Measure optimization for dynamical systems and control: overview and perspectives

Milan Korda

(LAAS, CNRS)

$$\dot{x} = f(x)$$

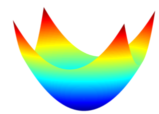
uncontrolled

$$\dot{x} = f(x, u)$$

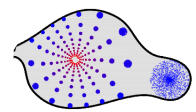
controlled

Questions:

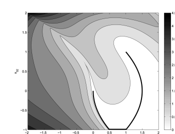
Stability



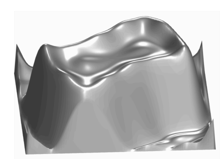
Reachability



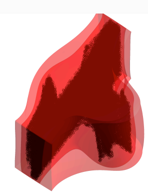
Optimal control



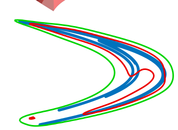
Region of attraction



Invariant sets



Invariant measures



⋮

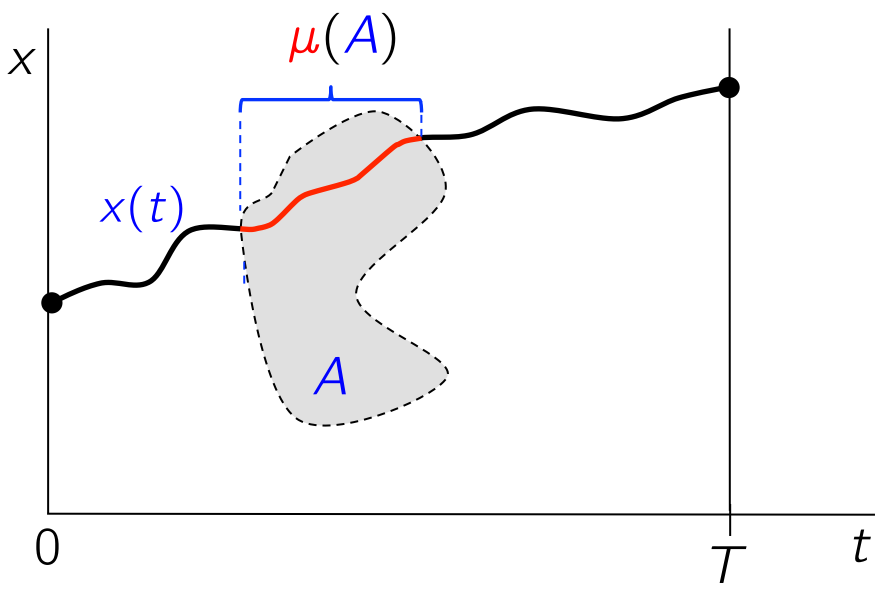
Occupation measures

$$\dot{x} = f(x)$$

Assumption: $x_0 \in \mathbb{R}^n$ given $\Rightarrow$ unique solution $t \mapsto x(t)$ on $[0, T]$

$$\dot{x} = f(x)$$

Occupation measure: $$\mu(A) = \int_0^T \mathbb{I}_A(t, x(t))\,\mathrm{d}t \qquad \forall A \in \mathcal{B}([0,T] \times \mathbb{R}^n)$$



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Key property:

$$\int_0^T g(t, x(t))\,\mathrm{d}t = \int_{[0,T]\times\mathbb{R}^n} g(t,x)\,\mathrm{d}\mu(t,x)$$

for all $g \in L_1([0,T] \times \mathbb{R}^n])$

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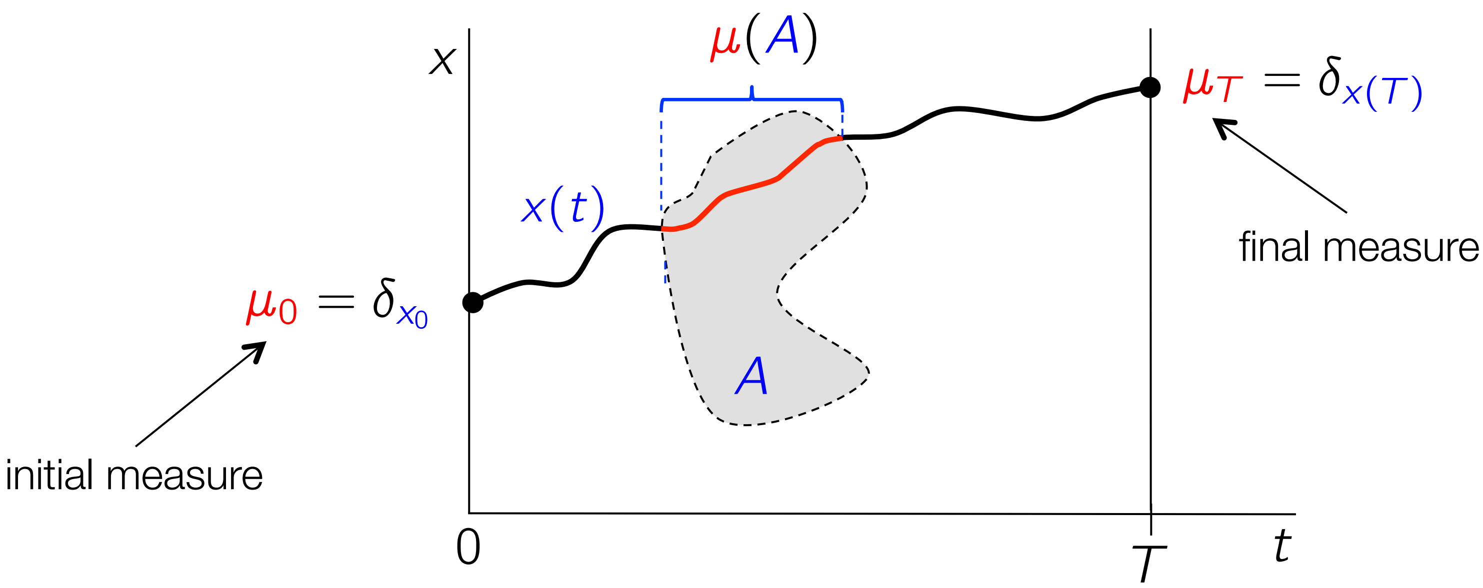

What is the relation between μ_0, μ, μ_T and f?

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Given a test function $\phi \in C^1([0,T] \times \mathbb{R}^n)$

$$\phi(T, x(T)) \quad - \quad \phi(0, x_0) \qquad = \qquad \int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\phi(t, x(t))\,\mathrm{d}t$$

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$$\int_{\mathbb{R}^n} \phi(T,\cdot)\,\mathrm{d}\mu_T \; - \; \int_{\mathbb{R}^n} \phi(0,\cdot)\,\mathrm{d}\mu_0 \; = \; \int_{[0,T]\times\mathbb{R}^n} \frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial x}\cdot f \; d\mu$$

for all $\phi \in C^1([0,T]\times\mathbb{R}^n)$

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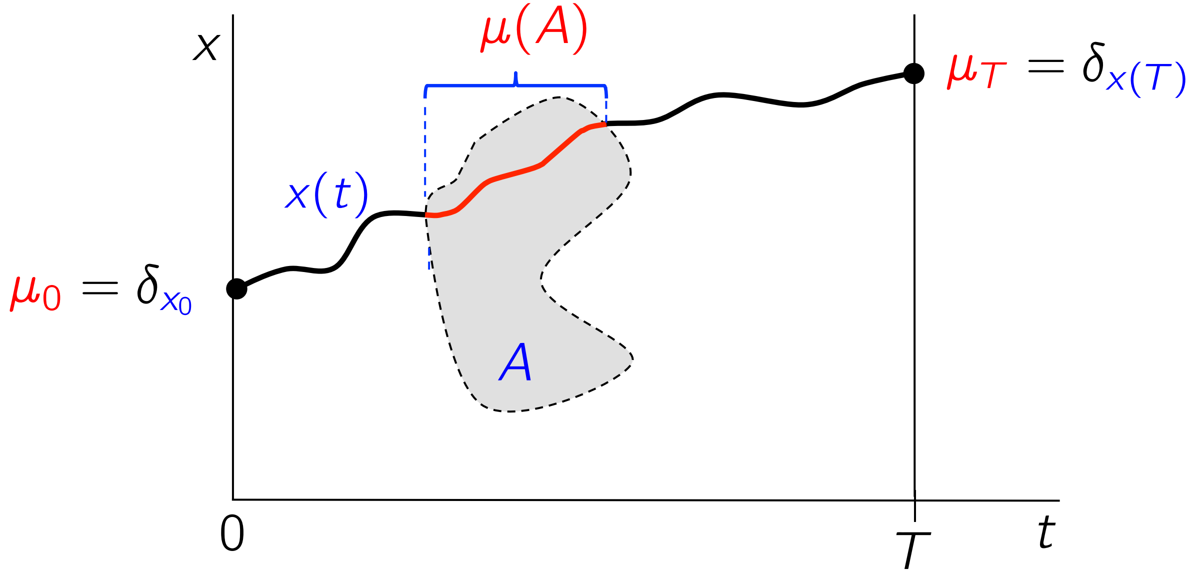
Fact: $\dot{x} = f(x)$ has a unique solution on $[0, T]$ $\Rightarrow$ $\mathcal{P}$ is a **singleton**



Infinite-dimensional linear program

Infinite-Dimensional **LP**

$$p = \inf_{\mu_0, \mu, \mu_T} / \sup \quad \int_{[0,T]\times\mathbb{R}^n} h(x,t)\,\mathrm{d}\mu(t,x)$$

$$\text{s.t.} \quad (\mu_0, \mu, \mu_T) \in \mathcal{P}$$

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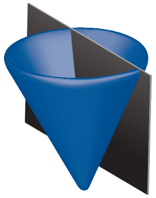
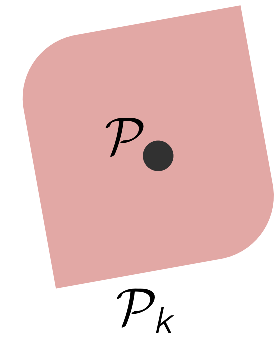


Finite-Dimensional **SDP** relaxation

$$
\begin{aligned}
p_k \quad = \quad & \inf_{\mu_0,\mu,\mu_T} / \sup \quad \int h \,\mathrm{d}\mu \\
& \text{s.t.} \quad (\mu_0, \mu, \mu_T) \in \mathcal{P}_k
\end{aligned}
$$

assumption: h and f polynomial



$\mathcal{P}_k$ **semidefinite programming** representable

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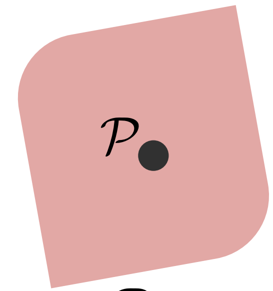

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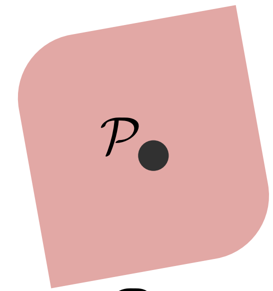

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$$\mathcal{M}_k(\mathbf{K})_+ \supset \mathcal{M}_{k+1}(\mathbf{K})_+ \supset \ldots \supset \mathcal{M}(\mathbf{K})_+$$

(Putinar, Schmudgen, Krivin-Stengle, Handelman,...)

