Certified and fast computation of supremum norms of approximation errors

Sylvain Chevillard   Mioara Joldes   Christoph Lauter

June 2, 2009

Outline

- Motivation
  - Correctly rounded elementary functions
  - Supremum norm of error functions
  - Previous approaches and difficulties
- Our approach
  - Automatic differentiation and Taylor models
  - Isolation of roots of polynomials
  - Enclosure of roots of functions
- Results & Conclusion

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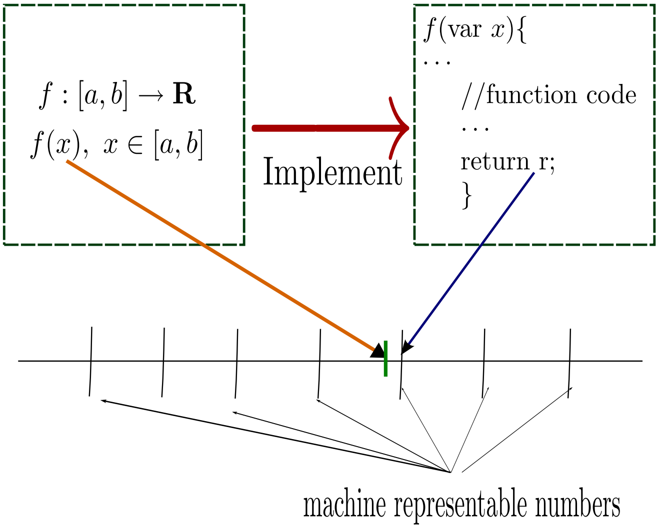
- Arenaire team develops the Correctly Rounded Libm (CRLibm)[1].

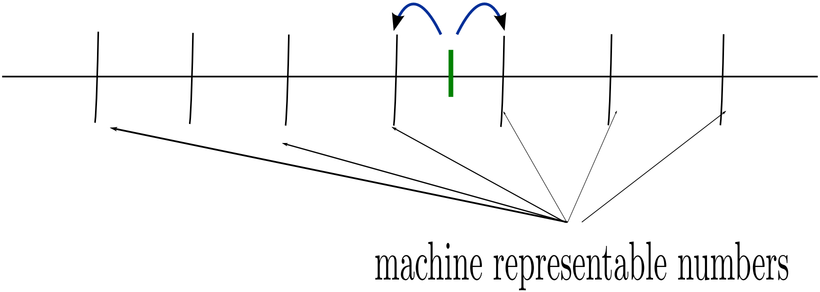
[1] http://lipforge.ens-lyon.fr/www/crlibm/

Correctly rounded functions



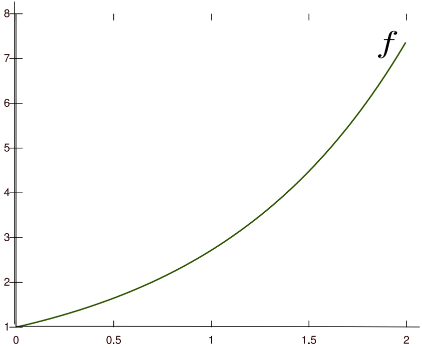
Correctly rounded functions

- Table Maker's Dilemma



- Increase working precision
- Worst cases search - V. Lefevre and J-M. Muller

Supremum Norms of Error Functions



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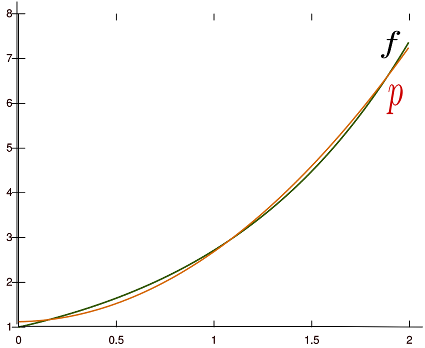

