h_{a,b}(x_1, x_2) = (1 + x_2 - ax_1^2, bx_1)$$

Goal: Given (a, b), prove existence of sinks (stable periodic orbits).

Method:

Hénon attractor

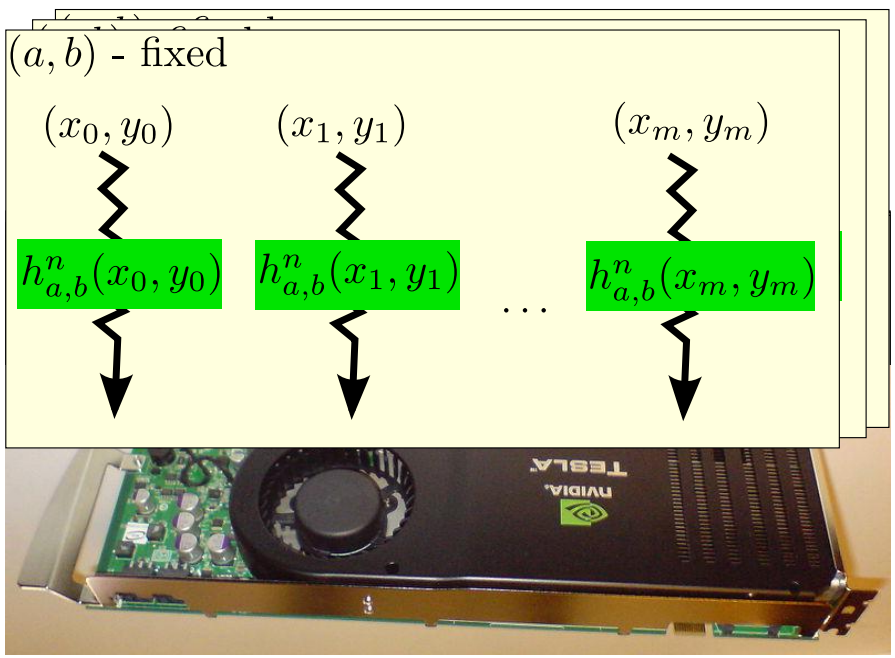
Hénon Map

$$h_{a,b}(x_1, x_2) = (1 + x_2 - ax_1^2, bx_1)$$

Goal: Given (a, b), prove existence of sinks (stable periodic orbits).

Method:

1. Find numerical approximation of sinks
   - Typical values: 10^6 parameters, 10^3 initial values, orbit length $10^6 \sim 10^9$.



Hénon attractor

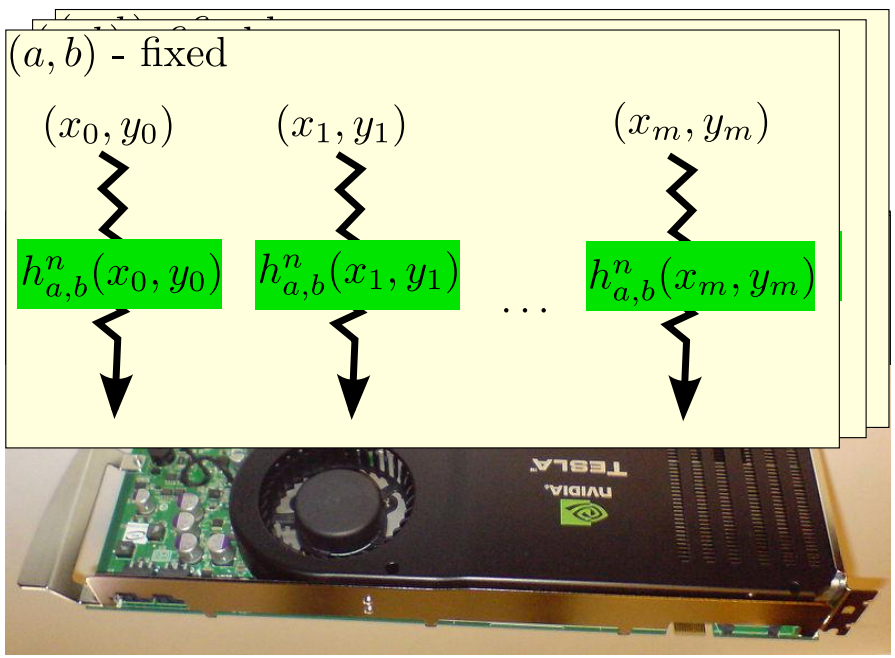
Hénon Map

$$h_{a,b}(x_1, x_2) = (1 + x_2 - ax_1^2, bx_1)$$

Goal: Given (a, b), prove existence of sinks (stable periodic orbits).

Method:

1. Find numerical approximation of sinks
   - Typical values: 10^6 parameters, 10^3 initial values, orbit length $10^6 \sim 10^9$.
   - chaotic $\rightarrow$ multiple precision



Hénon attractor

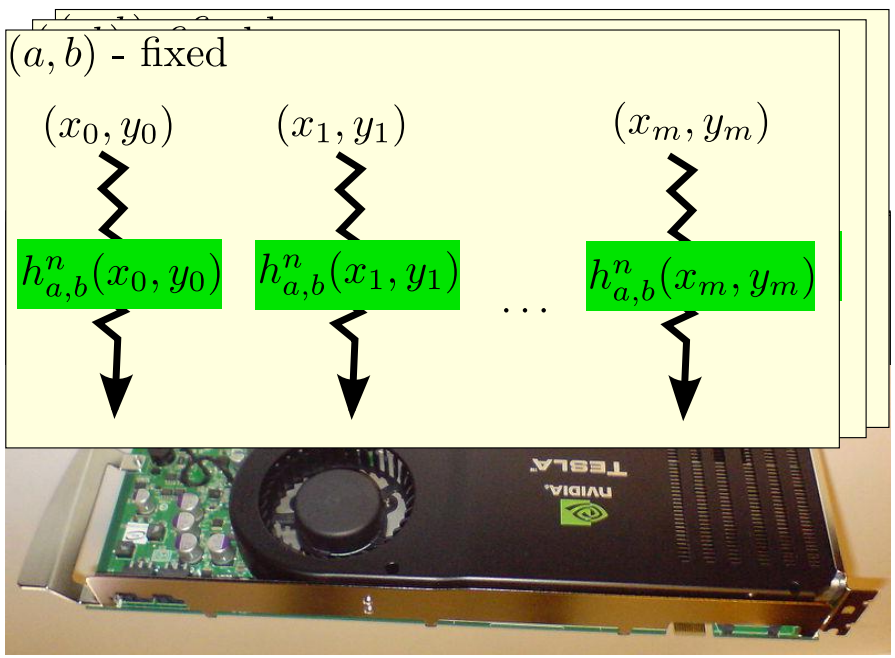
Hénon Map

$$h_{a,b}(x_1, x_2) = (1 + x_2 - ax_1^2, bx_1)$$

Goal: Given (a, b), prove existence of sinks (stable periodic orbits).

Method:

1. Find numerical approximation of sinks
   - Typical values: 10^6 parameters, 10^3 initial values, orbit length $10^6 \sim 10^9$.
   - chaotic $\rightarrow$ multiple precision
2. A posteriori validation of existence and stability with interval arithmetic



Hénon attractor

Hénon Map

$$h_{a,b}(x_1, x_2) = (1 + x_2 - ax_1^2, bx_1)$$

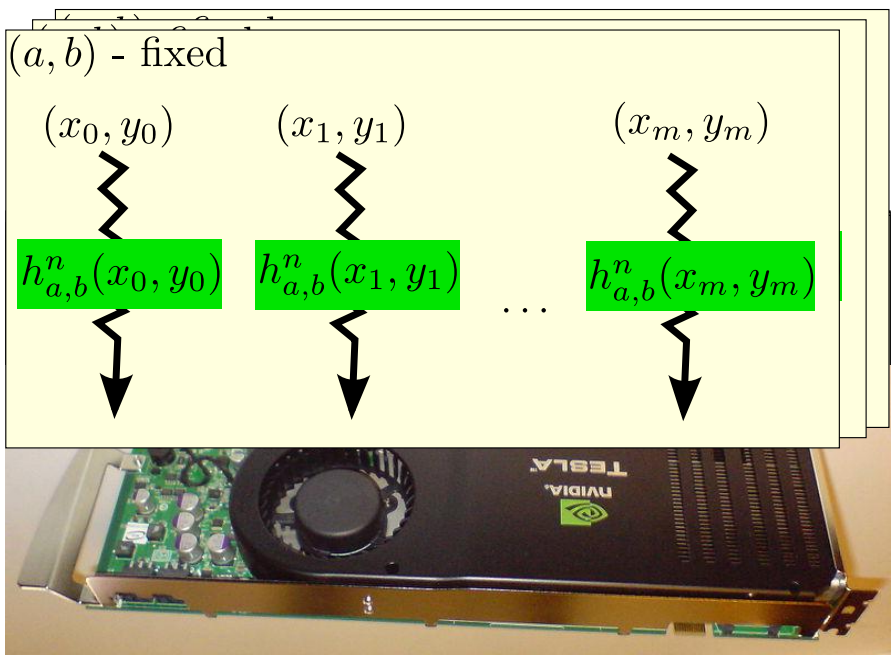
Goal: Given (a, b), prove existence of sinks (stable periodic orbits).