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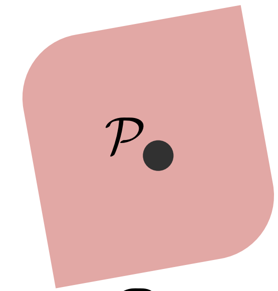

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(Putinar, Schmudgen, Krivin-Stengle, Handelman,…)

Convergence

(under a compactness assumption)

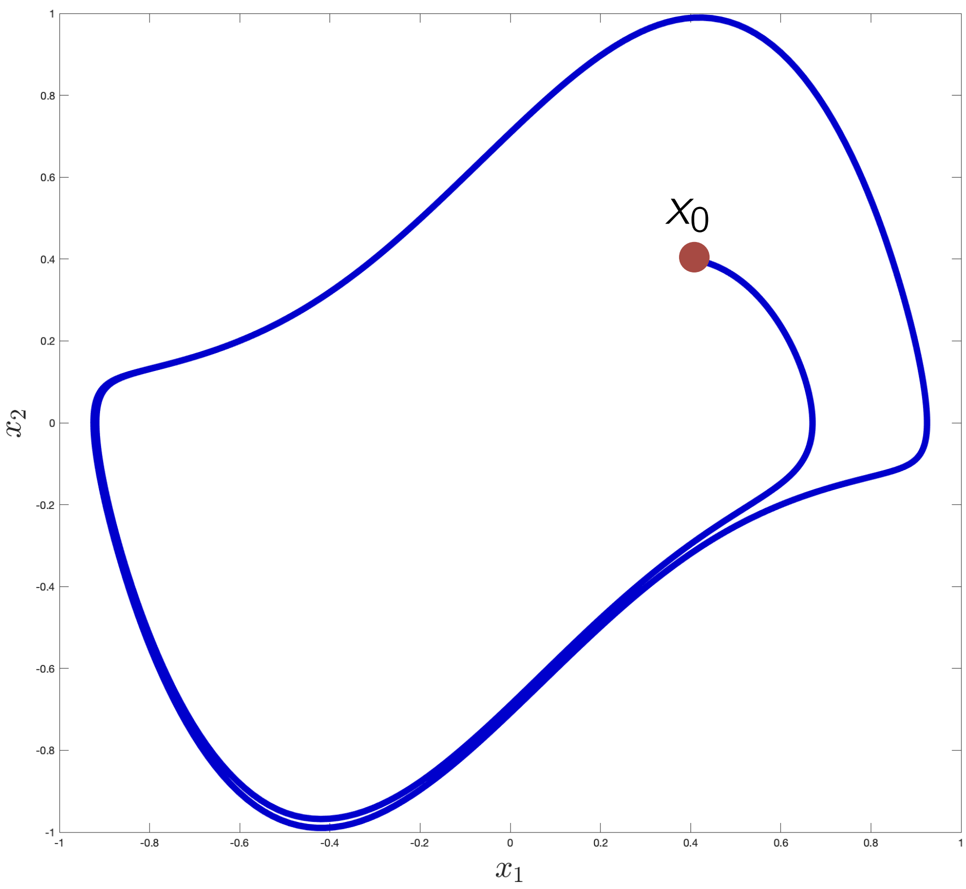
$p_k \nearrow p$ for "inf"

$p_k \searrow p$ for "sup"

$$p = \int_0^T h(t, x(t \,|\, x_0)) \, dt$$

Van der Pol oscillator

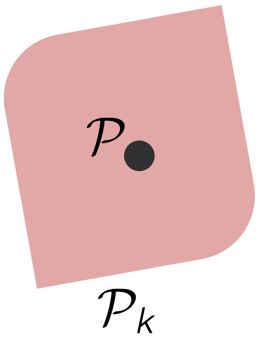
$$\dot{x}_1 = 2x_2$$
$$\dot{x}_2 = 10(0.21 - x_1^2)x_2 - 0.8x_1$$



$$p_k = \inf_{\mu_0,\mu,\mu_T} / \sup \int_{[0,T]\times\mathbb{R}^n} x_1^2 - x_2^2 \, d\mu(t,x)$$

$$\text{s.t.} \quad (\mu_0, \mu, \mu_T) \in \mathcal{P}_k$$

$\Rightarrow$ bounds on $\int_0^T x_1(t\,|\,x_0)^2 - x_2(t\,|\,x_0)^2 \, dt$



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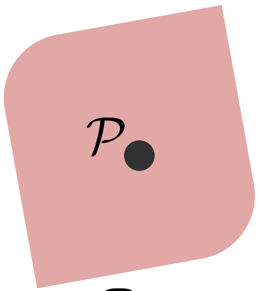
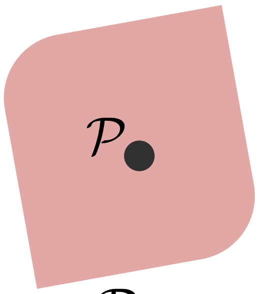
$$



moment	SDP	Exact
$\int 1$	1.0000	1.0000
$\int t$	0.5000	0.5000
$\int x_1$	-0.0605	-0.0632
$\int x_2$	-0.0438	-0.0434
$\int t^2$	0.3333	0.3333
$\int tx_1$	-0.0821	-0.0847
$\int tx_2$	-0.0208	0.0202
$\int x_1^2$	0.4359	0.4343
$\int x_1 x_2$	0.0020	0.0015
$\int x_2^2$	0.1600	0.1594
$\vdots$	$\vdots$	$\vdots$

$$p_k = \inf_{\mu_0,\mu,\mu_T} / \sup \int_{[0,T]\times\mathbb{R}^n} x_1^2 - x_2^2 \, d\mu(t,x)$$

$$\text{s.t.} \quad (\mu_0, \mu, \mu_T) \in \mathcal{P}_k$$



$\mathcal{P}_k$

$\Rightarrow$ bounds on $\int_0^T x_1(t\,|\,x_0)^2 - x_2(t\,|\,x_0)^2 \, dt$

$T = 1$

True $= 0.3313$

k	2	3	4
Lower bound	0.2401	0.3313	...
Upper bound	0.3326	0.3313	...

$$
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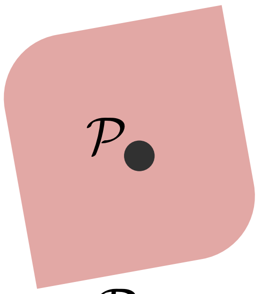

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True $= 0.3313$

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Lower bound	0.2401	0.3313	...
Upper bound	0.3326	0.3313	...

$T = 10$

True $= 0.2750$

k	2	3	4	5	6	7
Lower bound	-0.08	0.038	0.148	0.2664	0.2728	0.2739
Upper bound	0.484	0.329	0.294	0.2759	0.2752	0.2751

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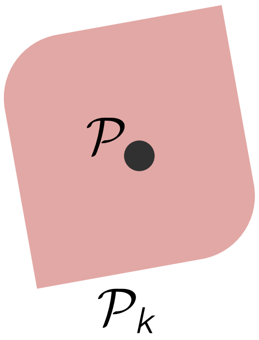

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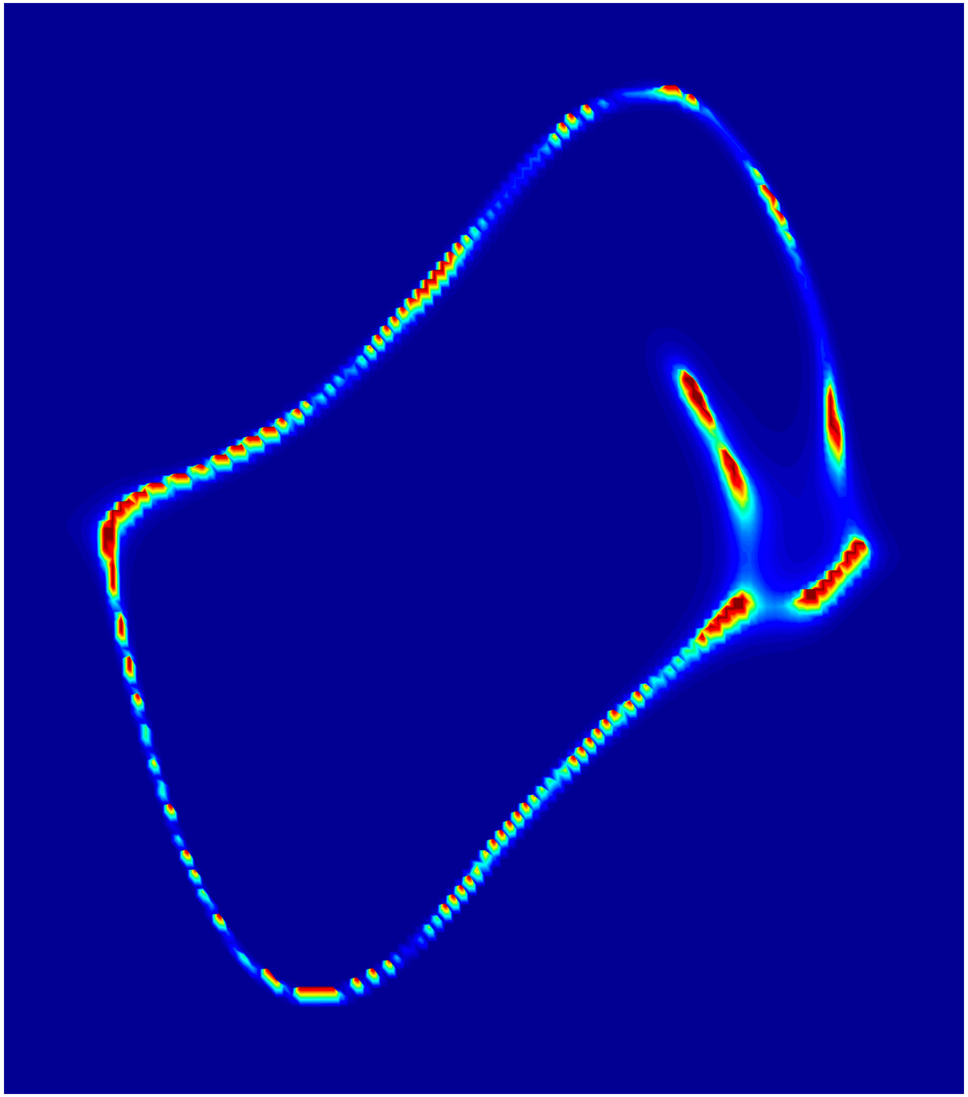
Trajectory recovery from approximate moments



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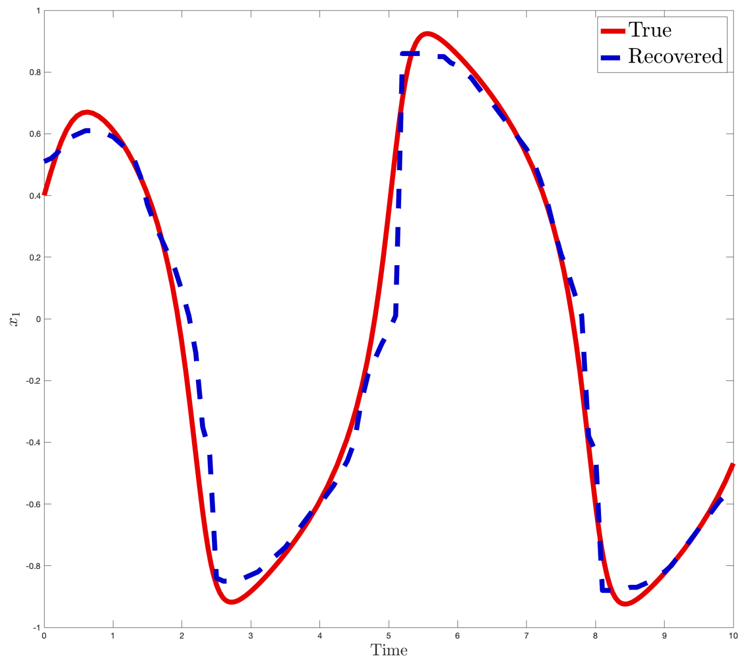

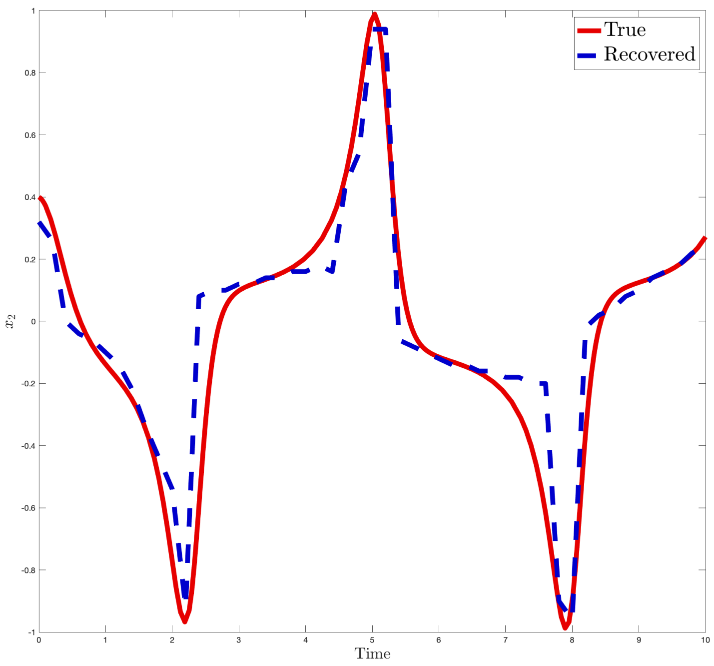

[Claeys, Sepulchre, 2014]