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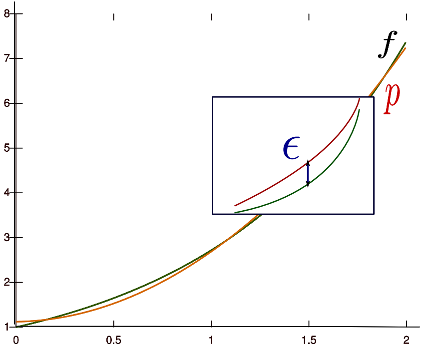

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$\varepsilon(x) = \frac{p(x)}{f(x)} - 1, \; x \in [a, b]$

Define $\|\varepsilon\|_\infty = \sup_{x \in [a,b]} |\varepsilon(x)|$

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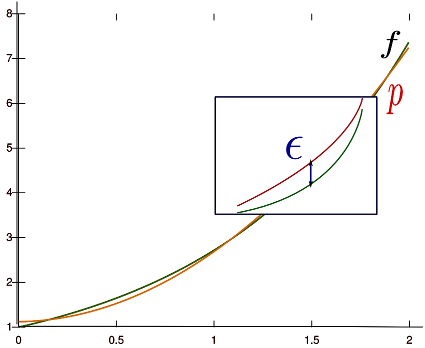

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- Compute a certified bound for the supremum norm of an error function
- “Quick and dirty“ supremum norms - another class of algorithms

Supremum Norms of Error Functions

- Error $\varepsilon(x) = f(x) - p(x) \quad \text{or} \quad \varepsilon(x) = \frac{p(x)}{f(x)} - 1, \quad x \in [a, b]$
- Define $\|\varepsilon\|_\infty = \sup_{x\in[a,\,b]} |\varepsilon(x)|$
- Purpose: Compute a certified bound for the supremum norm of an error function
- Given p and f find a narrow interval $\mathbf{r}$ such that $\|\varepsilon\|_\infty \in \mathbf{r}$.

Need for a fast and certified algorithm:

- Correctly rounded elementary functions
- For computing the minimum error between a function and thousands of polynomials with floating-point coefficients

Supremum norm of error functions

Example:

$f(x) = e^x, \; x \in [0, 1]$
$p(x) = \sum_{i=0}^{5} c_i x^i$ s.t. $\|f - p\|_\infty$ is as small as possible (Remez algorithm)
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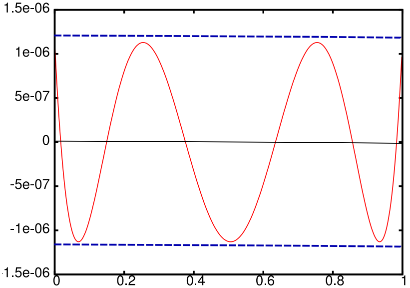

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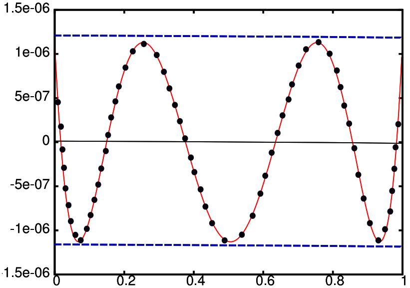

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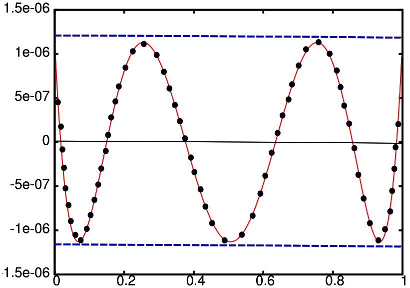

How to obtain a certified and tight bound?

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We obtain a certified enclosure: $f(x) \in F(X), \ \forall x \in [-1,2]$.