Method:

1. Find numerical approximation of sinks
   - Typical values: 10^6 parameters, 10^3 initial values, orbit length $10^6 \sim 10^9$.
   - chaotic $\rightarrow$ multiple precision
2. A posteriori validation of existence and stability with interval arithmetic

High parallelism $\rightarrow$
Graphics Processing Units
(GPUs)

Need for Multiple
Precison on GPUs



Why GPUs?

Typical values: 10^6 parameters, 10^3 initial values, 10^9 orbit iterations.

Why GPUs?

Typical values: 10^6 parameters, 10^3 initial values, 10^9 orbit iterations.
We need roughly 10^{18} evaluations of $h_{a,b}$.

Why GPUs?

Typical values: 10^6 parameters, 10^3 initial values, 10^9 orbit iterations.
We need roughly 10^{18} evaluations of $h_{a,b}$.

Timings: Computing a single orbit takes ca 11 seconds on an Intel Xeon E5-2620 @2.0GHz. This means that a full run, on a single thread, would take almost 350 years!

Why GPUs?

Typical values: 10^6 parameters, 10^3 initial values, 10^9 orbit iterations.
We need roughly 10^{18} evaluations of $h_{a,b}$.

Timings: Computing a single orbit takes ca 11 seconds on an Intel Xeon E5-2620 @2.0GHz. This means that a full run, on a single thread, would take almost 350 years!

Parallelization: Fortunately we can perform a large part of the computations in parallel.

Why GPUs?

Typical values: 10^6 parameters, 10^3 initial values, 10^9 orbit iterations.
We need roughly 10^{18} evaluations of $h_{a,b}$.

Timings: Computing a single orbit takes ca 11 seconds on an Intel Xeon E5-2620 @2.0GHz. This means that a full run, on a single thread, would take almost 350 years!

Parallelization: Fortunately we can perform a large part of the computations in parallel.

- OpenMP can parallelize the code – by adding one single line!

Why GPUs?

Typical values: 10^6 parameters, 10^3 initial values, 10^9 orbit iterations.
We need roughly 10^{18} evaluations of $h_{a,b}$.

Timings: Computing a single orbit takes ca 11 seconds on an Intel Xeon E5-2620 @2.0GHz. This means that a full run, on a single thread, would take almost 350 years!

Parallelization: Fortunately we can perform a large part of the computations in parallel.

- OpenMP can parallelize the code – by adding one single line!
- Using all 2×6 threads reduces the total time to 29 years.

Why GPUs?

Typical values: 10^6 parameters, 10^3 initial values, 10^9 orbit iterations.
We need roughly 10^{18} evaluations of $h_{a,b}$.

Timings: Computing a single orbit takes ca 11 seconds on an Intel Xeon E5-2620 @2.0GHz. This means that a full run, on a single thread, would take almost 350 years!

Parallelization: Fortunately we can perform a large part of the computations in parallel.

- OpenMP can parallelize the code – by adding one single line!
- Using all 2×6 threads reduces the total time to 29 years.
- A cluster of 64 such CPUs would still need 5.5 months.

Why GPUs?

Typical values: 10^6 parameters, 10^3 initial values, 10^9 orbit iterations.
We need roughly 10^{18} evaluations of $h_{a,b}$.

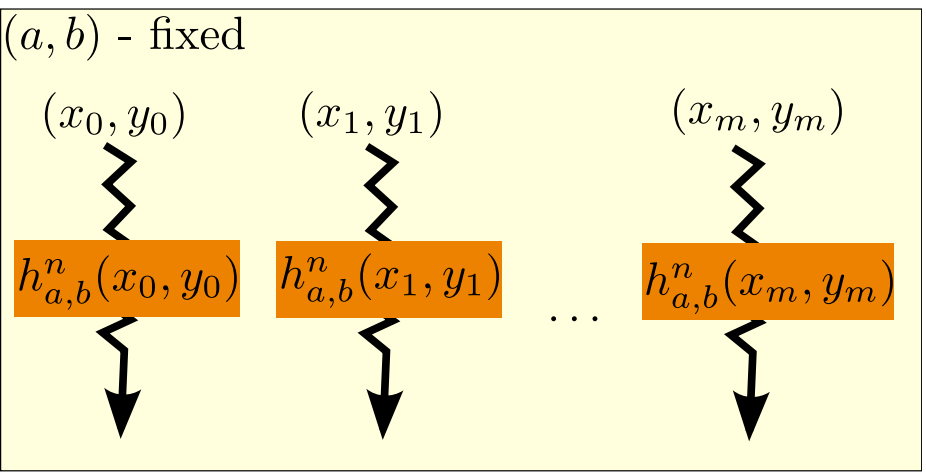
Timings: Computing a single orbit takes ca 11 seconds on an Intel Xeon E5-2620 @2.0GHz. This means that a full run, on a single thread, would take almost 350 years!

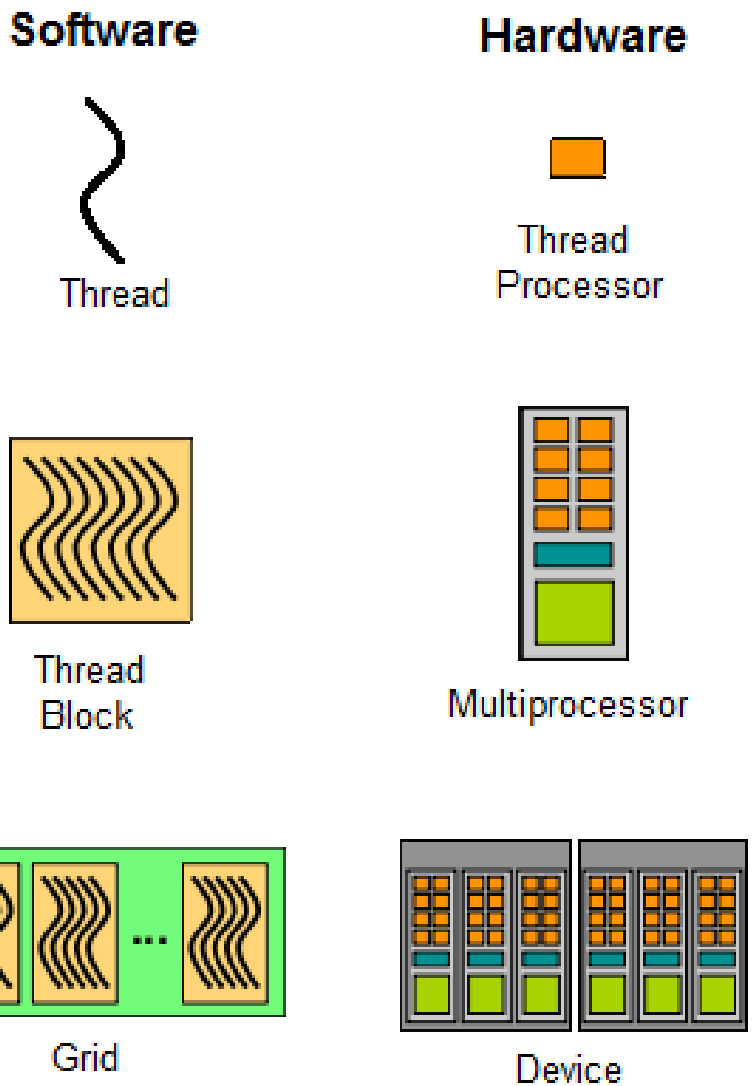
Parallelization: Fortunately we can perform a large part of the computations in parallel.

- OpenMP can parallelize the code – by adding one single line!
- Using all 2×6 threads reduces the total time to 29 years.
- A cluster of 64 such CPUs would still need 5.5 months.

We want to do this in less than one week!

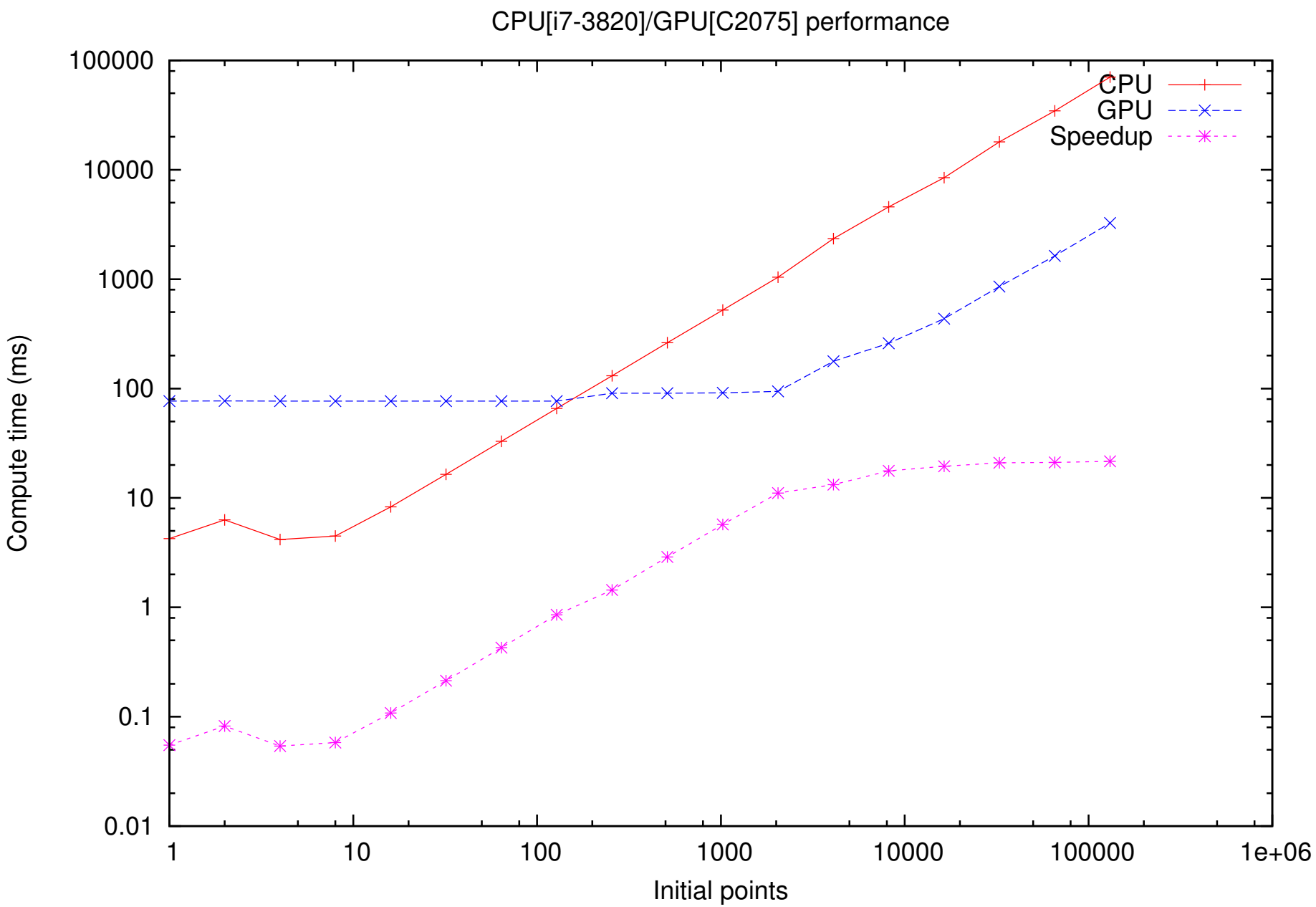
Hénon on GPU - Execution model in a nutshell





