Moments and convex optimization for analysis and control of nonlinear partial differential equations

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Problem setting – analysis

$$\begin{aligned} F(x, y(x), \mathcal{D}y(x)) &= 0 \quad \text{for} \quad x \in \Omega^\circ \\ G(x, y(x), \mathcal{D}y(x)) &= 0 \quad \text{for} \quad x \in \partial\Omega, \end{aligned}$$

$\Omega \subset \mathbb{R}^n$ compact

$y : \mathbb{R}^n \to \mathbb{R}^k$

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$$
J\big(y(\cdot)\big) := \int_\Omega L\,(x, y(x), \mathcal{D}y(x))\; dx + \int_{\partial\Omega} L_\partial\,(x, y(x), \mathcal{D}y(x))\; d\sigma(x)
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Example: $L(x, y, z) = y^2,\ \ L_\partial(x, y, z) \quad \Rightarrow \quad J(y(\cdot)) = \int_\Omega y(x)^2\, dx$

Problem setting – analysis

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$y : \mathbb{R}^n \to \mathbb{R}^k$

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Goal: establish bounds on

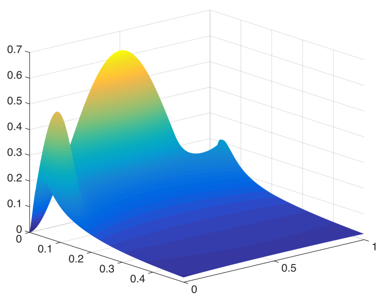
$$
\begin{aligned}
&\inf_y / \sup && J(y) \\
&\text{subject to} && y \text{ solves the PDE}
\end{aligned}
$$

Infinite dimensional linear programming

$$\min_{\mu} \langle g, \mu \rangle$$
$$\mathcal{A}\mu = b$$
$$\mu \in \mathcal{M}^+$$

Infinite-dimensional **LP**

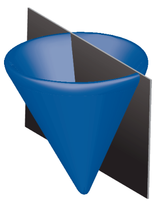
$\Nearrow$ $\Searrow$



$\inf / \sup_{y} \quad J(y)$

subject to $\quad y$ solves the PDE

$$\min_{\boldsymbol{y}} \langle g_N, \boldsymbol{y} \rangle$$
$$\mathcal{A}_N \boldsymbol{y} = b_N$$
$$\boldsymbol{y} \in \mathcal{M}_N^+$$



Convex semidefinite program

Infinite dimensional linear programming

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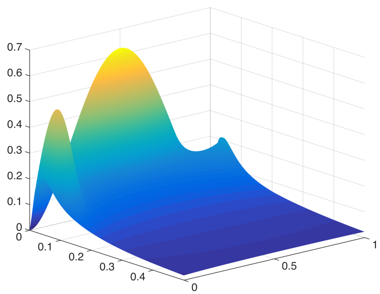
Infinite-dimensional **LP**

$\nearrow$ $\searrow$



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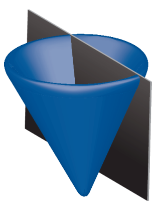

Convex semidefinite program

Long history: Global optimization (Lasserre, Parrilo, Nesterov,...)

Stability of ODEs (Rantzer, Vaydia,...)

Optimal control of ODEs (Young, Vinter, Lasserre, Henrion, Gaitsgory,...)

⋮

Occupation and boundary measures

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Occupation measure

$$\mu(A \times B \times C) = \int_\Omega \mathbb{I}_{A\times B\times C}\,(x, y(x), \mathcal{D}y(x))\; dx$$