Interval Arithmetic - Overestimation

$E = x + \varepsilon - x$

1. $\varepsilon \in [-100, 100],\ x \in [-10^{-2}, 10^{-2}]$

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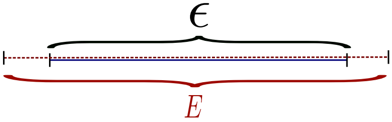

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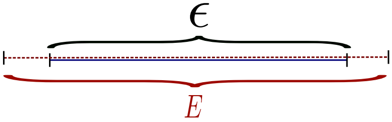

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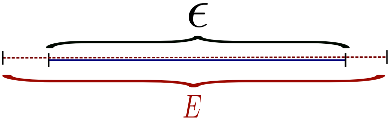

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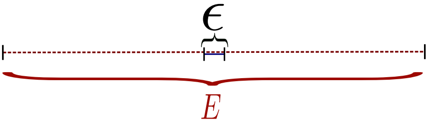

Interval Arithmetic - Overestimation

- When using IA, multiple occurrencies of same variable can not be detected. Dependency Phenomenon
- Overestimation can be reduced using smaller intervals.

Our Problem - Prone to High Dependency Phenomenon

Example:

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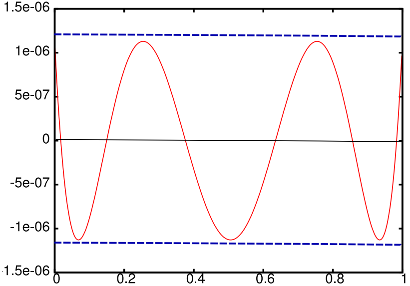

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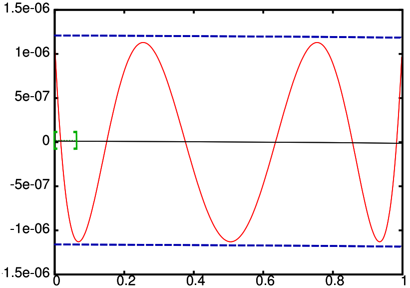

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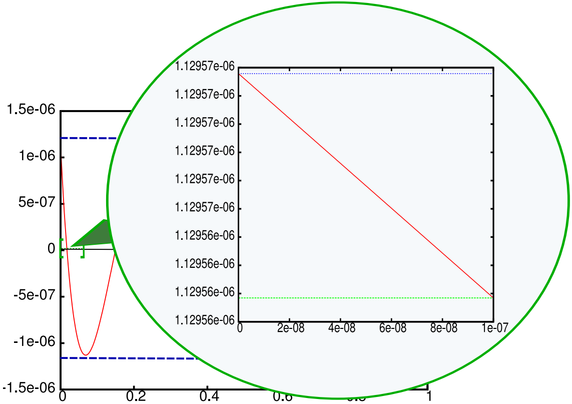

In this case, over $[0, 1]$ we need 10^7 intervals!

Previous Approaches

- Floating-point techniques - not “safe“
- Existent Interval Arithmetic methods - Not sufficient
- Global optimization software (eg. Globsol) - not tailored for our specific problem
- Chevillard and Lauter’s technique: interval arithmetic, tight bounding of the zeros of the derivative of the error function, removes false singularities (like $x = 0 \text{ for } \sin(x)/x$). High computation time for $\deg(p) > 10$.
- Techniques based on a high order Taylor expansion of the error function and a sufficiently close bounding of the remainder
- Certified polynomial approximations of analytic functions.

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- How to obtain a certified and tight bound for $\|\varepsilon\|_\infty = \|f - p\|_\infty$ AND $\|\varepsilon\|_\infty = \|p/f - 1\|_\infty$, over given interval $[a, b]$?

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[2] http://gforge.inria.fr/projects/mpfi/

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1 Automatic Differentiation/ Taylor Models

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1. Automatic Differentiation/ Taylor Models
2. Tightly bounding the polynomial difference - Roots isolation and refinement techniques

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How to compute (T, Δ)?