[Marx et al. 2019]

Arbitrary initial measure

$$\delta_T \otimes \mu_T - \delta_0 \otimes \mu_0 + \mathrm{div}(f\mu) = 0 \qquad (\mathcal{L})$$

$$\mathcal{P} = \{(\mu_0, \mu, \mu_T) \;\mid\; (\mu_0, \mu, \mu_T) \text{ satisfies } \mathcal{L} \text{ and } \mu_0 = \nu,\ \ \mu \in \mathcal{M}([0,T]\times\mathbb{R}^n)_+,\ \ \mu_T \in \mathcal{M}(\mathbb{R}^n)_+\}$$

Fact: $\dot{x} = f(x)$ has a unique solution on $[0, T]$ $\Rightarrow$ $\mathcal{P}$ is a **singleton**

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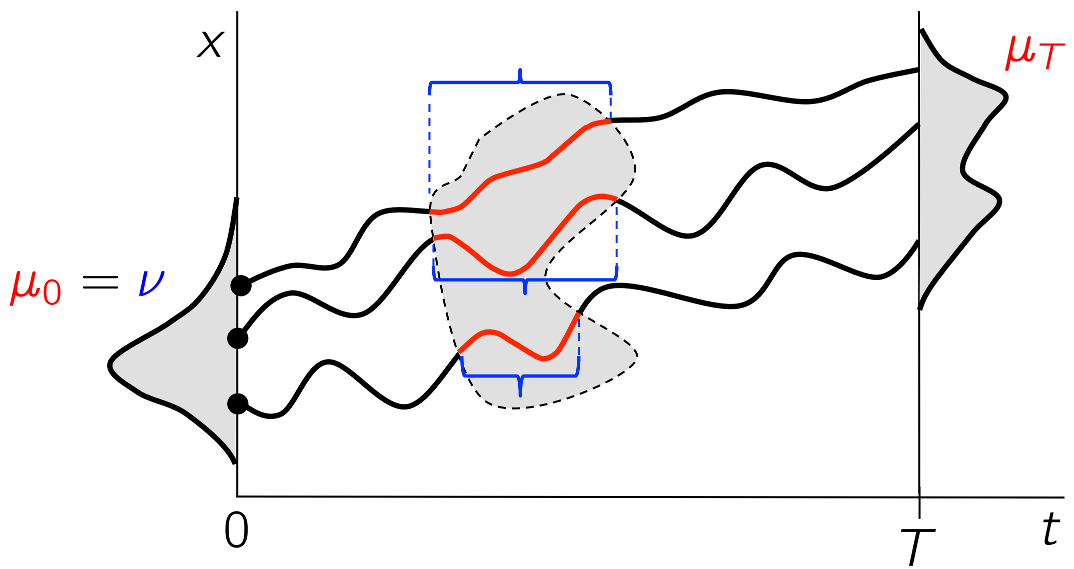

Finite-Dimensional **SDP**

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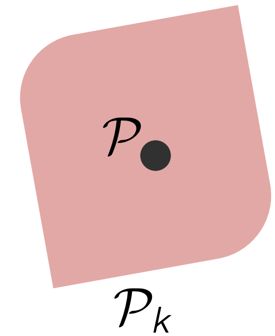

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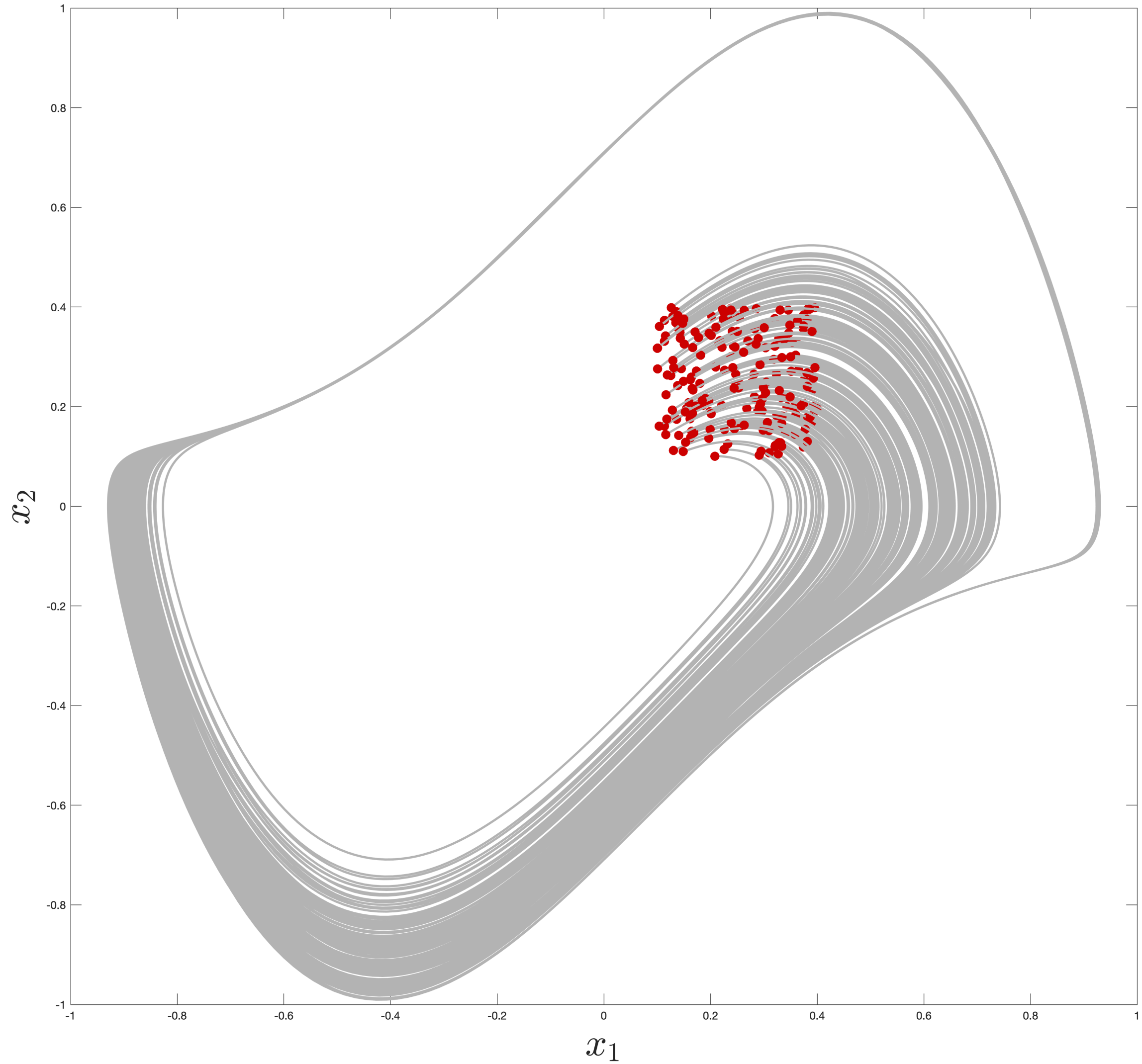
$p_k \nearrow p$ for "inf"

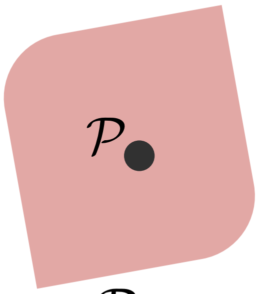
$p_k \searrow p$ for "sup"

$$p = \int_{\mathbb{R}^n} \int_0^T h(t, x(t \,|\, x_0)) \, dt \, d\nu(x_0)$$

ν = uniform on $[0.1, 0.3] \times [0.1, 0.3]$



$$
\begin{aligned}
p_k \quad = \quad & \inf_{\mu_0,\mu,\mu_T} / \sup \quad \int_{[0,T]\times\mathbb{R}^n} x_1^2 - x_2^2 \, d\mu(t,x) \\
& \text{s.t.} \quad (\mu_0, \mu, \mu_T) \in \mathcal{P}_k
\end{aligned}
$$



$\mathcal{P}_k$

$\Rightarrow$ bounds on $\int_{\mathbb{R}^n} \int_0^T x_1(t\,|\,x_0)^2 - x_2(t\,|\,x_0)^2 \, dt \, d\nu(x_0)$

$T = 1$

True $= 0.1817$

k	2	3	4	5	6
Lower bound	0.016	0.1780	0.1813	0.1816	0.1817
Upper bound	0.2152	0.1895	0.1831	0.1823	0.1821

$T = 10$

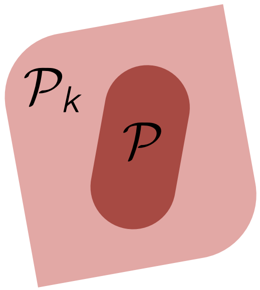
True $= 0.2554$

k	2	3	4	5	6	7
Lower bound	-0.088	0.0116	0.0769	0.1648	0.1935	0.2009
Upper bound	0.4858	0.3217	0.2854	0.2644	0.2622	0.2615

Free initial measure

Finite-Dimensional **SDP**

$$p_k = \inf_{\mu_0, \mu, \mu_T} \int_{[0,T]\times\mathbb{R}^n} h(x,t)\, \mathrm{d}\mu(t,x)$$
$$\text{s.t.} \quad (\mu_0, \mu, \mu_T) \in \mathcal{P}_k$$



$$\mathcal{P}_k = \Big\{ (\mu_0, \mu, \mu_T) \;\Big|\; (\mu_0, \mu, \mu_T) \text{ satisfies } \mathcal{L} \text{ for all } \phi \in \mathbb{R}[t,x]_{2k}, \; \int \mathrm{d}\mu_0 = 1 \text{ and }$$
$$\mu_0 \in \mathcal{M}_k(\mathbf{X}_0)_+, \;\; \mu \in \mathcal{M}_k([0,T]\times\mathbb{R}^n)_+, \;\; \mu_T \in \mathcal{M}_k(\mathbb{R}^n)_+ \Big\}$$

Convergence

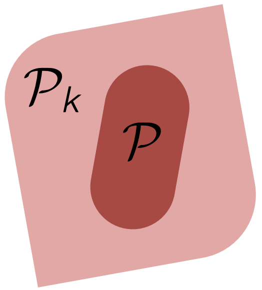
(under a compactness assumption)

$$p_k \nearrow p$$

$$p = \inf_{x_0 \in X_0} \int_0^T h(t, x(t\,|\,x_0))\, \mathrm{d}t$$

Finite-Dimensional **SDP**

$$p_k \quad = \quad \inf_{\mu_0,\mu,\mu_T} \quad \int_{[0,T]\times\mathbb{R}^n} h(x,t)\,\mathrm{d}\mu(t,x)$$
$$\text{s.t.} \quad (\mu_0,\mu,\mu_T) \in \mathcal{P}_k$$



$$\mathcal{P}_k = \Big\{(\mu_0,\mu,\mu_T) \;\Big|\; (\mu_0,\mu,\mu_T) \text{ satisfies } \mathcal{L} \text{ for all } \phi \in \mathbb{R}[t,x]_{2k},\; \int \mathrm{d}\mu_0 = 1 \text{ and}$$
$$\mu_0 \in \mathcal{M}_k(\mathbf{X}_0)_+,\quad \mu \in \mathcal{M}_k([0,T]\times\mathbf{X})_+,\quad \mu_T \in \mathcal{M}_k(\mathbf{X}_T)_+\Big\}$$