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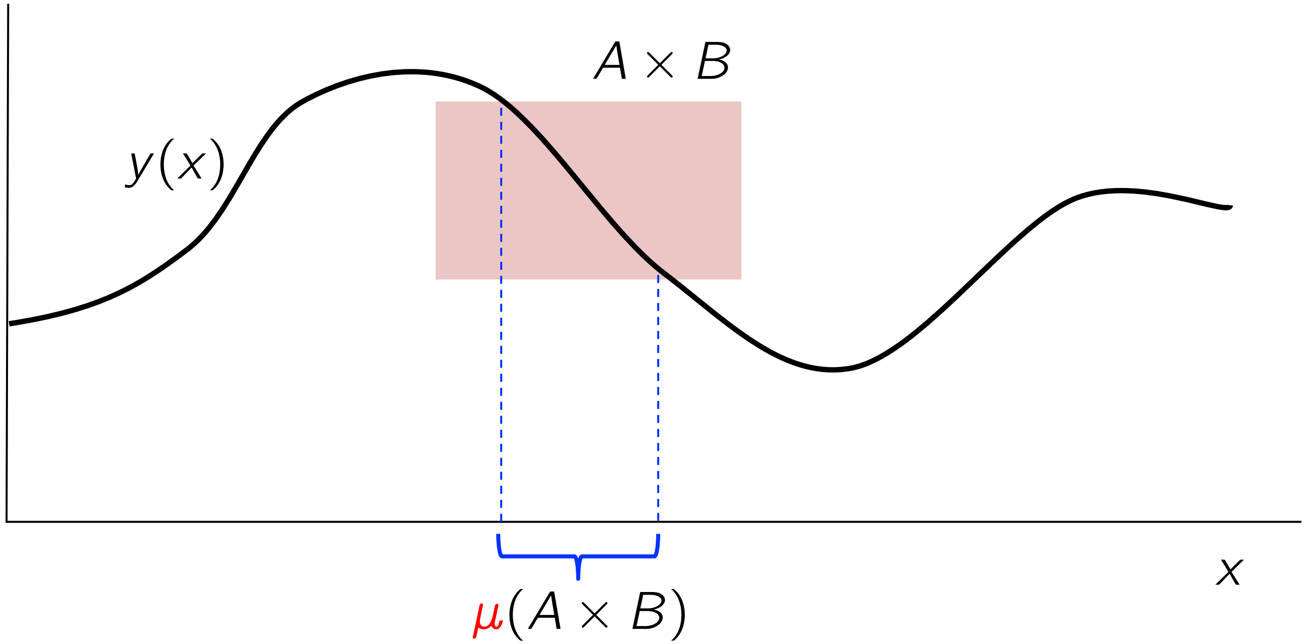

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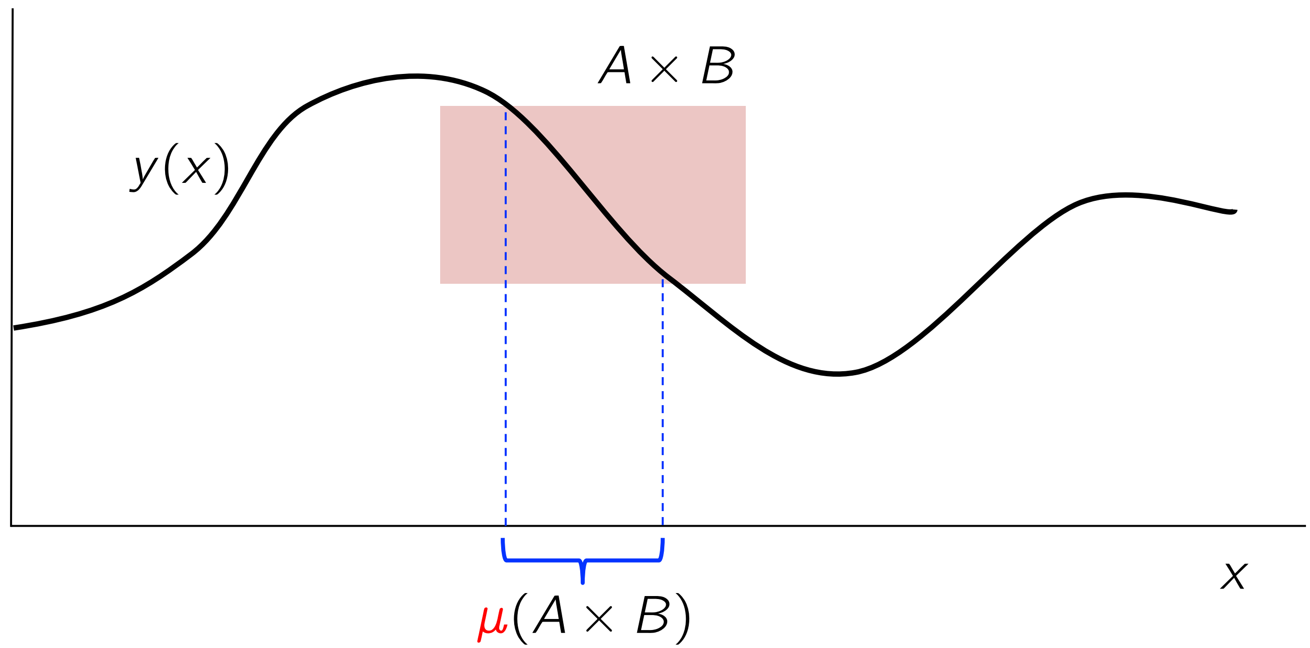
$$\mu(A \times B \times C) = \int_{\Omega} \mathbb{I}_{A\times B\times C}\,(x, y(x), \mathcal{D}y(x))\ dx$$