Compute (T, Δ) s.t $f(x) - T(x) \in \Delta, \forall x \in [a, b]$.
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- Taylor Formula with Lagrange form of remainder
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Taylor Formula with Lagrange form of remainder

Let $n \in \mathbb{N}$; Consider f an n times differentiable function over $[a, b]$ around x_0:

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$\Delta_n(x, \xi) = \dfrac{f^{(n)}(\xi)(x - x_0)^n}{n!}$, $x \in [a, b]$, ξ lies strictly between x and x_0

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- Compute coefficients of T: $\dfrac{f^{(i)}(x_0)}{i!}, i = \{0, \dots, n\}$
- Compute Δ as an enclosure for $\dfrac{f^{(n)}(\xi)(x - x_0)^n}{n!}$, when $\xi \in [a, b]$.

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- Can be easily extended to work with interval arithmetic
- Compute $\frac{f^{(i)}(x_0)}{i!}, i = \{0, \ldots, n\}$ or an enclosure for $\frac{f^{(i)}(\xi)}{i!}, i = \{0, \ldots, n\}$, when $\xi \in [a, b]$.

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Introduce a higher degree polynomial T: Use AD

$$T(x) = \sum_{i=0}^{8} \frac{f^{(i)}(1/2)}{i!}(x - 1/2)^i = \sum_{i=0}^{8} \frac{\exp(1/2)}{i!}(x - 1/2)^i$$

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Compute an enclosure of the remainder: Use AD

$$\Delta_9(x, \xi) = \underbrace{\frac{f^{(9)}(\xi)}{9!}}_{\in \frac{\exp([0,1])}{9!}} \times \underbrace{(x - 1/2)^9}_{\leq 2^{-9}}$$

$|\Delta_9(x, \xi)| \in 1.4630578142[1; 2] \times e - 8$

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Computing (T, Δ) - Taylor Models[3] at a Glance

Couple of the form (T, Δ), s.t. $f - T \in \Delta$.

Key idea: Easily define arithmetic operations $+, -, *$

Given: (T_1, Δ_1), (T_2, Δ_2), two TM's of order n.

- Addition: $(T_1 + T_2, \Delta_1 + \Delta_2)$
- Multiplication:
  $((T_1 T_2)_{0..n}, B((T_1 T_2)_{n+1..2n}) + \Delta_1 B(T_2) + \Delta_2 B(T_1))$

[3] Firstly introduced by M. Berz and K.Makino

Computing (T, Δ) - Taylor Models at a Glance

Features:

- Start with trivial TM: $(1, [0, 0])$.
- TM's for elementary functions are defined
- TM's can be recursively defined
- Propagation of error bounds combined with AD

Computing (T, Δ)

- Main idea:

$$\|f-p\|_\infty \leq \underbrace{\|f-T\|_\infty}_{\text{bounding a remainder}} + \underbrace{\|T-p\|_\infty}_{\text{bounding a polynomial}}$$

- Achieved so far:

$$\|\exp - p\|_\infty \leq \underbrace{\|\exp - T\|_\infty}_{\in\ 1.4630578142[1;2]\times e-8} + \underbrace{\|T-p\|_\infty}_{\text{bounding a polynomial}}$$

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Purpose: Tightly bound $\|T - p\|_\infty$ over the interval $[a, b]$

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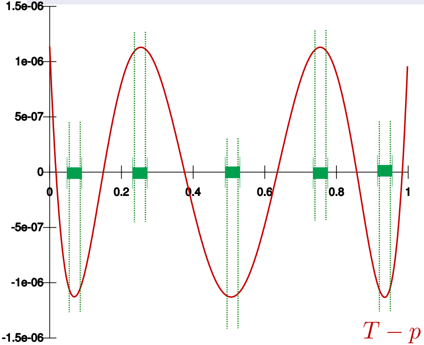

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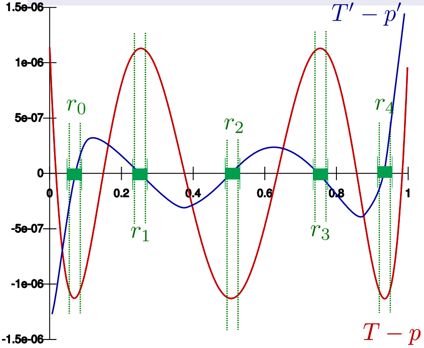