Constraints are **easy**

Convergence

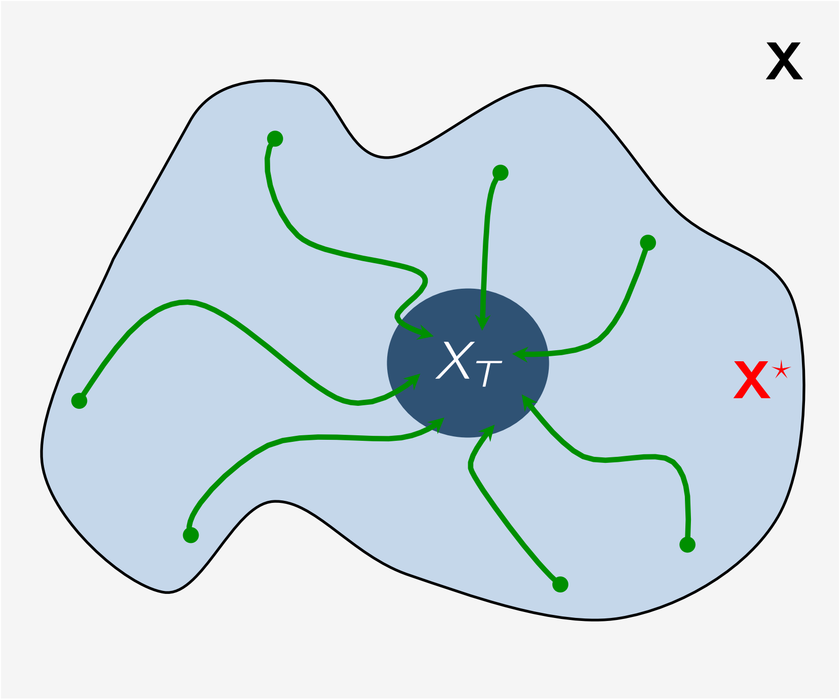
(under a compactness assumption)

$$p_k \nearrow p$$

$$p = \inf_{x_0\in X_0} \int_0^T h(t, x(t\,|\,x_0))\,\mathrm{d}t$$
$$\text{s.t.} \quad x(t) \in \mathbf{X} \;\forall\, t \in [0,T]$$
$$x(T) \in \mathbf{X}_T$$

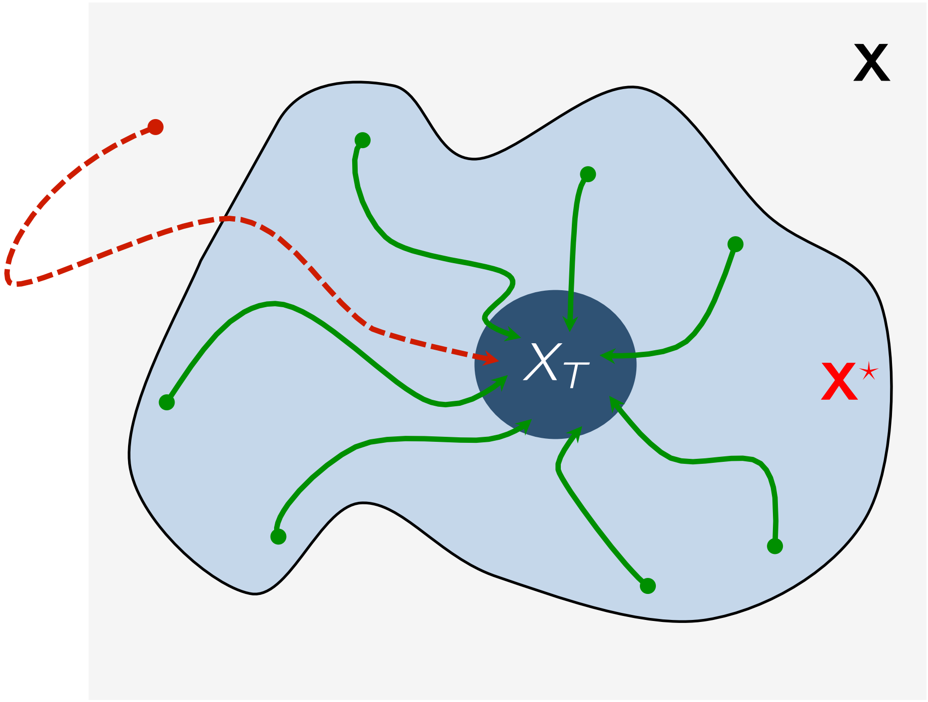
Region of attraction

Region of attraction



$$\mathbf{X}^\star = \Big\{ x_0 \;\Big|\; x(t\,|\,x_0) \in \mathbf{X} \,\forall t \in [0,T],\; x(T\,|\,x_0) \in \mathbf{X}_T,\; \dot{x}(t\,|\,x_0) = f(x(t\,|\,x_0)) \Big\}$$

Region of attraction



$$\mathbf{X}^\star = \left\{ x_0 \;\middle|\; x(t\,|\,x_0) \in \mathbf{X}\, \forall t \in [0,T],\; x(T\,|\,x_0) \in \mathbf{X}_T,\; \dot{x}(t\,|\,x_0) = f(x(t\,|\,x_0)) \right\}$$

Infinite-dimensional **LP** characterization of ROA

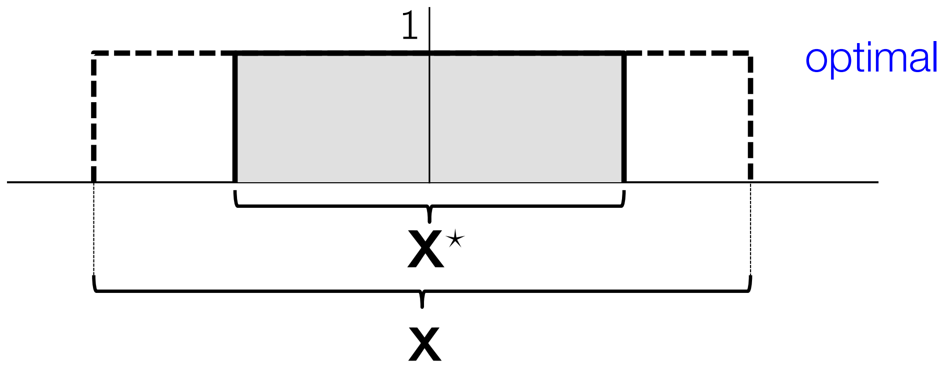
$$p \quad = \quad \sup_{\mu_0,\mu,\mu_T} \quad \int_{[0,T]\times\mathbb{R}^n} 1 \, \mathrm{d}\mu_0(x)$$

$$\text{s.t.} \quad (\mu_0, \mu, \mu_T) \in \mathcal{P}$$

$$\mathcal{P} = \Big\{(\mu_0, \mu, \mu_T) \quad \Big| \quad (\mu_0, \mu, \mu_T) \text{ satisfies } \mathcal{L}, \quad \mu_0 \le \lambda_X \text{ and}$$
$$\mu_0 \in \mathcal{M}(\mathbf{X})_+, \quad \mu \in \mathcal{M}([0,T]\times\mathbf{X})_+, \quad \mu_T \in \mathcal{M}(\mathbf{X}_T)_+\Big\}$$

[Henrion, K., 2014]

Claim : Optimal μ_0 equals to the restriction of λ_X to the ROA



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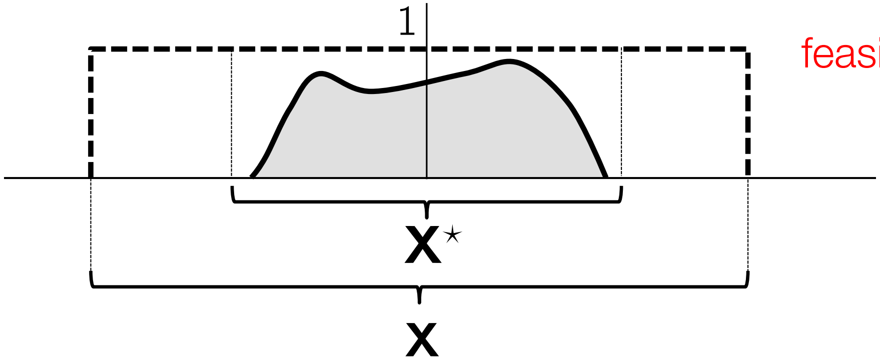
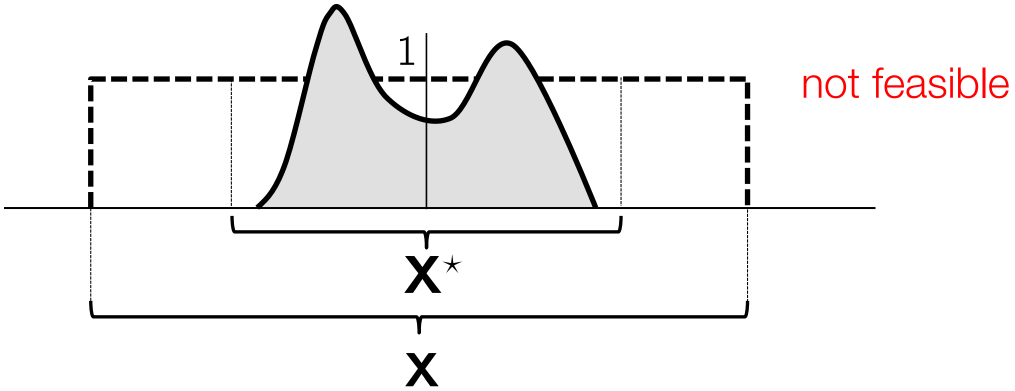


feasible but not optimal

Claim : Optimal μ_0 equals to the restriction of λ_X to the ROA



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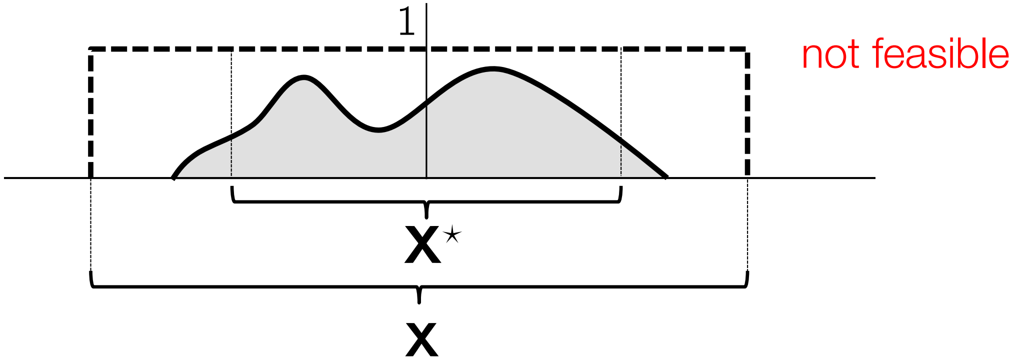

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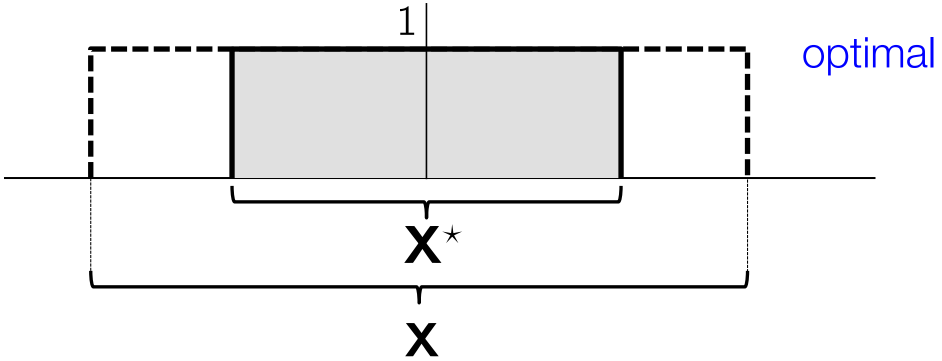


Dual LP

$$
\begin{array}{lll}
\inf\limits_{v \in \mathcal{C}^1, w \in \mathcal{C}} & \displaystyle\int_X w(x)\, \mathrm{d}x & \\
\text{s.t.} & \dfrac{\partial v}{\partial t} + \dfrac{\partial v}{\partial x} \cdot f \le 0, & \forall\, (t,x) \in [0,T] \times \mathbf{X} \\
& v(T,x) \ge 0, & \forall x \in \mathbf{X}_T \\
& w(x) \ge v(0,x) + 1, & \forall x \in \mathbf{X} \\
& w(x) \ge 0, & \forall x \in \mathbf{X}
\end{array}
$$

Dual LP

$$\inf_{v\in\mathcal{C}^1, w\in\mathcal{C}} \int_X w(x)\,\mathrm{d}x$$

$$\text{s.t.} \quad \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x}\cdot f \leq 0, \qquad \forall\,(t,x)\in[0,T]\times\mathbf{X}$$