Boundary measure

$$\mu_{\partial}(A \times B \times C) = \int_{\partial\Omega} \mathbb{I}_{A\times B\times C}\,(x, y(x), \mathcal{D}y(x))\ d\sigma(x),$$



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$$(\mu, \mu_{\partial}) \quad \longleftrightarrow \quad y(\cdot)$$

Occupation and boundary measures

Integration along solutions to the PDE $\rightarrow$ spatial integration w.r.t. μ and μ_∂

$$\int_\Omega h\,(x, y(x), \mathcal{D}y(x))\; dx = \int_{\Omega\times\mathrm{Y}\times\mathrm{Z}} h\,(x, y, z)\; d\mu(x, y, z)$$

for **all** $h \in L_\infty$

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$$\int_{\partial\Omega} h\,(x, y(x), \mathcal{D}y(x))\ \sigma(x) = \int_{\partial\Omega\times\mathrm{Y}\times\mathrm{Z}} h\,(x, y, z)\ d\mu_\partial(x, y, z)$$

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for **all** $h \in L_\infty$

$$\Rightarrow \quad J\big(y(\cdot)\big) = \int_\Omega L\,(x, y(x), \mathcal{D}y(x))\; dx + \int_{\partial\Omega} L_\partial\,(x, y(x), \mathcal{D}y(x))\; d\sigma(x)$$

$$= \int_\Omega L\,(x,y,z)\; d\mu(x,y,z) + \int_{\partial\Omega} L_\partial\,(x,y,z)\; d\mu_\partial(x,y,z)$$

When do μ and μ_∂ come from our PDE?

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$$\int_{\Omega \times \mathbf{Y} \times \mathbf{Z}} \phi(x,y) F(x,y,z)\, d\mu(x,y,z) = 0$$

$$\int_{\partial\Omega \times \mathbf{Y} \times \mathbf{Z}} \phi(x,y) G(x,y,z)\, d\mu_\partial(x,y,z) = 0$$

$$\int_{\partial\Omega \times \mathbf{Y} \times \mathbf{Z}} \phi(x,y) \eta(x)\, d\mu_\partial(x,y,z) - \int_{\Omega \times \mathbf{Y} \times \mathbf{Z}} \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} z\, d\mu(x,y,z) = 0.$$

$$\int_{\partial\Omega \times \mathbf{Y} \times \mathbf{Z}} \psi(x)\, d\mu_\partial(x,y,z) = \int_{\partial\Omega} \psi(x)\, \sigma(x)$$

for **all** $\phi \in C^\infty(\Omega \times \mathbf{Y})$ and $\psi \in C(\Omega)$

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System of **linear** equations in μ,μ_∂

Infinite-dimensional LP

$$\inf_{y(\cdot)} / \sup \quad J(y(\cdot))$$
$$\text{subject to} \quad y(\cdot) \text{ solves the PDE}$$

↘

$$\inf_{\mu,\mu_\partial} / \sup \quad \int_\Omega L\,(x,y,z)\; d\mu(x,y,z) + \int_{\partial\Omega} L_\partial\,(x,y,z)\; d\mu_\partial(x,y,z)$$

$$\int_{\Omega\times Y\times Z} \phi(x,y)F(x,y,z)\,d\mu(x,y,z) = 0 \;\; \forall\phi \in C(\Omega\times Y)$$

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$$\int_{\partial\Omega\times Y\times Z} \psi(x)\,d\mu_\partial(x,y,z) = \int_{\partial\Omega} \psi(x)\,\sigma(x) \;\; \forall\psi \in C(\Omega)$$

$$\mu \in \mathcal{M}(\Omega\times Y\times Z)_+, \;\; \mu_\partial \in \mathcal{M}(\partial\Omega\times Y\times Z)_+$$