- Tightly bound the roots of the derivative

(2)Bounding the polynomial difference

Purpose: Tightly bound $\|T - p\|_\infty$ over the interval $[a, b]$

Example:

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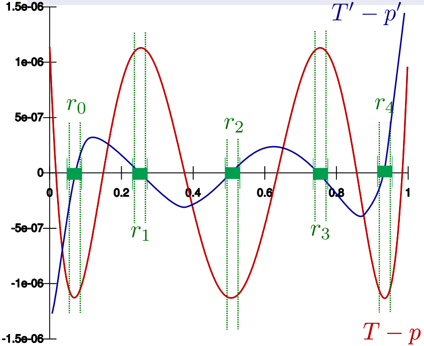

- Tightly bound the roots of the derivative
- Evaluate using interval arithmetic

(3)Isolation and refinement of roots of polynomials

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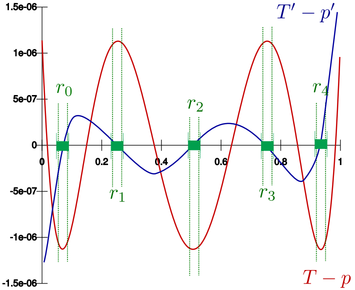
- Use dichotomy or Newton iteration process

(3)Bounding the polynomial difference



$r_0 \in 0.0068[5;6],\ r_1 \in 0.2544[3;4],$
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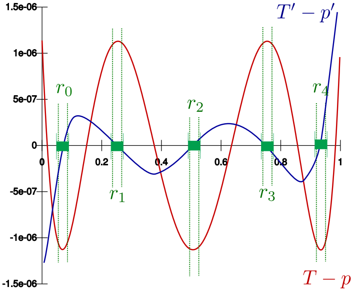

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$\|T - p\|_\infty \in \ 1.11486943[1;2] \times e - 6$

Our Approach - Absolute Error - Summary

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$$\|f-p\|_\infty \leq \underbrace{\|f-T\|_\infty}_{\in\ 1.4630578142[1;2]\times e-8} + \underbrace{\|T-p\|_\infty}_{\in\ 1.11486943[1;2]\times e-6}$$

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IA: $\|f-p\|_\infty \leq 0.4$, sampling: $\|f-p\|_\infty \simeq 1.1295e-6$.

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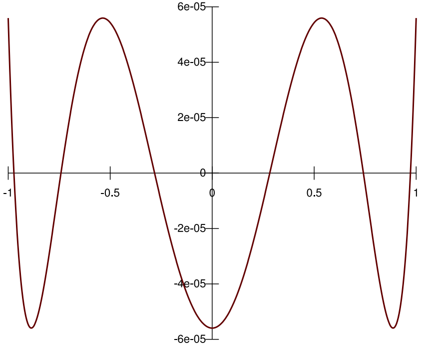

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Compute the Taylor Model for $\varepsilon(x)$, order 12.

TM for $f(x)$, remainder: $[-1.351322e-10; 1.351322e-10]$
TM for $\frac{1}{f(x)}$, remainder: $[-3.1634304e-2; 0.258216]$

Remainder bounds are unsatisfactory in our case.

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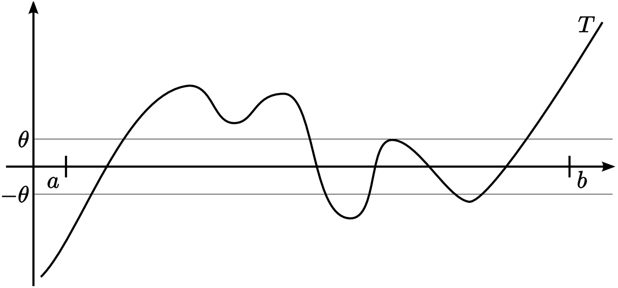