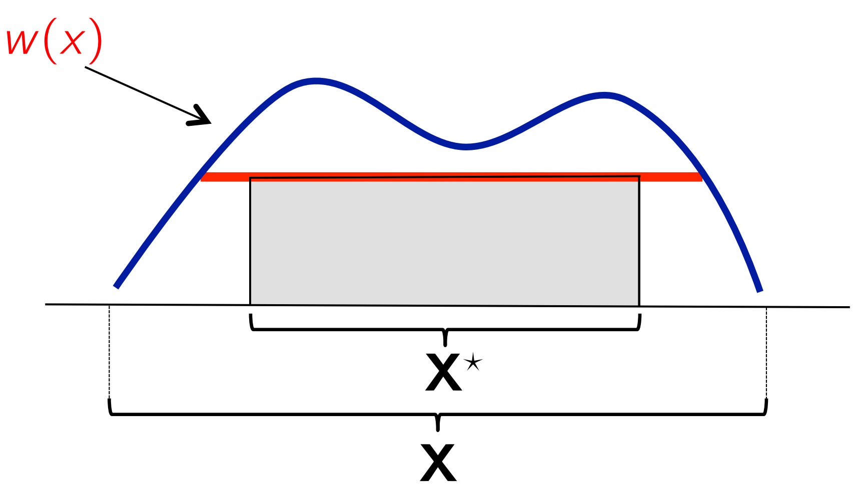
$$v(T,x) \geq 0, \qquad \forall\, x\in\mathbf{X}_T$$

$$w(x) \geq v(0,x)+1, \qquad \forall\, x\in\mathbf{X}$$

$$w(x) \geq 0, \qquad \forall\, x\in\mathbf{X}$$

Key obseravtion

$$w \geq \mathbb{I}_{X^\star} \quad \Rightarrow \quad \{x \mid w(x)\geq 1\} \supset \mathbf{X}^\star$$



Dual LP

$$
\begin{aligned}
\inf_{v\in\mathcal{C}^1, w\in\mathcal{C}} \quad & \int_X w(x)\,\mathrm{d}x \\
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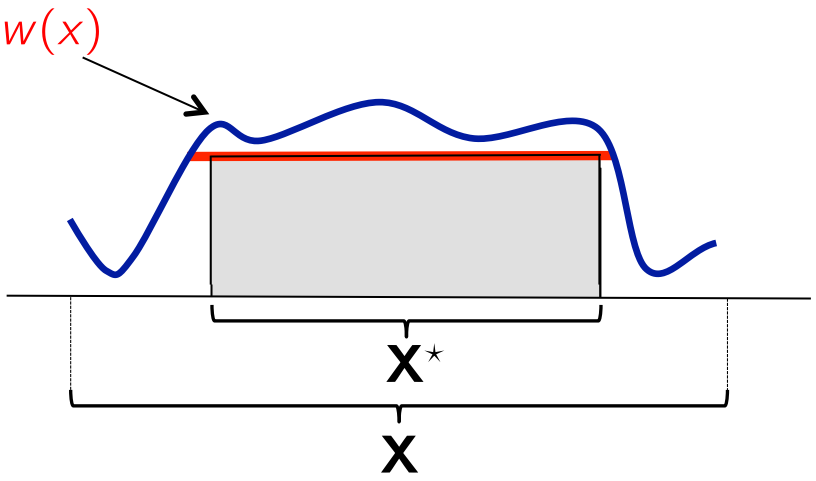

Dual LP

$$
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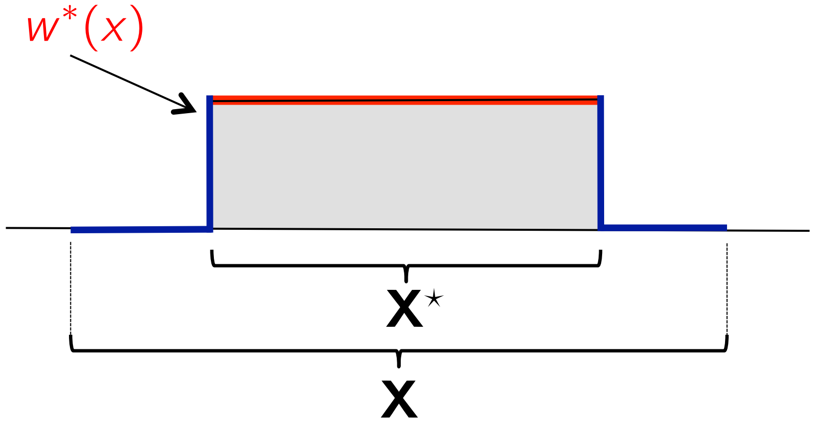

Dual LP

$$
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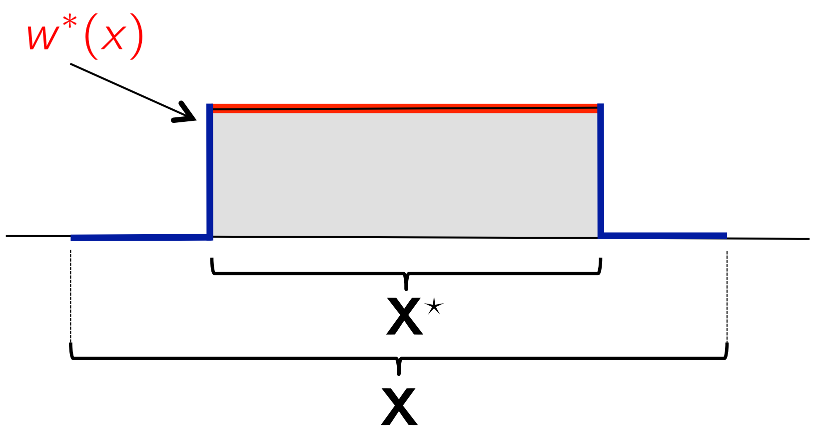
& w(x)\ge 0, & \forall x\in\mathbf{X}
\end{array}
$$

Theorem (SDP approximations):

$$w_k \searrow \mathbb{I}_{X^\star} \ \text{ in } \ L_1$$

$$\mathrm{vol}(\mathbf{X}_k \setminus \mathbf{X}^\star) \to 0$$

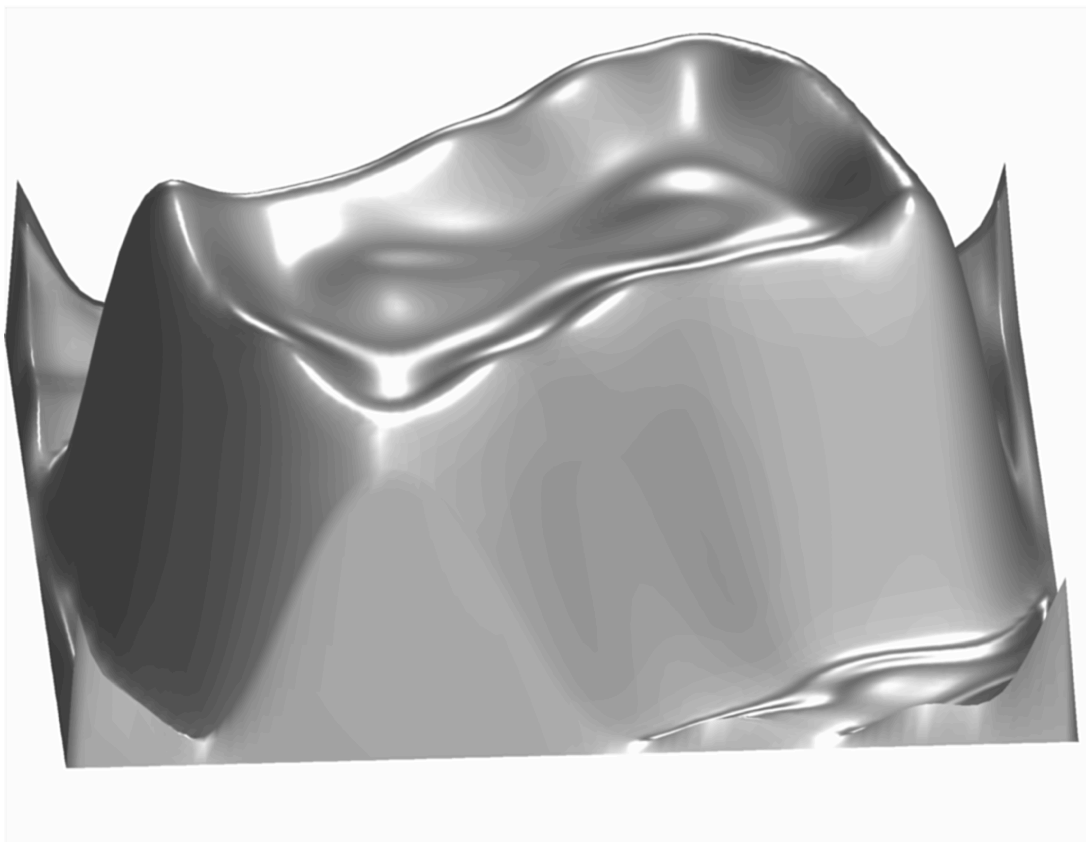


[Henrion, K., 2014]

Van der Pol (reverse time)

$\dot{x}_1 = -2x_2$
$\dot{x}_2 = 0.8x_1 + 10(x_1^2 - 0.21)x_2$
$X = [-1.2, -1.2]^2$
$X_T = \{x \mid \|x\|_2 \leq 0.01\},$

w_k



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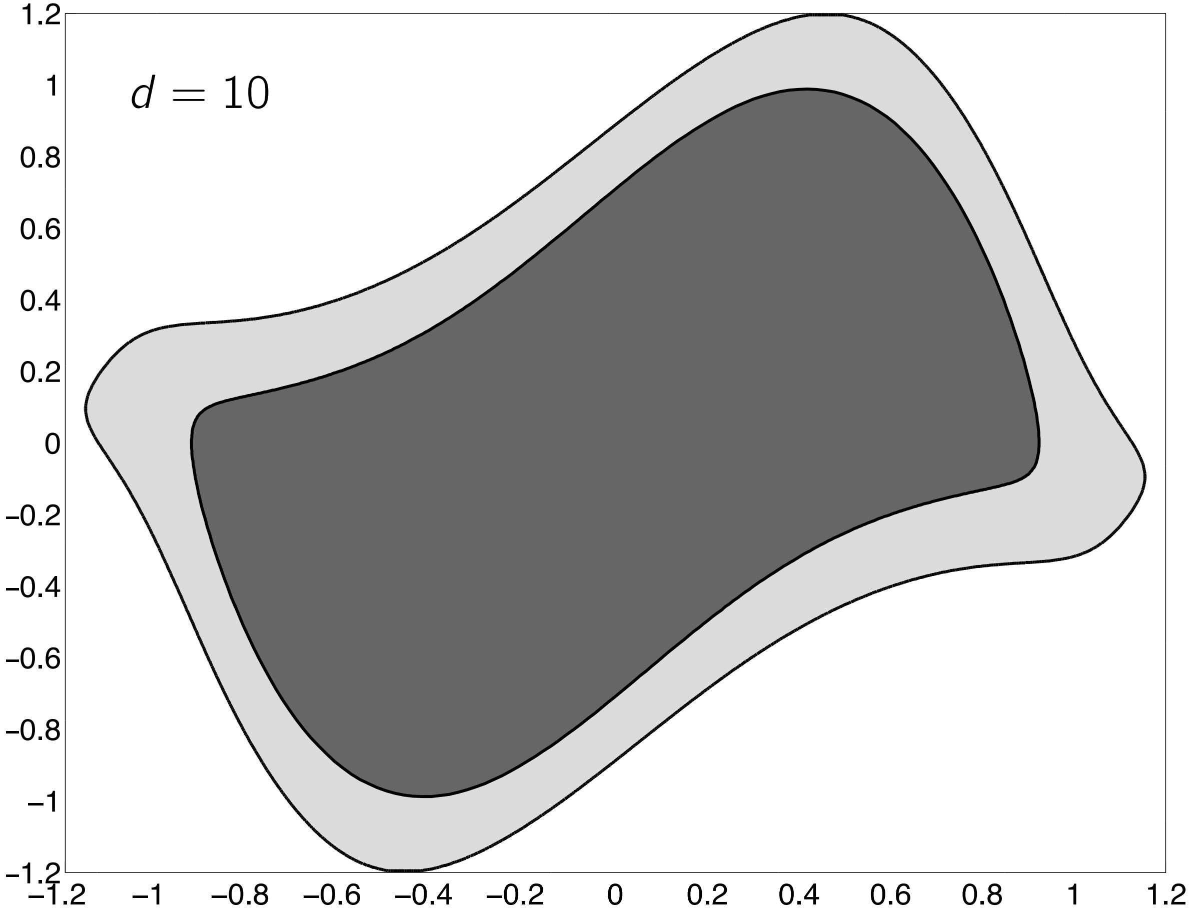