Example: Fermi Arch – up to 512 thread processors in 16 multiprocessors

Hénon on GPU - Speed-up



Hénon attractor

Hénon Map

$$h_{a,b}(x_1, x_2) = (1 + x_2 - ax_1^2, bx_1)$$

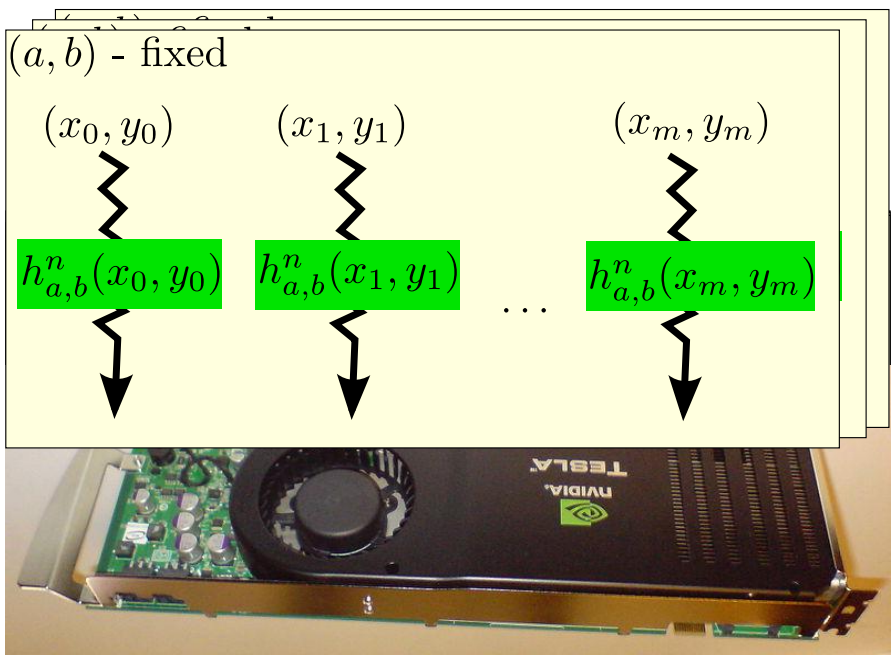
Goal: Given (a, b), prove existence of sinks (stable periodic orbits).

Method:

1. Find numerical approximation of sinks
   - many initial conditions($m \sim 2^9 \dots 2^{20}$); many iterations($n \sim 10^6 \dots 10^9$)/parameter
   - chaotic $\rightarrow$ multiple precision

High parallelism $\rightarrow$
Graphics Processing Units
(GPUs)

Need for Multiple
Precison on GPUs



Hénon attractor

Hénon Map

$$h_{a,b}(x_1, x_2) = (1 + x_2 - ax_1^2, bx_1)$$

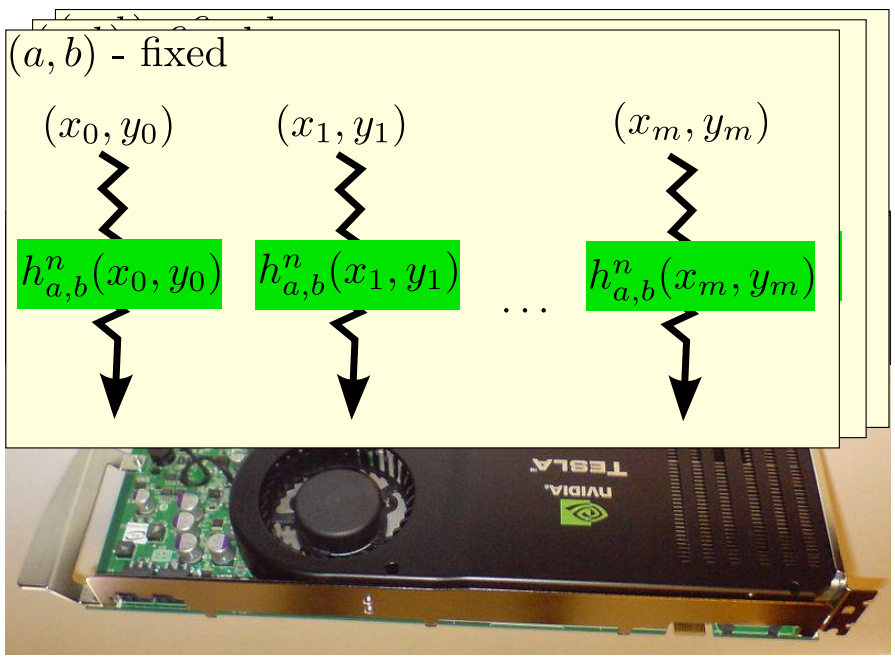
Goal: Given (a, b), prove existence of sinks (stable periodic orbits).

Method:

1. Find numerical approximation of sinks
   - many initial conditions($m \sim 2^9 \dots 2^{20}$); many iterations($n \sim 10^6 \dots 10^9$)/parameter
   - chaotic $\rightarrow$ multiple precision
2. A posteriori validation of existence and stability with interval arithmetic

High parallelism $\rightarrow$
Graphics Processing Units
(GPUs)

Need for Multiple
Precison on GPUs



Rigorous a posteriori proof of existence: interval Newton operator [Neumaier 1990]

Theorem

Let $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$, $F \in \mathcal{C}^1(D)$, $\boldsymbol{v} \in \mathbb{ID}$, $\hat{v} \in \boldsymbol{v}$.

$$N(\boldsymbol{v}) := \hat{v} - F'(\boldsymbol{v})^{-1} F(\hat{v})$$

If $N(\boldsymbol{v})$ is well defined, then the following statements hold:

(1) if $\boldsymbol{v}$ contains a zero x^* of F, then so does $N(\boldsymbol{v}) \cap \boldsymbol{v}$;

(2) if $N(\boldsymbol{v})) \cap \boldsymbol{v} = \emptyset$, then $\boldsymbol{v}$ has no zeros of F;

(3) if $N(\boldsymbol{v})) \subseteq \boldsymbol{v}$, then $\boldsymbol{v}$ contains a unique zero of F;

Proof.

(1) Follows from Mean Value Theorem;

(2) Contra-positive of (1);

(3) Existence from Brouwer's fixed point theorem; uniqueness from non-vanishing F'.

□

$\mathbb{ID}$ is the set of all real intervals included in D.

Validation of periodic orbits using Newton operator [Galias 2001], [Galias & Tucker, 2011, 2013]

- Let $v = (z_0, z_1, \ldots, z_{p-1})$ be the computer generated period p orbit.

Validation of periodic orbits using Newton operator [Galias 2001], [Galias & Tucker, 2011, 2013]

- Let $v = (z_0, z_1, \ldots, z_{p-1})$ be the computer generated period p orbit.

Remark

Let $v = (z_0, z_1, \ldots, z_{p-1})$ and

$$[F(v)]_k = z_{(k+1) \bmod p} - h(z_k), k = 0, \ldots, p-1.$$

Then $F(v) = 0$ iif $h^p(z_0) = z_0$.

$\longrightarrow$ existence of zeros of $\mathrm{Id} - h^p \iff$ existence of zeros of F;
$\longrightarrow$ reduced overestimation using F [Galias 2001].

Validation of periodic orbits using Newton operator [Galias 2001], [Galias & Tucker, 2011, 2013]

- Let $v = (z_0, z_1, \ldots, z_{p-1})$ be the computer generated period p orbit.

Remark

Let $v = (z_0, z_1, \ldots, z_{p-1})$ and

$$[F(v)]_k = z_{(k+1) \bmod p} - h(z_k), k = 0, \ldots, p-1.$$

Then $F(v) = 0$ iif $h^p(z_0) = z_0$.