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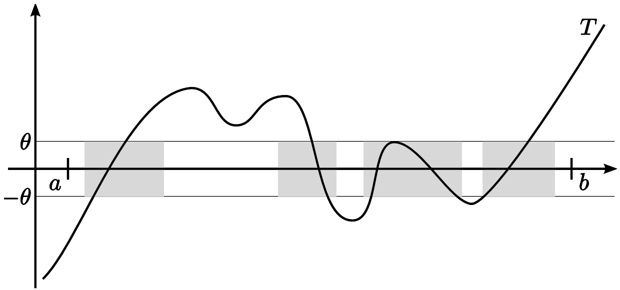

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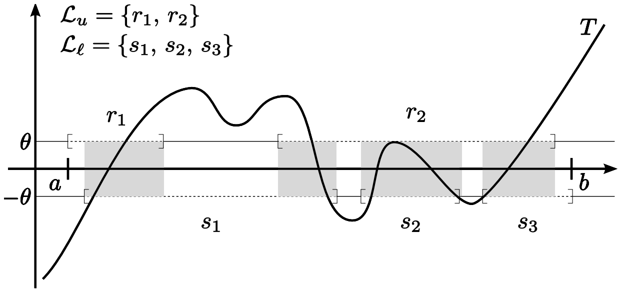

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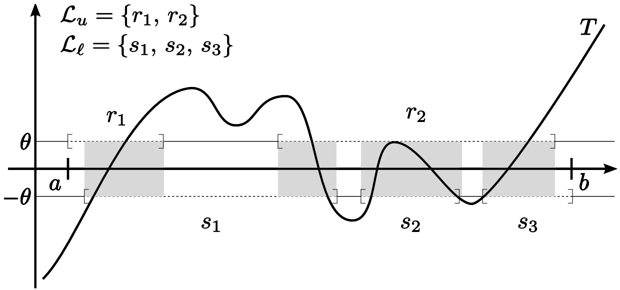
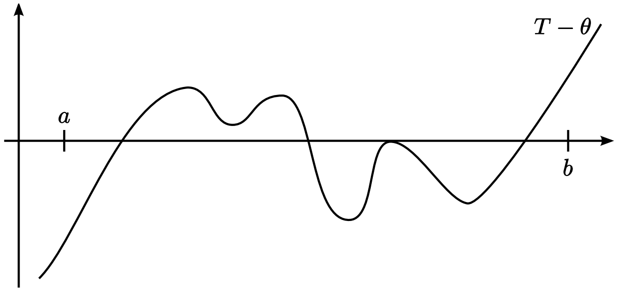


Finding enclosures of roots of a function

- Compute a list of intervals where $T - \theta \leq 0$, T is a polynomial



Finding enclosures of roots of a function

- Compute a list of intervals where $T - \theta \leq 0$, T is a polynomial
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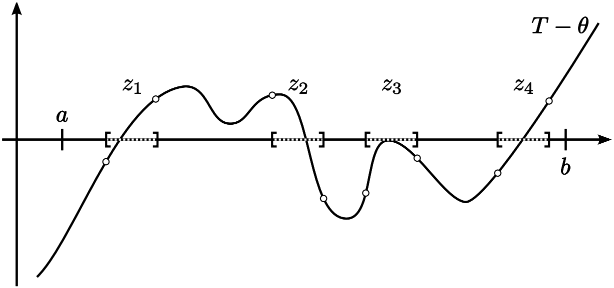

Finding enclosures of roots of a function

- Compute a list of intervals where $T - \theta \leq 0$, T is a polynomial
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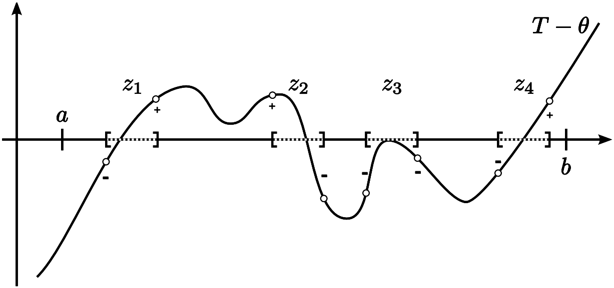
- Compute sign changes