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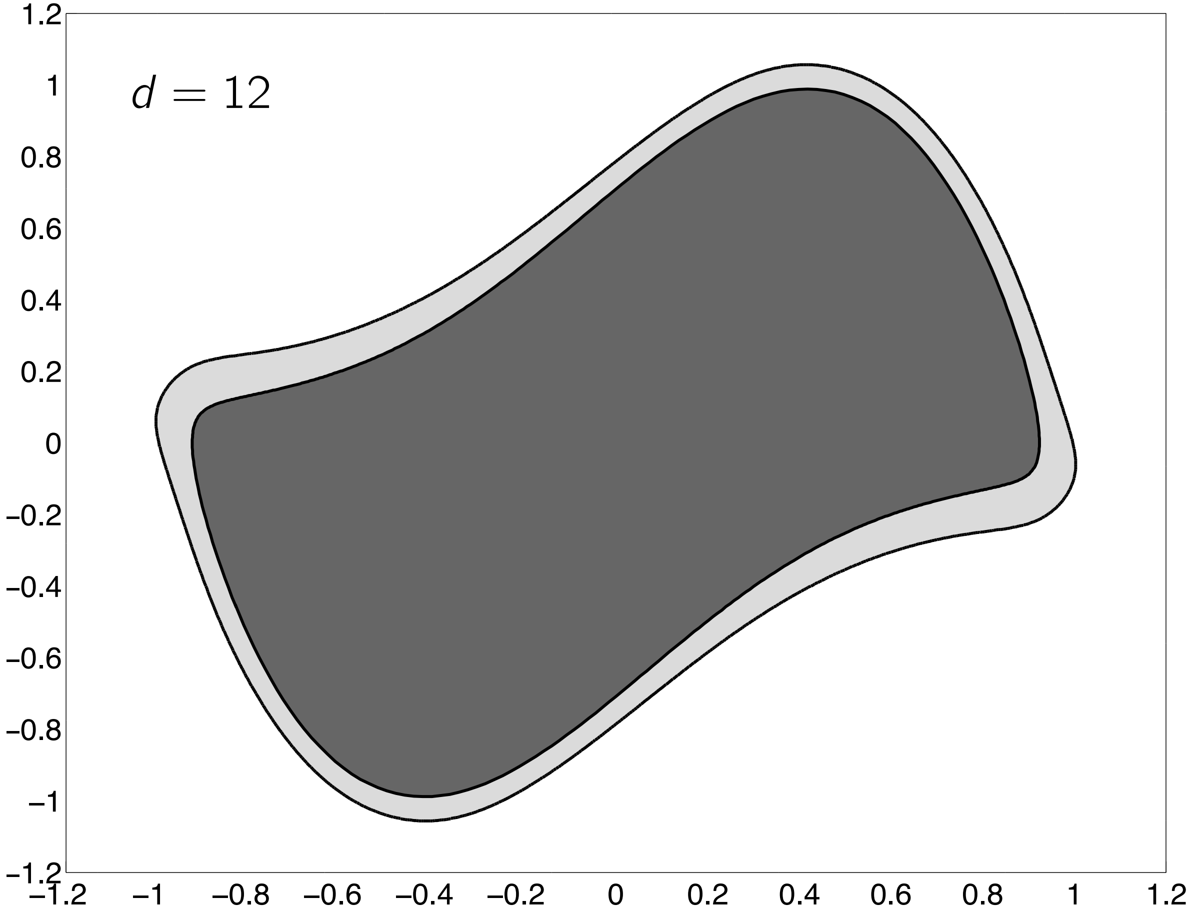

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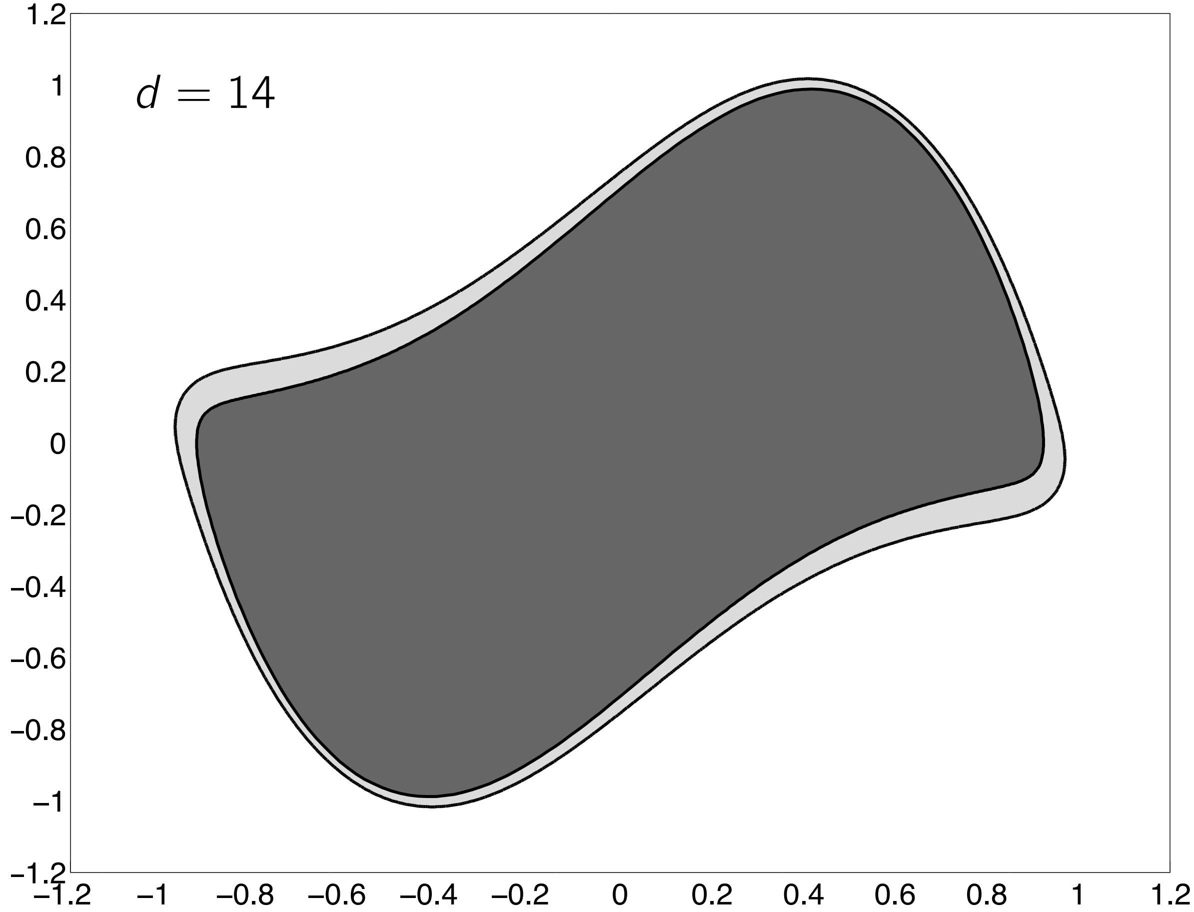

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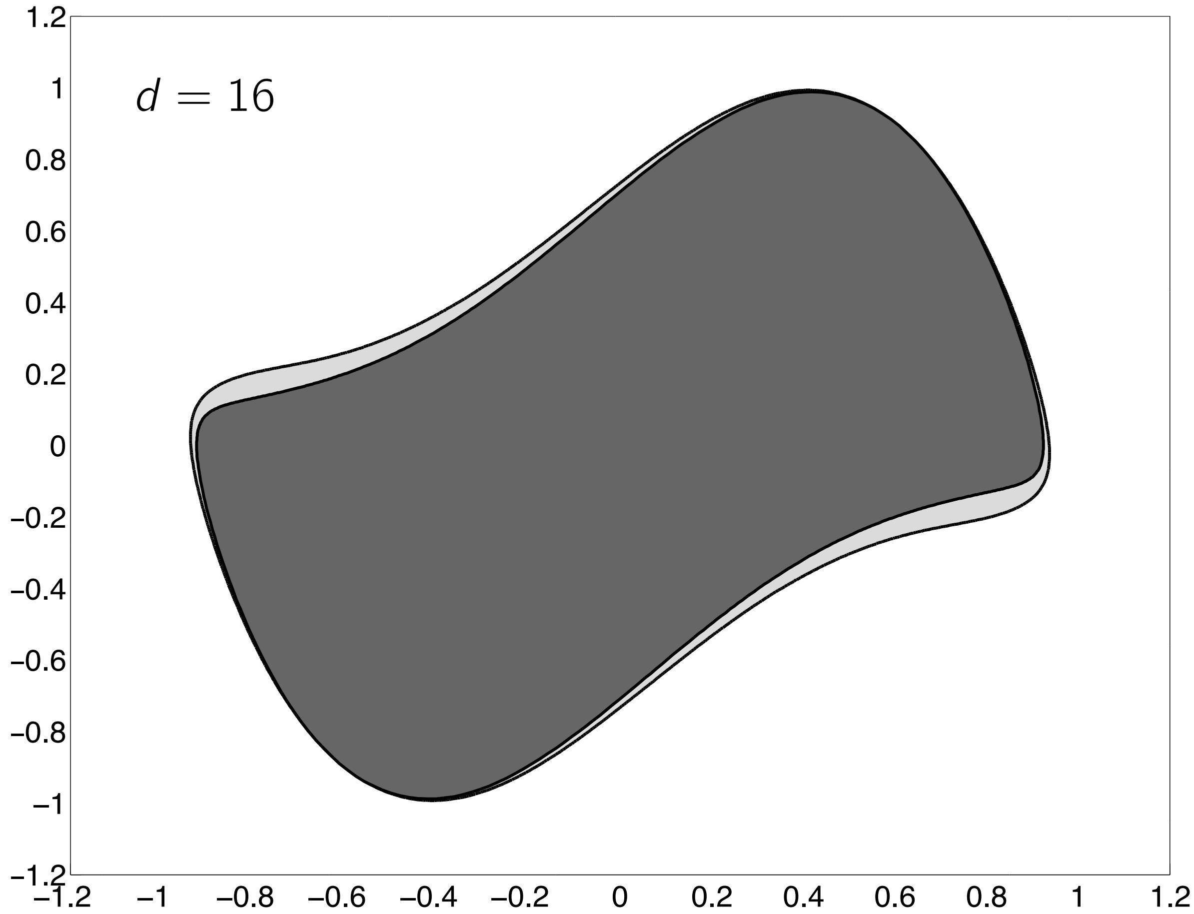

Adding control

Adding control is easy

$$\dot{x} = f(x, u)$$

Occupation measure: $\mu(A) = \int_0^T \mathbb{I}_A(t, x(t), u(t))\,\mathrm{d}t \qquad \forall\, A \in \mathcal{B}([0, T] \times \mathbb{R}^n \times \mathbb{R}^m)$