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$$\inf / \sup_{\mu,\mu_\partial} \quad \big\langle (\mu, \mu_\partial), c \big\rangle$$

$$\text{s.t.} \quad \mathcal{A}(\mu, \mu_\partial) = b$$

$$(\mu, \mu_\partial) \in \mathcal{K}$$

Infinite-dimensional LP

$$\inf_{y(\cdot)} / \sup \quad J(y(\cdot))$$

subject to $y(\cdot)$ solves the PDE

optimal value p

$$\inf_{\mu,\mu_\partial} / \sup \quad \langle (\mu, \mu_\partial), c \rangle$$

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optimal value p_{LP}

Lemma: p_{LP} is a lower/upper bound on p

Relaxation gap

When do we have $p = p_{\mathrm{LP}}$?

- Proven for ODEs

- Proven for scalar hyperbolic conservation laws in *[Marx et. al, 2018]*

  $\rightarrow$ additional constraints added to the LP - “entropy inequalities”

- General case **open**

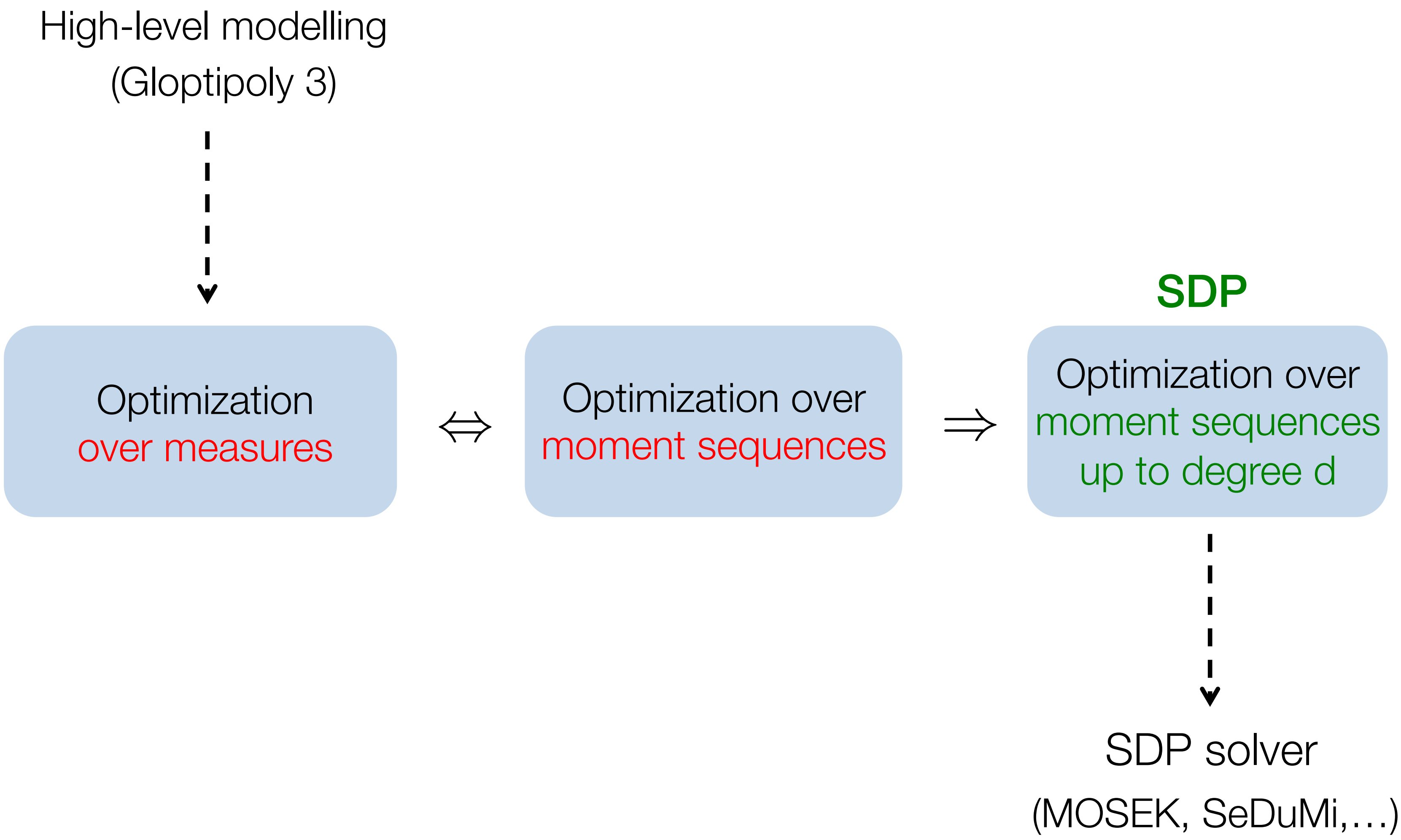
Finite-dimensional approximation

Optimization over measures $\Leftrightarrow$ Optimization over moment sequences $\Rightarrow$ Optimization over moment sequences up to degree d

($\Leftrightarrow$: Compactness)

SDP: Optimization over moment sequences up to degree d

Finite-dimensional approximation



Finite-dimensional approximation

SDP

Optimization over measures	$\Leftrightarrow$	Optimization over moment sequences	$\Rightarrow$	Optimization over moment sequences up to degree d

Theorem: $p_k \leq p_{\mathrm{LP}} \leq p$ and $\lim_{k\to\infty} p_k = p_{\mathrm{LP}}$

Numerical examples – Burgers' equation

$$\frac{\partial y}{\partial x_1} + y\frac{\partial y}{\partial x_2} = 0 \qquad \Omega = [0,5] \times [0,1]$$

- $y(0, x_2) = 10(x_2(1 - x_2))^2$
- $y(x_1, 0) = y(x_1, 1)$

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Case 1: $J(y(\cdot)) = \int_0^5 \int_0^1 y(x_1, x_2)^2 \, dx_2 dx_1$