$\longrightarrow$ existence of zeros of $\mathrm{Id} - h^p$ $\iff$ existence of zeros of F;
$\longrightarrow$ reduced overestimation using F [Galias 2001].

- Choose $r > 0$ s.t. $\boldsymbol{v} = ([z_0 - r, z_0 + r], \ldots, [z_{p-1} - r, z_{p-1} + r])$.

Validation of periodic orbits using Newton operator [Galias 2001], [Galias & Tucker, 2011, 2013]

- Let $v = (z_0, z_1, \ldots, z_{p-1})$ be the computer generated period p orbit.

Remark

Let $v = (z_0, z_1, \ldots, z_{p-1})$ and

$$[F(v)]_k = z_{(k+1) \bmod p} - h(z_k), k = 0, \ldots, p-1.$$

Then $F(v) = 0$ iif $h^p(z_0) = z_0$.

$\longrightarrow$ existence of zeros of $\mathrm{Id} - h^p$ $\iff$ existence of zeros of F;
$\longrightarrow$ reduced overestimation using F [Galias 2001].

- Choose $r > 0$ s.t. $\boldsymbol{v} = ([z_0 - r, z_0 + r], \ldots, [z_{p-1} - r, z_{p-1} + r])$.
- Verify that $N(\boldsymbol{v}) \subseteq \boldsymbol{v}$.

Floating-Point (FP) Multiple Precision on Graphics Processing Units (GPUs)

High parallelism + Multiple precision $\rightarrow$ Need for Multiple Precision on GPUs

GPUs:

- used for scientific computing: GP-GPU programs

Floating-Point (FP) Multiple Precision on Graphics Processing Units (GPUs)

High parallelism + Multiple precision $\rightarrow$ Need for Multiple Precision on GPUs

GPUs:

- used for scientific computing: GP-GPU programs
- Languages: OpenCL, CUDA (NVIDIA)

Floating-Point (FP) Multiple Precision on Graphics Processing Units (GPUs)

High parallelism + Multiple precision $\rightarrow$ Need for Multiple Precision on GPUs

GPUs:

- used for scientific computing: GP-GPU programs
- Languages: OpenCL, CUDA (NVIDIA)
- Support single and double precision FP conformant with IEEE-754 standard

Floating-Point (FP) Multiple Precision on Graphics Processing Units (GPUs)

High parallelism + Multiple precision $\rightarrow$ Need for Multiple Precision on GPUs

GPUs:

- used for scientific computing: GP-GPU programs
- Languages: OpenCL, CUDA (NVIDIA)
- Support single and double precision FP conformant with IEEE-754 standard
- Support for all rounding modes for basic operations $(+, -, *, /, \sqrt{})$

Floating-Point (FP) Multiple Precision on Graphics Processing Units (GPUs)

High parallelism + Multiple precision $\rightarrow$ Need for Multiple Precision on GPUs

GPUs:

- used for scientific computing: GP-GPU programs
- Languages: OpenCL, CUDA (NVIDIA)
- Support single and double precision FP conformant with IEEE-754 standard
- Support for all rounding modes for basic operations $(+, -, *, /, \sqrt{})$
- Dynamic rounding mode without any penalties:

  `__dadd_rn(a,b)=RN(a+b)`

Floating-Point (FP) Multiple Precision on Graphics Processing Units (GPUs)

High parallelism + Multiple precision $\rightarrow$ Need for Multiple Precision on GPUs

GPUs:

- used for scientific computing: GP-GPU programs
- Languages: OpenCL, CUDA (NVIDIA)
- Support single and double precision FP conformant with IEEE-754 standard
- Support for all rounding modes for basic operations $(+, -, *, /, \sqrt{})$
- Dynamic rounding mode without any penalties:

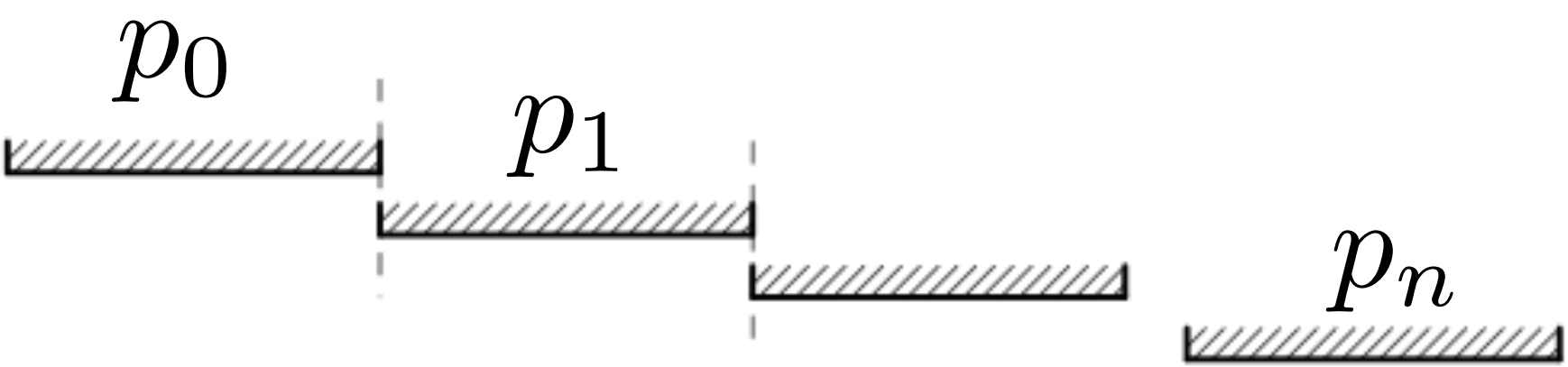
  `__dadd_rn(a,b)=RN(a+b)`
- The FMA (Fused Multiply-Add):

  `__dfma_rn (a,b,c)= RN(A *B + C)`

Our approach: multiple-term expansions, i.e.

a number is expressed as a non-evaluated sum of FP numbers
$N = p_0 + \ldots + p_n$.

The expansions are required to be non-overlapping.



Previous works: Zimmer, et al. (C - XSC Library with Kulisch long accumulator), Dekker71, Priest91, Shewchuk96, Bailey01, Nievergelt04, Rump05

Non-overlapping FP Sequences

Precision-$2p$ vs. double-double

π in double-double

$$p_1 = 11.0010010000111111011010101000100010000101101000110000_2,$$

and

$$p_2 = 1.0001101001100010011000110011000101000101110000000111_2 \times 2^{-53}.$$

$p_1 + p_2 \leftrightarrow 107$ bits FP approx.

Non-overlapping FP Sequences

Precision-2p vs. double-double

π in double-double

$$p_1 = 11.001001000011111101101010100010001000010110100011000_2,$$

and

$$p_2 = 1.000110100110001001100011001100010100010111000000011 1_2 \times 2^{-53}.$$

$p_1 + p_2 \leftrightarrow 107$ bits FP approx.

$\ln(12)$ in double-double

$$\ell_1 = 10.011111000010001011010111100110100111001111001111101_2,$$

and

$$\ell_2 = -1.100110001110010000001111100001011011110101111001011 1_2 \times 2^{-55}.$$

$\ell_1 + \ell_2 \leftrightarrow 109$ bits FP approx.

Error-Free Transformations: 2Sum & 2Prod

Error-Free Transformations: 2Sum & 2Prod

Example 2Sum:

$$
\begin{aligned}
a &= 1.\underbrace{0\ldots0}_{50}11 \times 2^0 \\
b &= 0.\underbrace{0\ldots00}_{51}111 \times 2^0 = 1.11 \times 2^{-52}
\end{aligned}
$$

Error-Free Transformations: 2Sum & 2Prod

Example 2Sum:

$$
\begin{aligned}
a &= 1.\underbrace{0\ldots0}_{50}11 \times 2^0 \\
b &= 0.\underbrace{0\ldots00}_{51}111 \times 2^0 = 1.11 \times 2^{-52} \\
s &= 1.\underbrace{0\ldots0}_{49}101 \times 2^0 \\
t &= -0.\underbrace{0\ldots00}_{53}1 \times 2^0 = -1 \times 2^{-54} \\
a + b = s + t &= 1.\underbrace{0\ldots0}_{49}10011 \times 2^0
\end{aligned}
$$

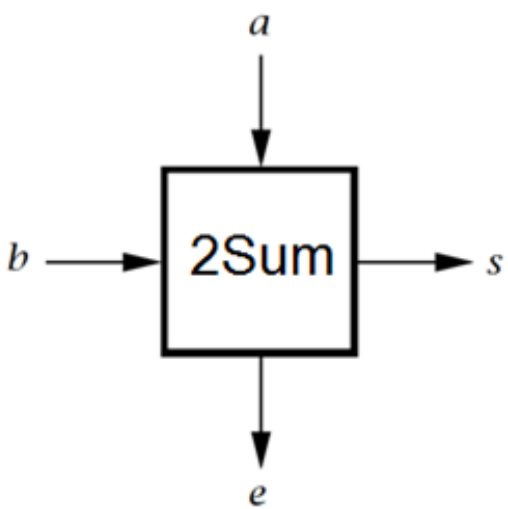
Error-Free Transformations: 2Sum & 2Prod

Assume the floating-point system being used has radix $\beta \leq 3$, subnormal numbers available, and provides correct rounding with rounding to nearest.