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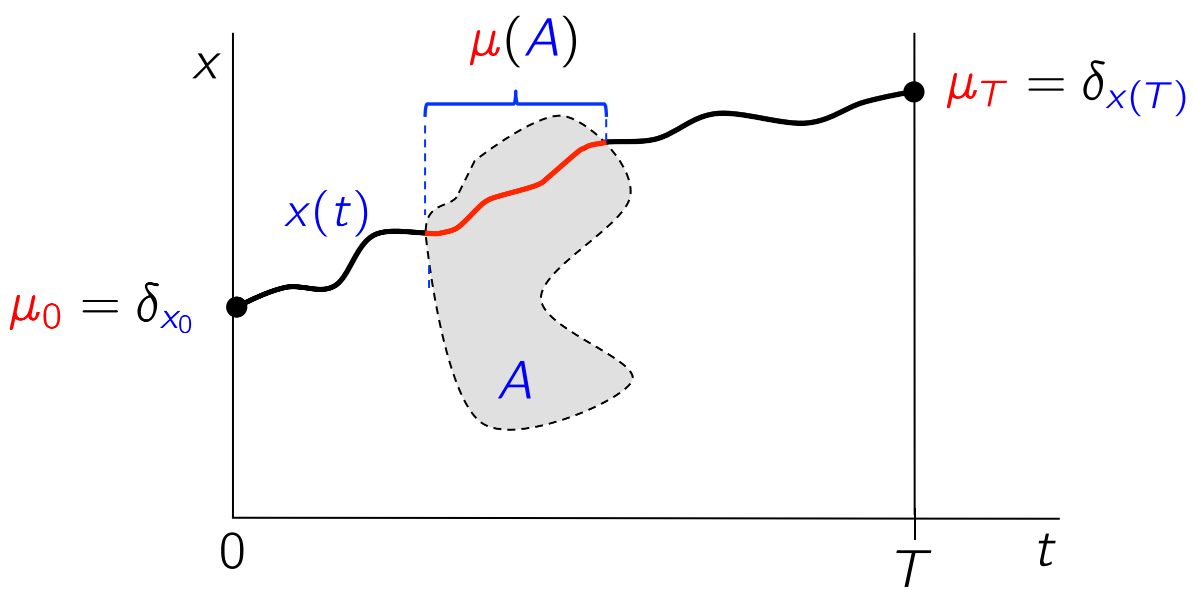

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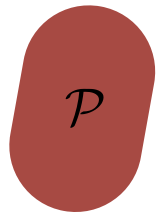
Occupation measure: $\mu(A) = \int_0^T \mathbb{I}_A(t, x(t), u(t))\,\mathrm{d}t \qquad \forall A \in \mathcal{B}([0,T] \times \mathbb{R}^n \times \mathbb{R}^m)$

$$\int_{\mathbb{R}^n} \phi(T,\cdot)\,\mathrm{d}\mu_T \; - \; \int_{\mathbb{R}^n} \phi(0,\cdot)\,\mathrm{d}\mu_0 \; = \; \int_{[0,T]\times\mathbb{R}^n\times\mathbb{R}^m} \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x} \cdot f \, d\mu \qquad (\mathcal{L})$$

for all $\phi \in C^1([0,T] \times \mathbb{R}^n)$

$$\mathcal{P} = \Big\{(\mu_0, \mu, \mu_T) \;\; \Big| \;\; (\mu_0, \mu, \mu_T) \text{ satisfies } \mathcal{L} \text{ and }$$
$$\mu_0 = \nu, \;\; \mu \in \mathcal{M}([0,T] \times \mathbf{X} \times \mathbf{U})_+, \;\; \mu_T \in \mathcal{M}(\mathbb{R}^n)_+ \Big\}$$



$\mathcal{P}$ **not** a **singleton**

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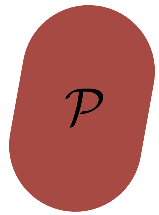

$\mathcal{P}$ **not** a **singleton**

What does $\mathcal{P}$ look like?

$\mathcal{P}$ contains **superpositions** of occupation measures associated to

$$\dot{x} \in \mathrm{conv}(f(x, \mathbf{U}))$$

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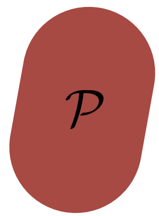

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What does $\mathcal{P}$ look like?

$\mathcal{P}$ contains **superpositions** of occupation measures associated to

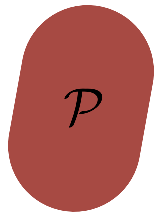
$$\dot{x} \in \mathrm{conv}(f(x, \mathbf{U}))$$

$$\bar{\mu}(A) = \int_{\mathcal{C}([0,T];X)} \int_0^T \mathbb{I}_A(t, x(t))\, \mathrm{d}t\, \mathrm{d}\sigma(x(\cdot))$$

with σ supported on trajectories of $\dot{x} \in \mathrm{conv}(f(x, \mathbf{U}))$

[Rubio, 1976]
[Vinter, Lewis, 1978]
[Vinter, 1993]
[Henrion, K., 2014]

$$\mathcal{P} = \{(\mu_0, \mu, \mu_T) \;\mid\; (\mu_0, \mu, \mu_T) \text{ satisfies } \mathcal{L} \text{ and}$$
$$\mu_0 = \nu,\;\; \mu \in \mathcal{M}([0,T] \times \mathbf{X} \times \mathbf{U})_+,\;\; \mu_T \in \mathcal{M}(\mathbb{R}^n)_+\}$$



$\mathcal{P}$ **not** a **singleton**

What does $\mathcal{P}$ look like?

$\mathcal{P}$ contains **superpositions** of occupation measures associated to

Note: Trajectories of $\dot{x} = f(x, u),\, u \in \mathbf{U}$

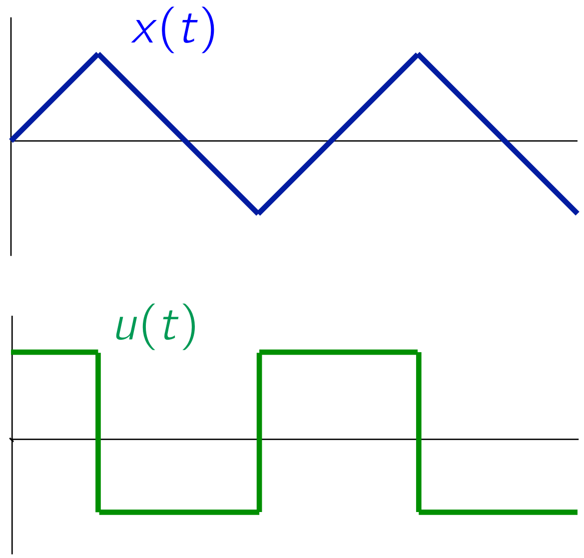
dense in the set of trajectories of

$\dot{x} \in \mathrm{conv}(f(x, \mathbf{U}))$

Example: $\dot{x} = u, \quad \mathbf{U} = \{-1, +1\}$

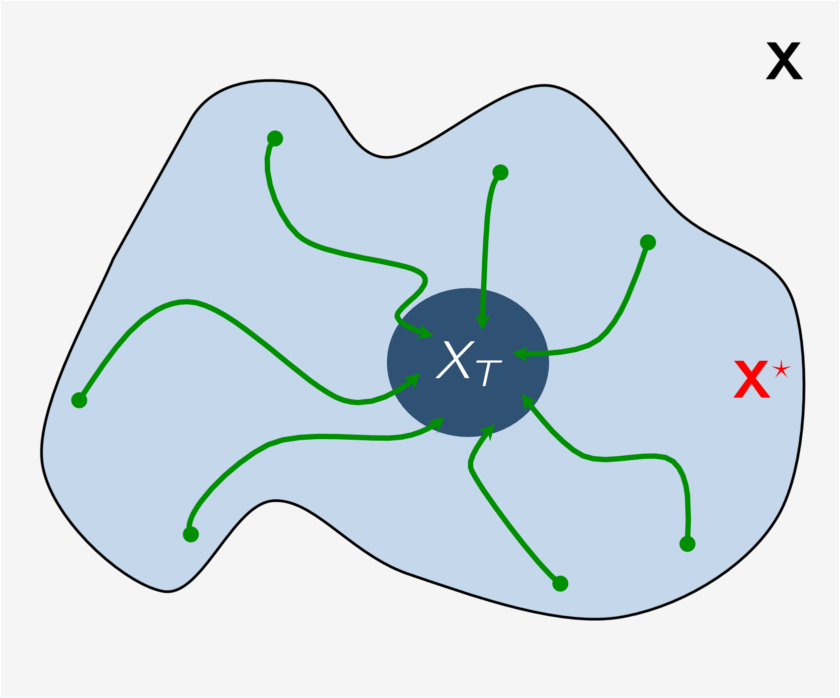
$$\mathrm{conv}(f(x, \mathbf{U})) = [-1, 1]$$

$\Rightarrow$ trajectory $x(t) = 0$ satisfies $\dot{x} \in \mathrm{conv}(f(x, \mathbf{U}))$ but not $\dot{x} = u$, $u \in \{-1, -1\}$



Region of attraction with control

Region of attraction



$\mathbf{X}^\star$ = the set of all states that can be steered to $\mathbf{X}_T$ using admissible control inputs

Infinite-Dimensional **LP** for ROA

$$\sup_{\mu_0,\mu,\mu_T} \quad \int_{[0,T]\times\mathbb{R}^n} 1\,\mathrm{d}\mu_0(x)$$
$$\text{s.t.} \quad (\mu_0,\mu,\mu_T)\in\mathcal{P}$$

$$\mathcal{P} = \Big\{(\mu_0,\mu,\mu_T) \;\;|\;\; (\mu_0,\mu,\mu_T) \text{ satisfies } \mathcal{L}\,, \quad \mu_0 \le \lambda_X \text{ and}$$
$$\mu_0 \in \mathcal{M}(\mathbf{X})_+,\;\; \mu \in \mathcal{M}([0,T]\times\mathbf{X}\times\mathbf{U})_+,\;\; \mu_T \in \mathcal{M}(\mathbf{X}_T)_+\Big\}$$

Theorem (SDP approximations):

$$p_k \searrow \mathrm{vol}(\mathbf{X}^\star)$$

$$w_k \searrow \mathbb{I}_{X^\star} \;\text{ in }\; L_1$$

$$\mathrm{vol}(\mathbf{X}_k \setminus \mathbf{X}^\star) \to 0$$

[Henrion, K., 2014]

Brockett integrator (ROA known semi-analytically)

$$\dot{x}_1 = u_1$$
$$\dot{x}_2 = u_2$$
$$\dot{x}_3 = u_1 x_2 - u_2 x_1$$
$$\mathbf{X} = \{x \mid ||x||_\infty \leq 1\}$$
$$\mathbf{U} = \{u \mid ||u||_2 \leq 1\}$$
$$\mathbf{X}_T = \{0\}, \ \ T = 1$$

Brockett integrator (ROA known semi-analytically)

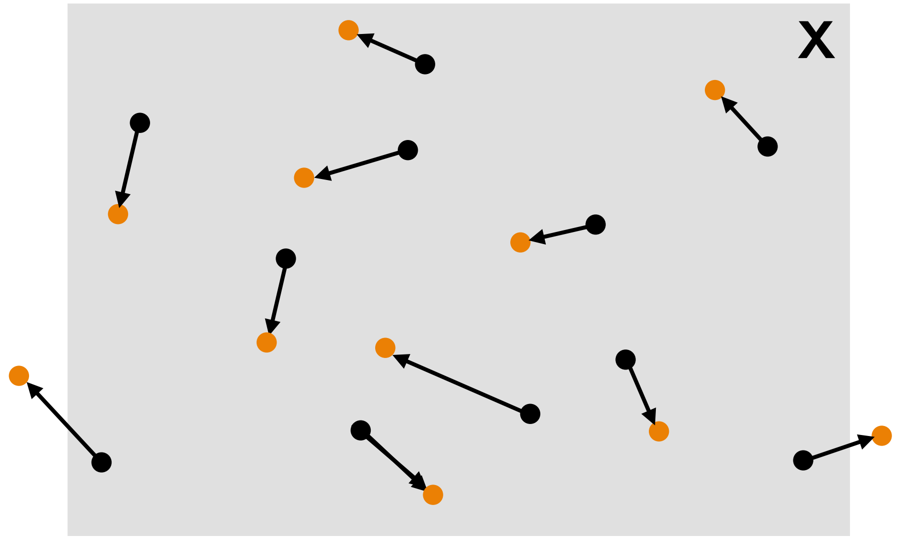
$$\dot{x}_1 = u_1$$
$$\dot{x}_2 = u_2$$
$$\dot{x}_3 = u_1 x_2 - u_2 x_1$$
$$\mathbf{X} = \{x \mid ||x||_\infty \le 1\}$$
$$\mathbf{U} = \{u \mid ||u||_2 \le 1\}$$
$$\mathbf{X}_T = \{0\}, \;\; T = 1$$