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Analytical solution: $J(y(\cdot)) = \frac{50}{63} \approx 0.79365079365$

SDP bounds:
(d = 4)

$p_{\mathrm{LB}} \approx 0.79365079357$

$p_{\mathrm{UB}} \approx 0.79365080188$

Numerical examples – Burgers' equation

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Case 2: $J(y(\cdot)) = \int_0^5 \int_0^1 x_2^2 y(x_1,x_2)^2\, dx_2 dx_1$

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Case 2: $J(y(\cdot)) = \int_0^5 \int_0^1 x_2^2 y(x_1, x_2)^2 \, dx_2 dx_1$

d	4	6	8
Lower bound (SeDuMi)	0.206	0.263	0.276
Upper bound (SeDuMi)	0.380	0.297	0.283
Parse time (Gloptipoly 3)	2.91s	3.41 s	6.23 s
SDP solve time (SeDuMi / MOSEK)	2.62 / 1.63 s	2.61 / 1.32 s	20.67 / 7.05 s

Control – problem setup

$$
\begin{aligned}
F(x, y(x), \mathcal{D}y(x)) &= C(x, y(x))u(x) \quad &\text{for} \quad & x \in \Omega^\circ \\
G(x, y(x), \mathcal{D}y(x)) &= C_\partial(x, y(x))u_\partial(x) \quad &\text{for} \quad & x \in \partial\Omega,
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$$

$$
\begin{aligned}
J(y(\cdot)) := &\int_\Omega L\,(x, y(x), \mathcal{D}y(x))\; dx + \int_{\partial\Omega} L_\partial\,(x, y(x), \mathcal{D}y(x))\; d\sigma(x) \\
&+ \int_\Omega L_u\,(x, y(x))\, u(x)\, dx + \int_{\partial\Omega} L_{u_\partial}\,(x, y(x))\, u\, d\sigma(x)
\end{aligned}
$$

Goal: solve (at least approximately)

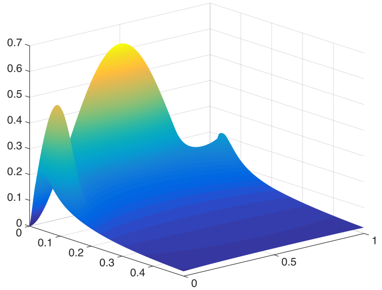
$$
\begin{aligned}
\inf_{y, u, u_\partial} \quad & J(y, u, u_\partial) \\
\text{subject to} \quad & y \text{ solves the PDE}(u, u_\partial) \\
& 0 \le u \le 1 \text{ on } \Omega^\circ \\
& 0 \le u_\partial \le 1 \text{ on } \partial\Omega
\end{aligned}
$$