Finding enclosures of roots of a function

- Compute a list of intervals where $T - \theta \leq 0$, T is a polynomial
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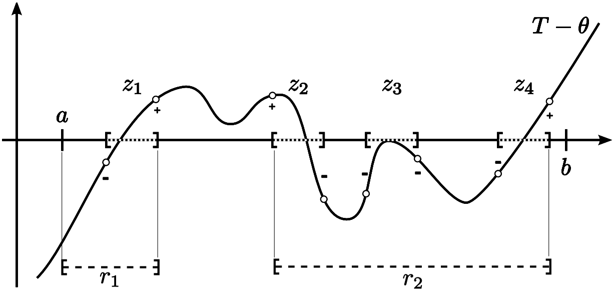
- Compute sign changes
- Find suitable intervals



Our example - Relative error

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Compute Taylor Model for $\tau = p' f - p f'$, order 12.

Remainder: $[-1.67352e-7; 1.67352e-7]$

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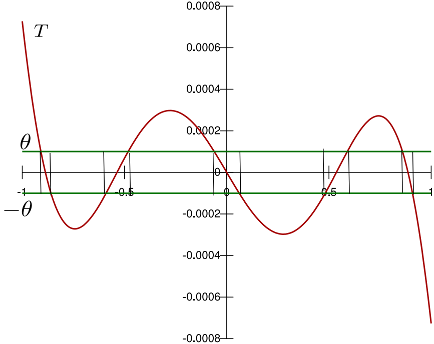

$r_0 \in -0.886[9;8],\ r_1 \in -0.536[6;4],$
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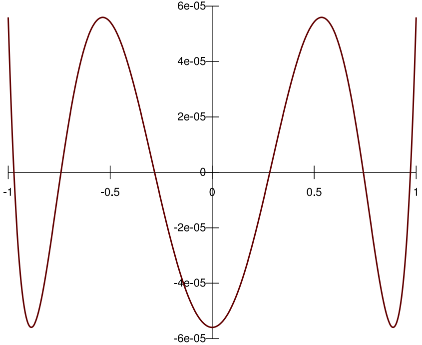

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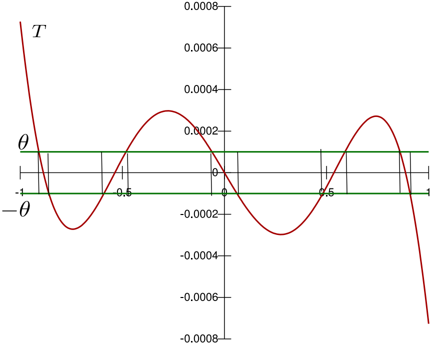
$\|\varepsilon\|_\infty \in 5.59[3/4] \times e - 5$.

Our Approach - Summary

- Purpose: fast and safely compute the supremum norm $\|f - p\|_\infty$ and $\|p/f - 1\|_\infty$ over an interval $[a, b]$
- Enclose all the zeros of the first derivative of the error
- Evaluate the approximation error on these small intervals only, using IA
- Use a Taylor Model based approach to overcome the dependency
- Enclose the zeros of a function using our algorithm

Our example - Relative error - How small should the remainder be?

$\tau(x) - T(x) \in [-\theta, \theta], \forall x \in I$
$\tau(x) = 0 \implies T(x) \in [-\theta, \theta]$



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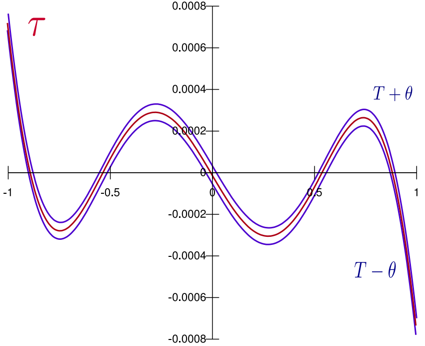