Theorem 1 (2Sum algorithm)

Let a and b be floating-point numbers. Algorithm 2Sum computes two floating-point numbers s and t that satisfy the following:

- $s + t = a + b$ *exactly;*
- $s = RN(a + b)$.



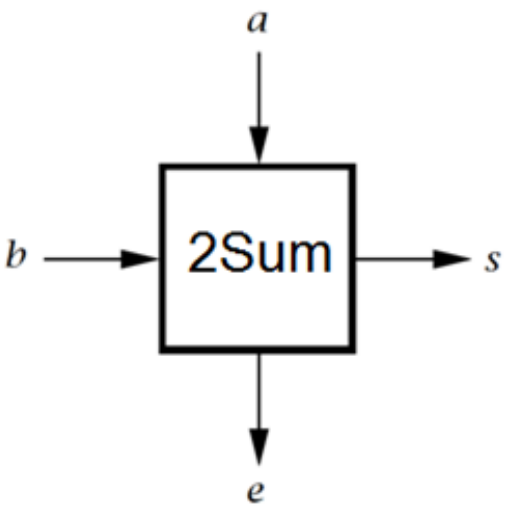
Error-Free Transformations: 2Sum & 2Prod

Assume the floating-point system being used has radix $\beta \leq 3$, subnormal numbers available, and provides correct rounding with rounding to nearest.

Theorem 1 (2Sum algorithm)

Let a and b be floating-point numbers. Algorithm 2Sum computes two floating-point numbers s and t that satisfy the following:

- $s + t = a + b$ *exactly;*
- $s = RN(a + b)$.

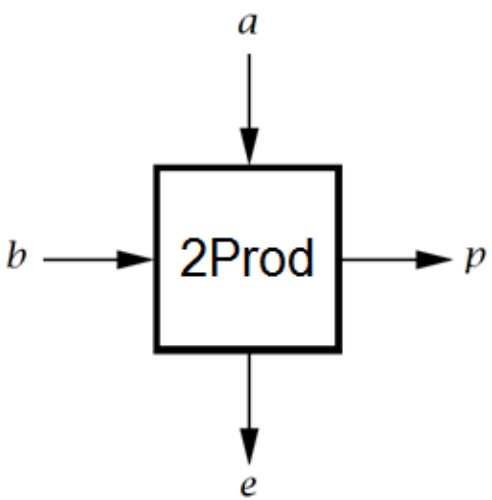



```
/* Computes RN(a+b) and t=(a+b)-RN(a+b).  */
__device__ static __forceinline__
  double 2Sum(double a, double b, double &t) {
  double s = __dadd_rn(a,b);
  double aa = __dadd_rn(s,-b);
  double bb = __dadd_rn(s,-aa);
  double da = __dadd_rn(a,-aa);
  double db = __dadd_rn(b,-bb);
  t= __dadd_rn(da,db);
  return s;
}
```

Theorem 2 (2ProdFMA algorithm)

Let a and b be floating-point numbers, such that ab does not overflow and $e_a + e_b \geq e_{min} + p - 1$. Algorithm 2ProdFMA computes two floating-point numbers p and e that satisfy the following:

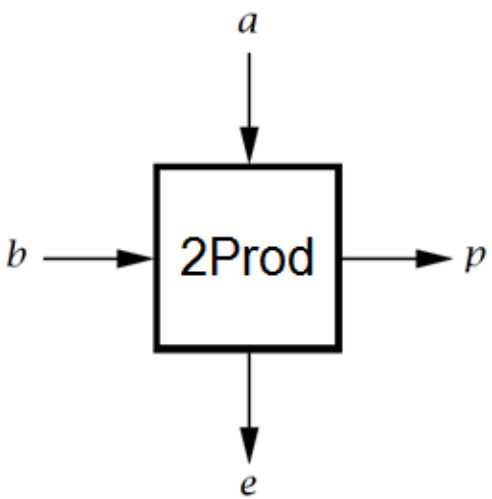
- $p + e = a * b$ *exactly;*
- $p =$ *RN*$(a * b)$.



Theorem 2 (2ProdFMA algorithm)

Let a and b be floating-point numbers, such that ab does not overflow and $e_a + e_b \geq e_{min} + p - 1$. Algorithm 2ProdFMA computes two floating-point numbers p and e that satisfy the following:

- $p + e = a * b$ *exactly;*
- $p = RN(a * b)$.

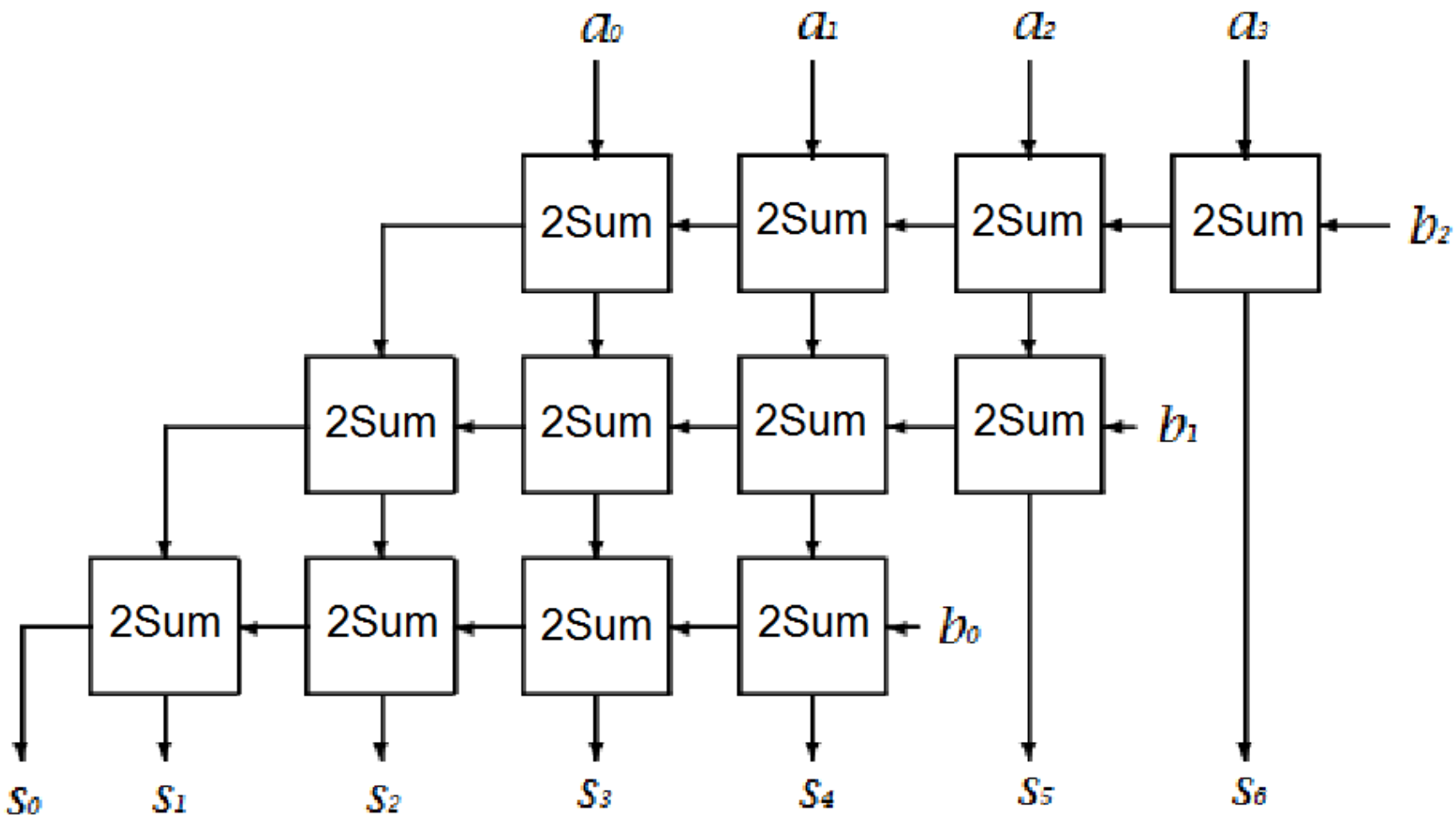



```
/* Computes p=RN(a*b) and e=(a*b)-p, using an FMA.
__device__ static __forceinline__
  double 2ProdFMA(double a, double b, double &e) {
  double p = __dmul_rn(a, b);
  e = __fma_rn(a, b, -p);
  return p;
}
```

Shewchuck's ExpansionSum

Input: (a_i), (b_i) non-overlapping expansions

Output:(s_i) non-overlapping expansion, s.t. $\sum_{i=1}^{n+m} s_i = \sum_{i=1}^{n} a_i + \sum_{i=1}^{m} b_i$



"Unfair" Comparison with MPFR

- Timings (seconds) for 10^6 multiplications on Intel i7-3820 @3.60GHz

double-double	MPFR $2 * 53$ bits
2.06	3.59

quad-double	MPFR $4 * 53$ bits
2.70	4.50

Speed-up : 1.6X

"Unfair" Comparison with MPFR

- Timings (seconds) for 10^6 multiplications on Intel i7-3820 @3.60GHz

double-double	MPFR $2 * 53$ bits
2.06	3.59

quad-double	MPFR $4 * 53$ bits
2.70	4.50

Speed-up : 1.6X

- Hénon : performance [10^6 iterations/sec] for
Intel i7-3820 @3.60GHz vs Nvidia C2075 @1.15GHz.