Infinite dimensional linear programming

$$\min_{\mu} \langle g, \mu \rangle$$
$$\mathcal{A}\mu = b$$
$$\mu \in \mathcal{M}^{+}$$

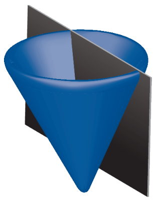
Infinite-dimensional **LP**

$\nearrow$ $\searrow$



Nonlinear PDE
with **control**

$$\min_{\boldsymbol{y}} \langle g_N, \boldsymbol{y} \rangle$$
$$\mathcal{A}_N \boldsymbol{y} = b_N$$
$$\boldsymbol{y} \in \mathcal{M}_N^{+}$$



Convex semidefinite program

Control measures

$$\nu(A \times B) := \int_{\Omega} \mathbb{I}_{A\times B}(x, y(x))\, u(x) dx$$

$\Leftrightarrow$ $\nu \ll \bar{\mu}$ with density u

Constraints easy to impose: $\nu \le \bar{\mu} \quad \Leftrightarrow \quad u \in [0, 1]$

Control measures

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$$\nu_{\partial}(A \times B) := \int_{\partial\Omega} \mathbb{I}_{A\times B}\,(x, y(x))\; u_{\partial}(x) dx$$

$\Leftrightarrow$

$\nu \ll \bar{\mu}$ with density u

$\nu_{\partial} \ll \bar{\mu}_{\partial}$ with density u

Constraints easy to impose: $\nu \leq \bar{\mu} \quad \Leftrightarrow \quad u \in [0, 1]$

$\nu_{\partial} \leq \bar{\mu}_{\partial} \quad \Leftrightarrow \quad u_{\partial} \in [0, 1]$

Infinite-dimensional LP – with control

$$\inf_{y,u,u_\partial} \quad J(y,u,u_\partial)$$

subject to y solves the PDE(u, u_∂)

$$0 \le u \le 1 \text{ on } \Omega^\circ$$

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$$\inf / \sup_{\mu,\mu_\partial,\nu,\nu_\partial} \quad \langle (\mu,\mu_\partial,\nu,\nu_\partial), c \rangle$$

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$$(\mu,\mu_\partial,\nu,\nu_\partial) \in \mathcal{K}$$

Infinite-dimensional LP – with control

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subject to $\quad y$ solves the PDE(u, u_∂)

$$0 \le u \le 1 \text{ on } \Omega^\circ$$

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optimal value p

$$\inf / \sup_{\mu,\mu_\partial,\nu,\nu_\partial} \quad \langle (\mu,\mu_\partial,\nu,\nu_\partial), c \rangle$$

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optimal value p_{LP}

Lemma: p_{LP} is a lower bound on p

Controller extraction

SDP

Optimization over measures $\Leftrightarrow$ Optimization over moment sequences $\Rightarrow$ Optimization over moment sequences up to degree d

Feedback controller

$$u(x_1, x_2) = \kappa(y(x_1, x_2), x_1, x_2)$$

$$u_\partial(x_1, x_2) = \kappa_\partial(y(x_1, x_2), x_1, x_2)$$

Numerical examples with control

$$\frac{\partial y}{\partial x_1} + y\frac{\partial y}{\partial x_2} = u(x_1, x_2) \qquad \Omega = [0,3] \times [0,1]$$

- $y(0, x_2) = 10(x_2(1 - x_2))^2$
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Goal: minimize $\int_0^3 \int_0^1 y(x_1, x_2)^2 dx_2 dx_1$