Degree: 12

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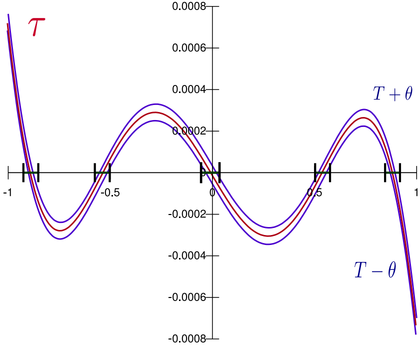

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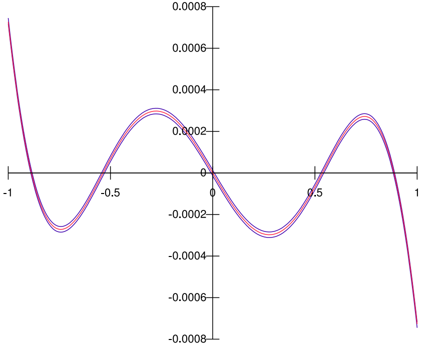

Degree: 14

Our example - Relative error - How small should the remainder be?

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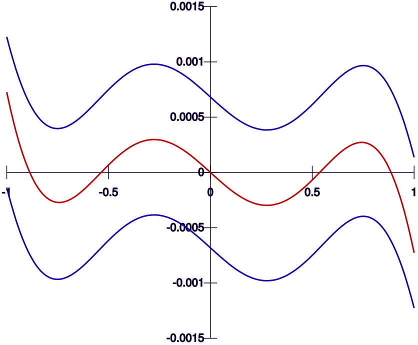

Degree: 8

Our example - Relative error - How small should the remainder be?

- Heuristics
- Start with a value close to the approx. max value of τ
- In our example: $\theta_i = 6.44e - 4$

θ	degree T	$\|\varepsilon\|_\infty$
θ_i	8	$[5.59e-5; 1.84]$
$\theta_i/2$	9	$[5.59e-5; 1e-3]$
$\theta_i/10$	10	$[5e-5; 8e-5]$
$\theta_i/10^2$	11	$0.5[5; 7]e-4$
$\theta_i/10^3$	12	$0.559[3; 4]e-4$
$\theta_i/10^4$	13	$0.55935[2; 4]e-4$
$\theta_i/10^5$	14	$0.5593528[2; 4]e-4$

Results

Experiments were made on an Intel Pentium D 3.00GHz with a 2GB RAM.

f	$[a,\, b]$	d_p[1]	m[2]	acc[3]	time[4]
$\exp(x) - 1$	$[-0.25,\, 0.25]$	5	r	37.6	412
$\log_2(1 + x)$	$[-2^{-9},\, 2^{-9}]$	7	r	83.3	2, 186
$\cos(x)$	$[-0.5,\, 0.25]$	15	r	19.5	2, 235
$\exp(x)$	$[-0.125,\, 0.125]$	25	r	42.3	7, 753
$\sin(x)$	$[-0.5,\, 0.5]$	9	a	21.5	520
$\exp(\cos(x)^2 + 1)$	$[1,\, 2]$	15	r	25.5	10, 984
$\tan(x)$	$[0.25,\, 0.5]$	10	r	26.0	1, 072
$x^{2.5}$	$[1,\, 2]$	7	r	15.5	1, 362

[1] Degree of p

[2] Error mode considered: a=absolute, r=relative

[3] Accuracy

[4] Timings in ms

Conclusion

- Safe and fast algorithm for bounding the supremum norm of the error functions
- Combination and reusal of various techniques (TM, AD, polynomial roots isolation, interval arith)
- Absolute and Relative errors handled
- Faster and more accurate than other current approaches
- Future works:
  - Formal proof (AD, isolation of roots, multiple precision interval arithmetic are needed in the proof checker)
  - Generalization of the algorithm to multivariate functions.

Questions

Thank you for your attention!

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