MPFR 106 bits +omp	double-double +omp	double-double +gpu
11.8	115.3	3804.5

MPFR 212 bits +omp	quad-double +omp	quad-double +gpu
10.1	27.5	894.0

Found Hénon sinks [Galias & Tucker, 2013]

Closest to classical parameters, so far: $a = 1.399999486944, b = 0.3$.

a	p	d	r	λ_1
1.399922051	25	$5.522-12$	$2.473-12$	-0.00132
1.39997174948	30	$1.354-11$	$3.561-12$	-0.01887
1.3999769102	18	$3.207-09$	$1.014-09$	-0.05306
1.39998083519	24	$1.703-11$	$7.384-12$	-0.02819
1.399984477	20	$8.875-10$	$4.076-10$	-0.05099
1.39999492185	22	$3.686-11$	$1.531-11$	-0.09600
1.3999964733062	39	$2.784-13$	$1.115-13$	-0.03547
1.399999486944	33	$1.110-12$	$6.901-13$	-0.01843
1.40000929916	25	$1.118-11$	$5.128-12$	-0.08379
1.4000227433	21	$2.262-10$	$7.901-11$	-0.05612
1.40002931695	27	$5.782-11$	$2.646-11$	-0.01140
1.40006377472	27	$8.692-11$	$3.810-11$	-0.05636
1.40006667358	24	$6.278-11$	$2.646-11$	-0.01112
1.4000843045	27	$9.400-11$	$4.572-11$	-0.06870
1.40009110518	22	$3.493-11$	$1.531-11$	-0.02157
1.4000967515	26	$2.463-10$	$1.365-10$	-0.13233

Conclusion

- Finding sinks near $(a, b) = (1.4, 0.3)$ is hard.
- Using a massively parallel search is necessary.

Conclusion

- Finding sinks near $(a, b) = (1.4, 0.3)$ is hard.
- Using a massively parallel search is necessary.
- GPU multiprecision library can be very efficient; careful implementation: avoid dynamic memory allocation on the GPU; use templates; use in-place algorithms to spare local memory; avoid memory spill/loads.
- Haddock Cluster with 28 compute nodes with dual Tesla M2075 GPUs can do our job in less than one week!

Conclusion

- Finding sinks near $(a, b) = (1.4, 0.3)$ is hard.
- Using a massively parallel search is necessary.
- GPU multiprecision library can be very efficient; careful implementation: avoid dynamic memory allocation on the GPU; use templates; use in-place algorithms to spare local memory; avoid memory spill/loads.
- Haddock Cluster with 28 compute nodes with dual Tesla M2075 GPUs can do our job in less than one week!
- Short period sinks can look like strange attractors.

Conclusion

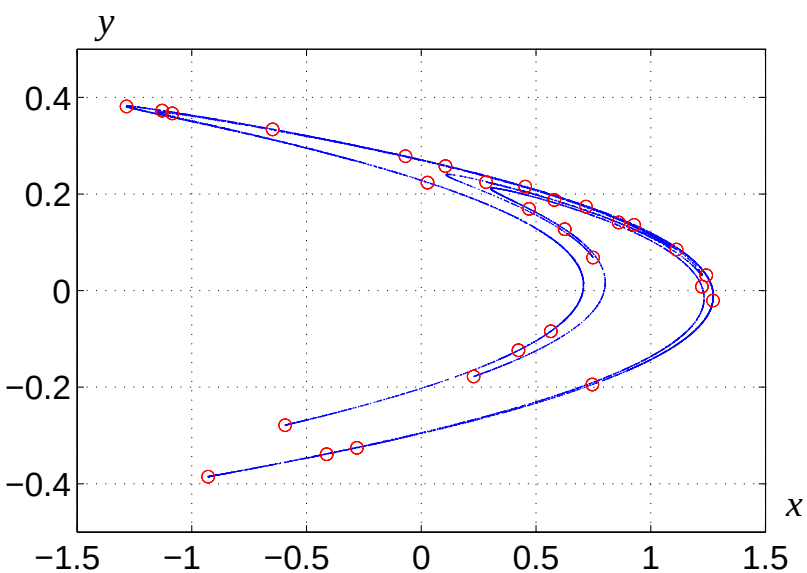
- Finding sinks near $(a, b) = (1.4, 0.3)$ is hard.
- Using a massively parallel search is necessary.
- GPU multiprecision library can be very efficient; careful implementation: avoid dynamic memory allocation on the GPU; use templates; use in-place algorithms to spare local memory; avoid memory spill/loads.
- Haddock Cluster with 28 compute nodes with dual Tesla M2075 GPUs can do our job in less than one week!
- Short period sinks can look like strange attractors.

Conclusion

- Finding sinks near $(a, b) = (1.4, 0.3)$ is hard.
- Using a massively parallel search is necessary.
- GPU multiprecision library can be very efficient; careful implementation: avoid dynamic memory allocation on the GPU; use templates; use in-place algorithms to spare local memory; avoid memory spill/loads.
- Haddock Cluster with 28 compute nodes with dual Tesla M2075 GPUs can do our job in less than one week!
- Short period sinks can look like strange attractors.

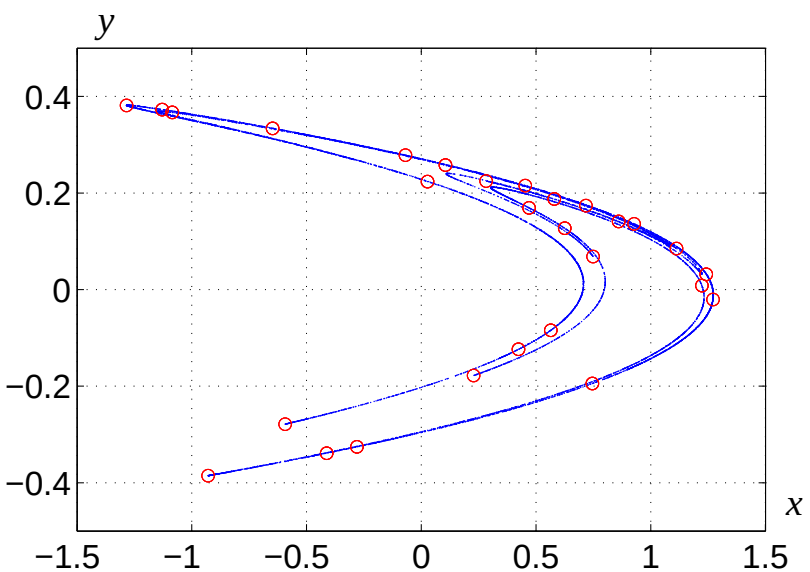
Conclusion

- Finding sinks near $(a, b) = (1.4, 0.3)$ is hard.
- Using a massively parallel search is necessary.
- GPU multiprecision library can be very efficient; careful implementation: avoid dynamic memory allocation on the GPU; use templates; use in-place algorithms to spare local memory; avoid memory spill/loads.
- Haddock Cluster with 28 compute nodes with dual Tesla M2075 GPUs can do our job in less than one week!
- Short period sinks can look like strange attractors.



Conclusion

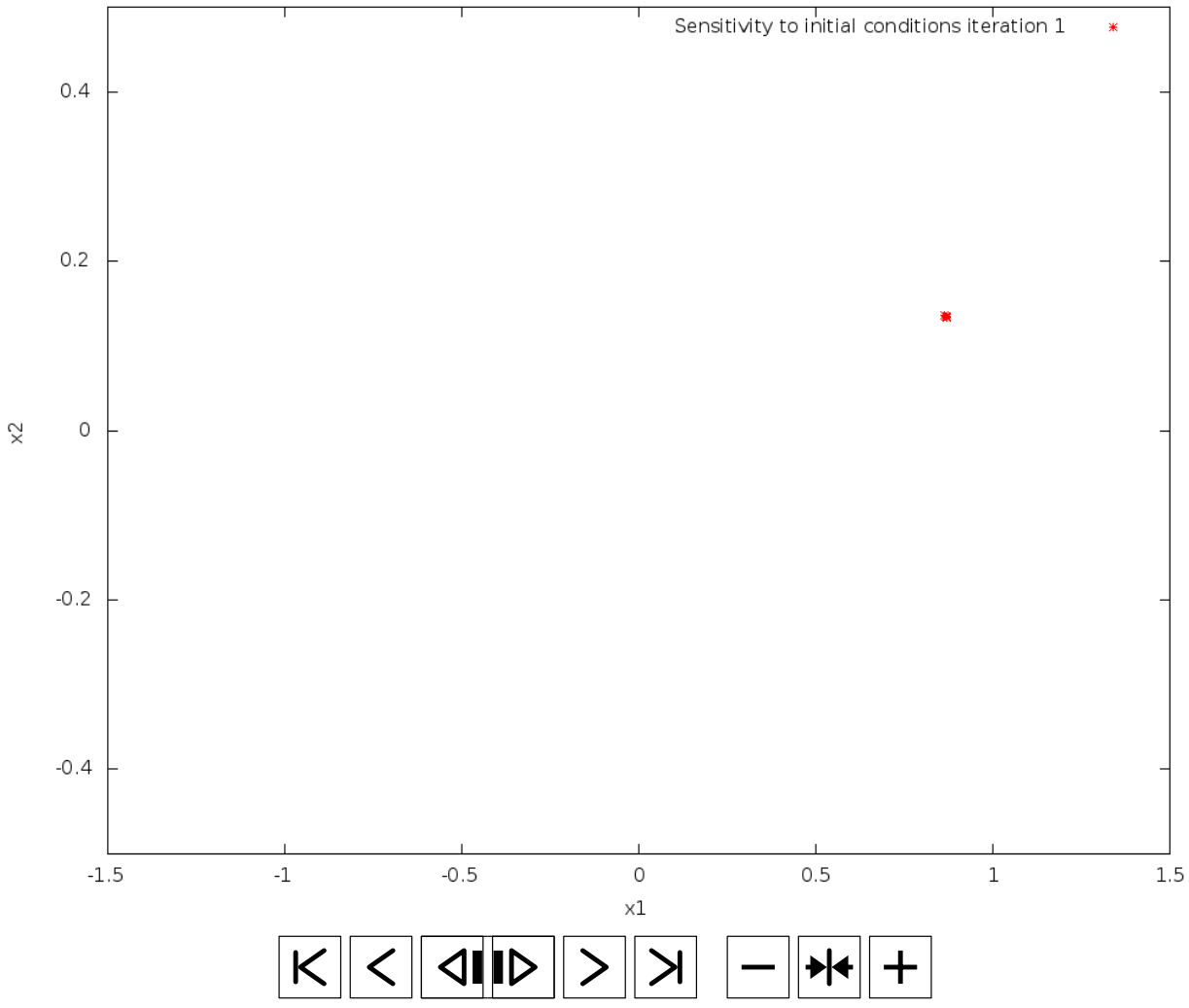
- Finding sinks near $(a, b) = (1.4, 0.3)$ is hard.
- Using a massively parallel search is necessary.
- GPU multiprecision library can be very efficient; careful implementation: avoid dynamic memory allocation on the GPU; use templates; use in-place algorithms to spare local memory; avoid memory spill/loads.
- Haddock Cluster with 28 compute nodes with dual Tesla M2075 GPUs can do our job in less than one week!
- Short period sinks can look like strange attractors.