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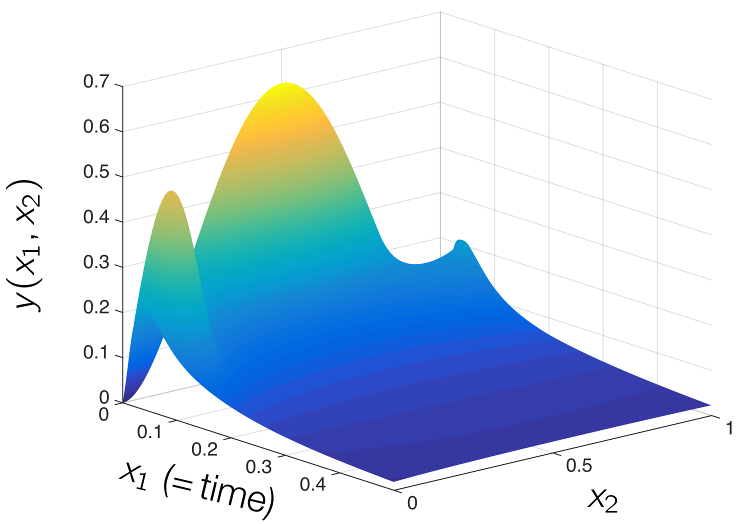
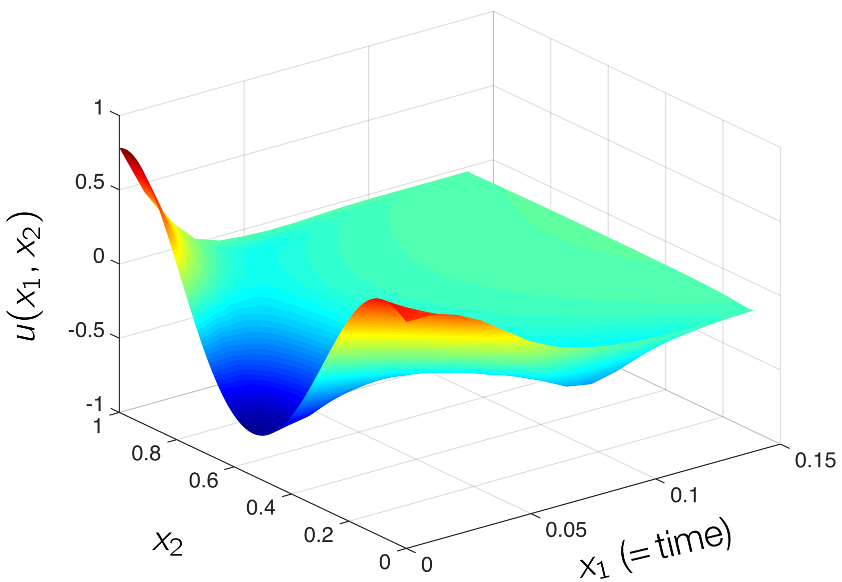
SDP $\rightarrow$ polynomial controller of degree 3





Computational complexity

Measures supported on subsets of $\mathbb{R}^n$ of dimension

$$n \;=\; \dim(x) \;+\; \dim(y) \;+\; \#\text{ derivatives appearing nonlinarly}$$

Largest SDP block of size: $N = \binom{n+d/2}{n}$

n	4	6	8	10	12	17	21	24
$d = 4$	15	28	45	66	91	171	253	325
$d = 6$	35	84	165	286	455	1140	2024	2925
$d = 8$	70	210	495	1001	1820	5985	12650	20475

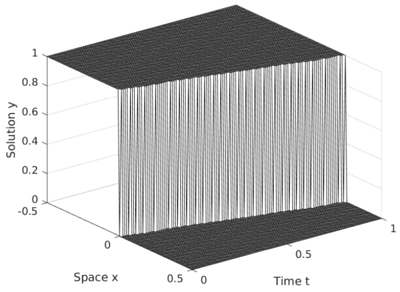
Complexity reduction: Sparsity

Degree bounding

Conclusion

- **Convex** optimization based approach to analysis and control of PDEs
  - $\rightarrow$ Off-the-shelf software available

[Marx et al, 2018]



- No spatio-temporal gridding
  - $\rightarrow$ **Discontinuities** (e.g., shocks) well resolved

- Solutions represented by measures supported on their **graphs**

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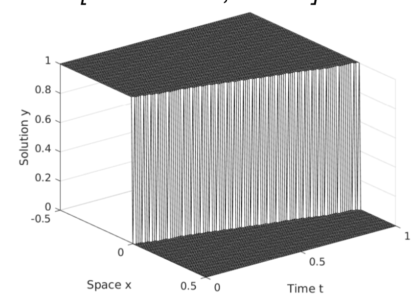
- **Convex** optimization based approach to analysis and control of PDEs
  - $\rightarrow$ Off-the-shelf software available

[Marx et al, 2018]



- No spatio-temporal gridding
  - $\rightarrow$ **Discontinuities** (e.g., shocks) well resolved

- Solutions represented by measures supported on their **graphs**

Future directions

- Absence of relaxation gap

- More extensive numerical experiments

Preprint: `https://arxiv.org/abs/1804.07565`