Hénon Attractor - Sensitivity to initial conditions

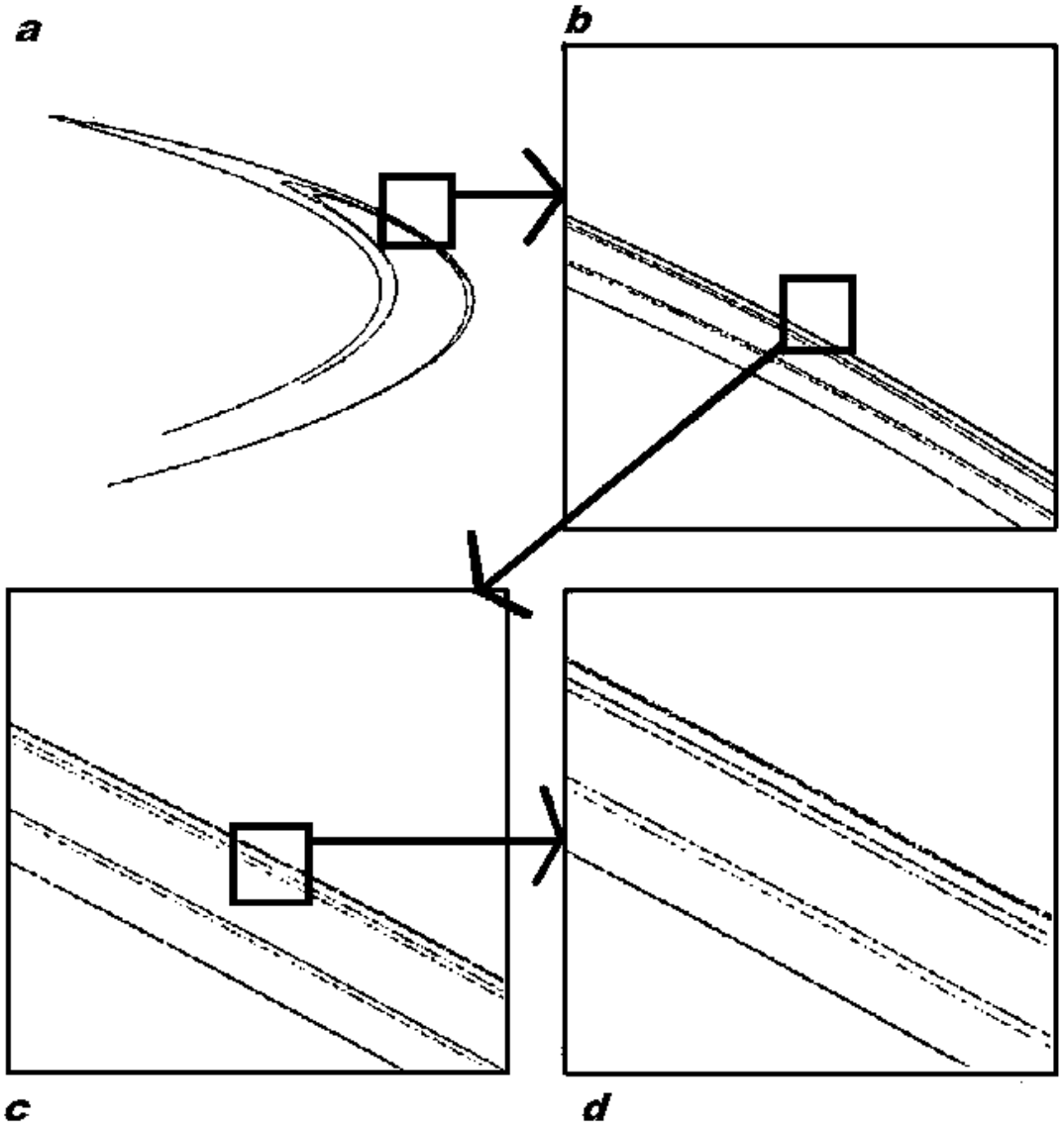
Hénon Map

$$h_{a,b}(x_1, x_2) = (1 + x_2 - ax_1^2, bx_1)$$



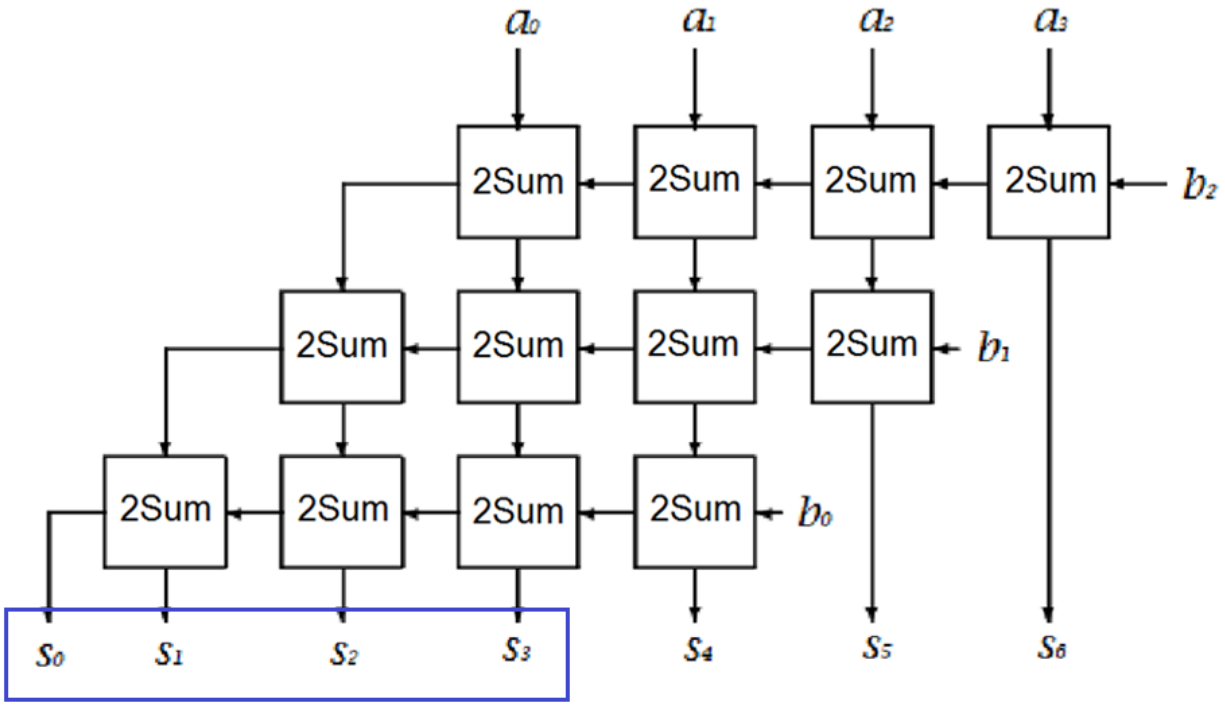
Back

Hénon Attractor - Fractal Dimension

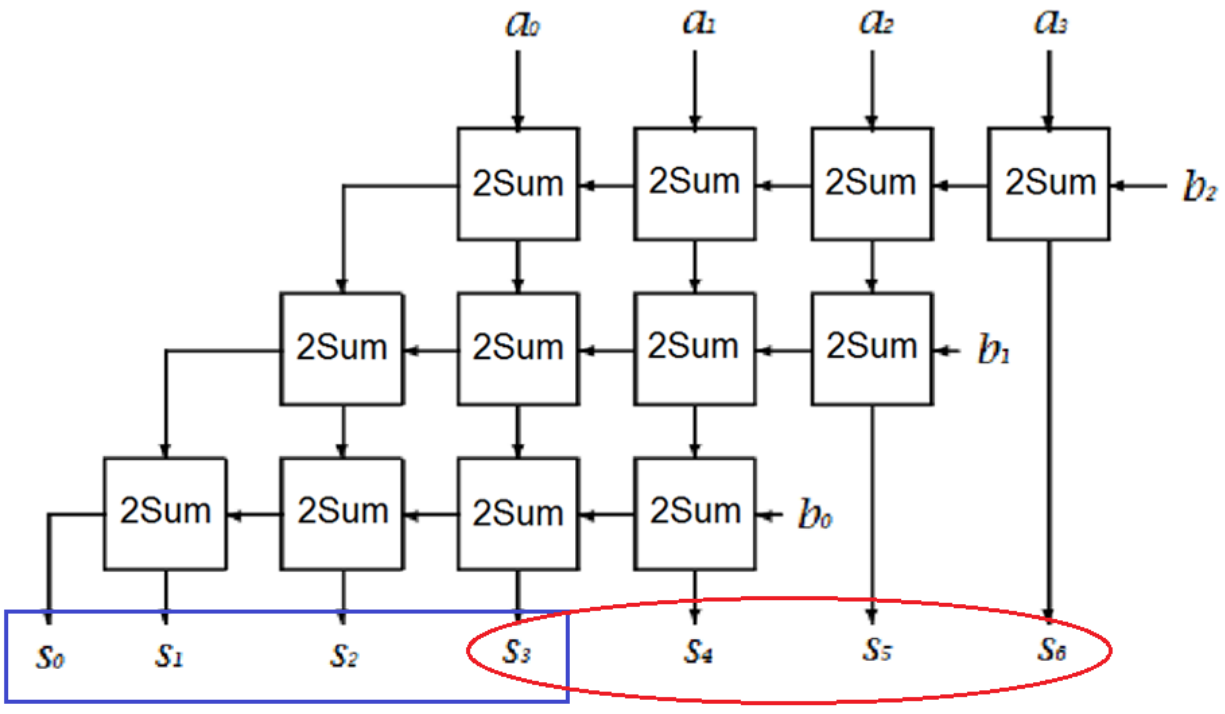


Back

Shewchuck's ExpansionSum



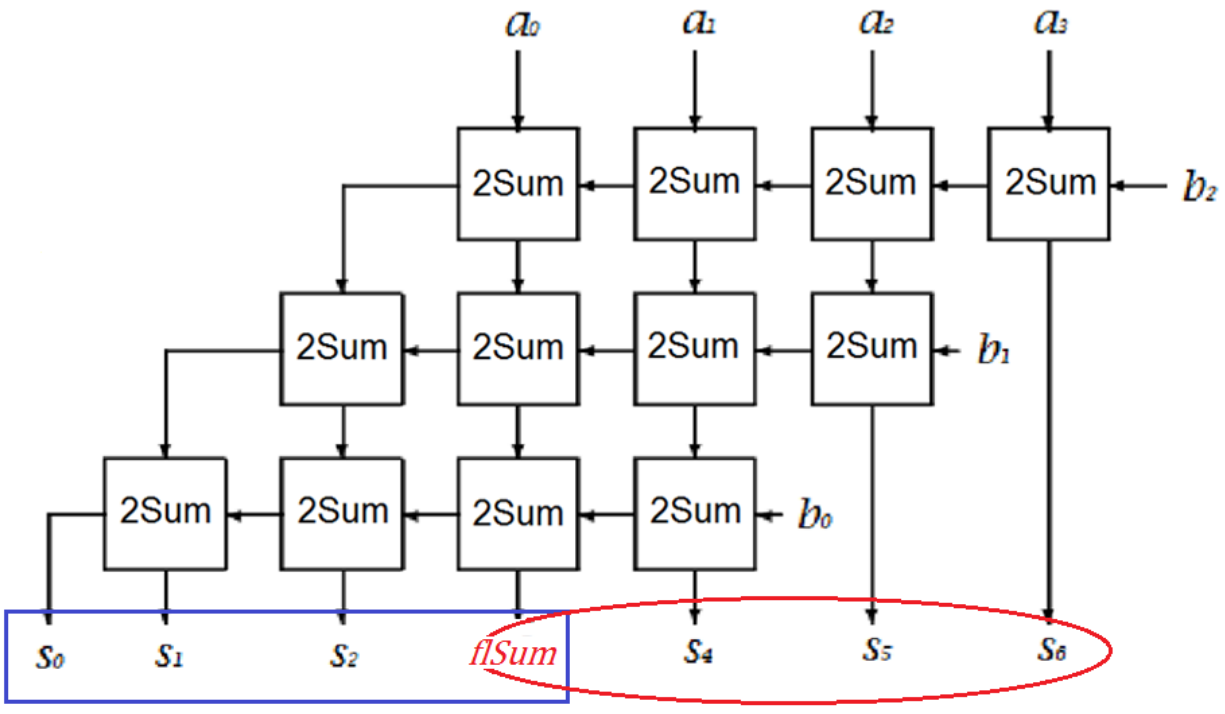
Shewchuck's ExpansionSum



$$u = \mathsf{RU}\left(\sum_{i=r-1}^{n+m-1} s_i\right); \qquad flSum = \mathsf{RN}\left((u+d)/2\right);$$

$$d = \mathsf{RD}\left(\sum_{i=r-1}^{n+m-1} s_i\right); \qquad \varepsilon = \max\left(\mathsf{RU}\left(flSum - d\right), \mathsf{RU}\left(u - flSum\right)\right);$$

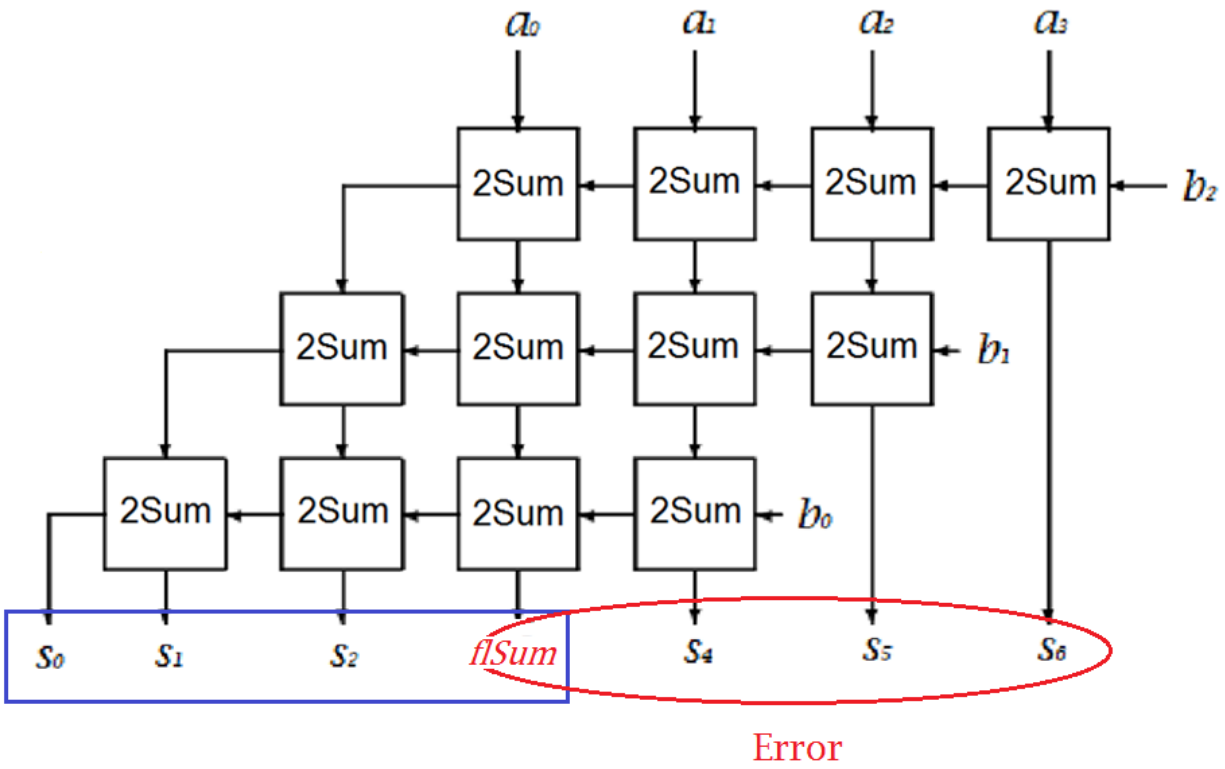
Shewchuck's ExpansionSum



$$u = \mathsf{RU}\left(\sum_{i=r-1}^{n+m-1} s_i\right); \qquad flSum = \mathsf{RN}\left((u+d)/2\right);$$

$$d = \mathsf{RD}\left(\sum_{i=r-1}^{n+m-1} s_i\right); \qquad \varepsilon = \max\left(\mathsf{RU}\left(flSum - d\right), \mathsf{RU}\left(u - flSum\right)\right);$$

Shewchuck's ExpansionSum



$$u = \mathsf{RU}\left(\sum_{i=r-1}^{n+m-1} s_i\right); \qquad flSum = \mathsf{RN}\left((u+d)/2\right);$$

$$d = \mathsf{RD}\left(\sum_{i=r-1}^{n+m-1} s_i\right); \qquad \varepsilon = \max\left(\mathsf{RU}\left(flSum - d\right), \mathsf{RU}\left(u - flSum\right)\right);$$

Interval arithmetic support: $(\sum_{i=1}^{r} s_i, \varepsilon)$ with very little overhead.