$k = 5$

Data

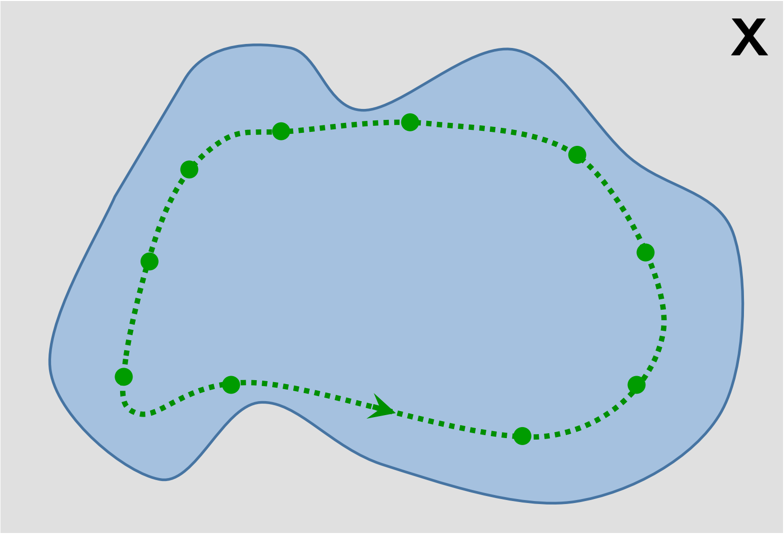
$$x^+ = f(x)$$

f **not known**, only **data** $\{x_i, x_i^+\}_{i=1}^K$ available



$$x^+ = f(x)$$

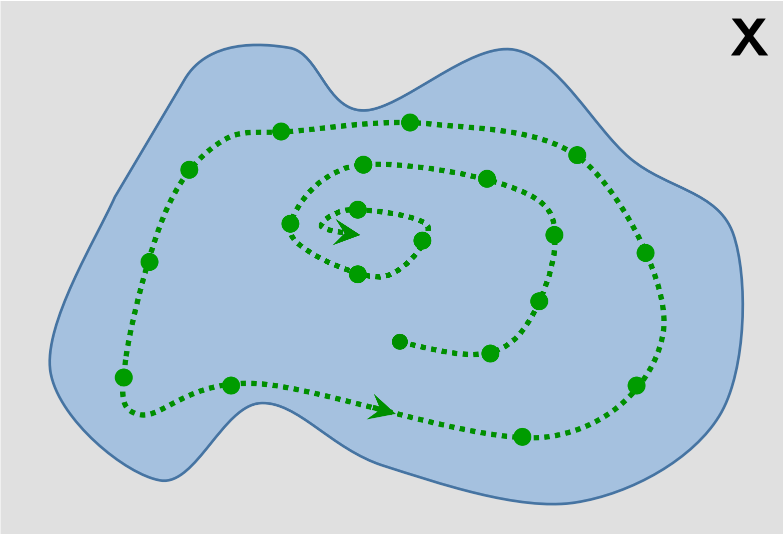
f **not known**, only **data** $\{x_i, x_i^+\}_{i=1}^K$ available



Maximum positively invariant set

$$x^{+} = f(x)$$

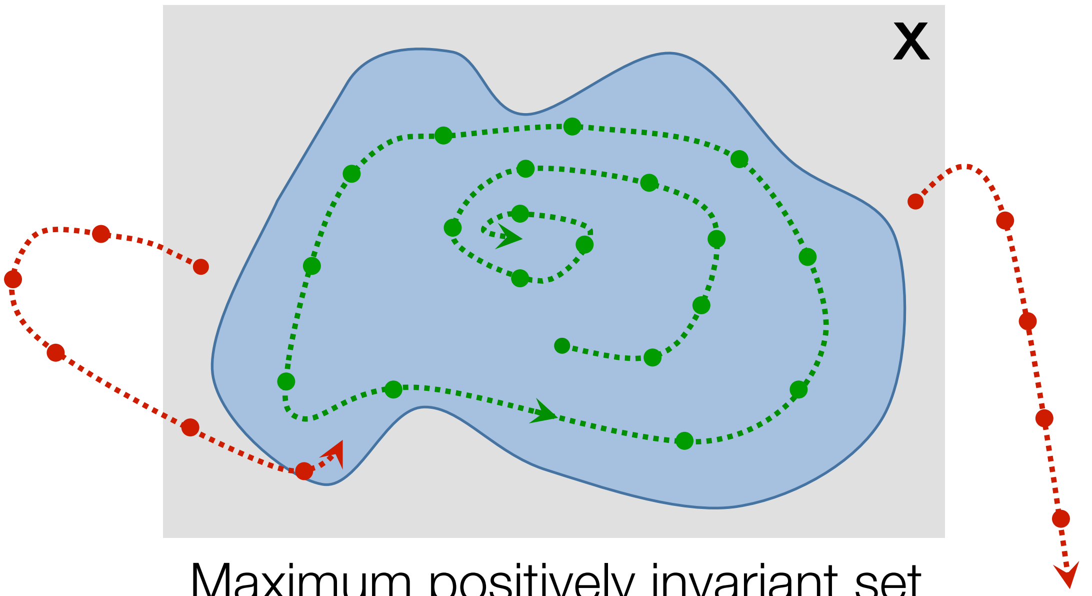
f **not known**, only **data** $\{x_i, x_i^{+}\}_{i=1}^{K}$ available



Maximum positively invariant set

$$x^+ = f(x)$$

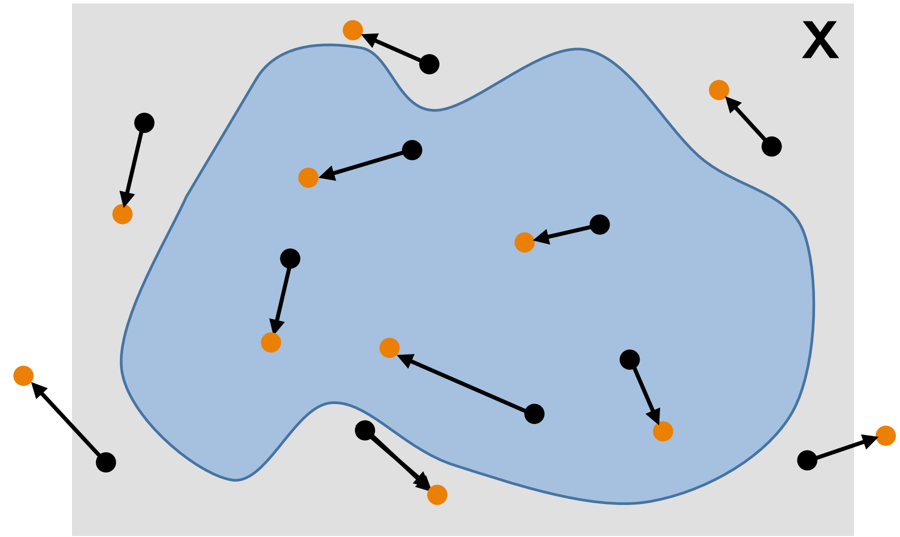
f **not known**, only **data** $\{x_i, x_i^+\}_{i=1}^K$ available



Maximum positively invariant set

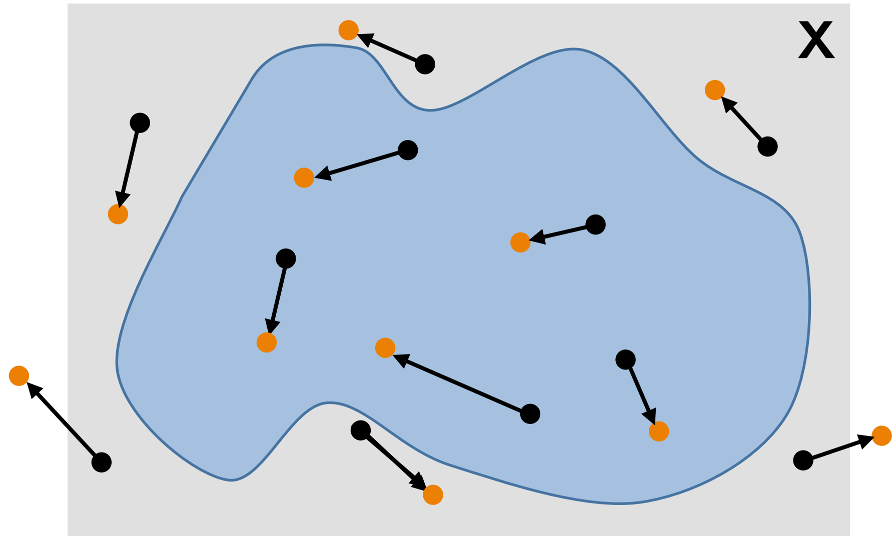
$$x^+ = f(x)$$

f **not known**, only **data** $\{x_i, x_i^+\}_{i=1}^K$ available



$$x^+ = f(x, u)$$

f **not known**, only **data** $\{x_i, x_i^+\}_{i=1}^K$ available



Maximum **controlled** invariant set

Linear programming formulation

$$\sup_{v \in \mathcal{C}(\mathsf{X})} \int_{\mathsf{X}} v(x)\, dx$$

$$\text{s.t.} \quad v \leq \mathrm{dist}_{\mathsf{X}} \circ f + \alpha v \circ \mathrm{proj}_{\mathsf{X}} \circ f \quad \text{on} \quad \mathbf{X}$$

Linear programming formulation

$$\sup_{v \in \mathcal{C}(X)} \int_X v(x)\, dx$$

$$\text{s.t.} \quad v \le \underbrace{\mathrm{dist}_X \circ f}_{\ell} + \alpha v \circ \underbrace{\mathrm{proj}_X \circ f}_{\bar{f}} \quad \text{on} \quad \mathbf{X}$$

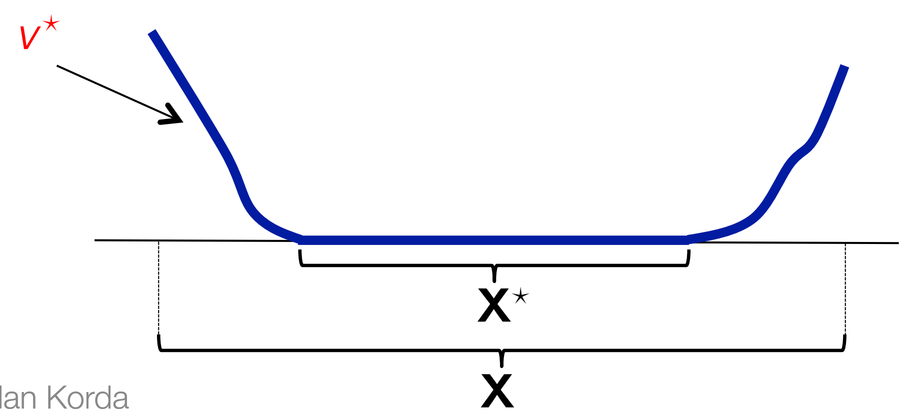
Linear programming formulation

$$\sup_{v \in \mathcal{C}(\mathbf{X})} \int_{\mathbf{X}} v(x)\,dx$$

$$\text{s.t.} \quad v \le \underbrace{\mathrm{dist}_{\mathbf{X}} \circ f}_{\ell} + \alpha v \circ \underbrace{\mathrm{proj}_{\mathbf{X}} \circ f}_{\bar{f}} \quad \text{on} \quad \mathbf{X}$$

Crucial fact

$$v^\star(x) = \sum_{k=0}^{\infty} \alpha^k \ell\big(\bar{f}^{(k)}(x)\big) \begin{cases} = 0 & \text{if} \quad x \in \mathbf{X}^\star \\ > 0 & \text{if} \quad x \notin \mathbf{X}^\star \end{cases}$$



Remark: $\mathbf{X}$ invariant under $\bar{f}$

Linear programming formulation

$$\sup_{v \in \mathcal{C}(\mathbf{X})} \int_{\mathbf{X}} v(x)\, dx$$

$$\text{s.t.} \quad v \leq \underbrace{\mathrm{dist}_{\mathbf{X}} \circ f}_{\ell} + \alpha v \circ \underbrace{\mathrm{proj}_{\mathbf{X}} \circ f}_{\bar{f}} \quad \text{on} \quad \mathbf{X}$$

Crucial fact

$$v^\star(x) = \sum_{k=0}^{\infty} \alpha^k \ell\left(\bar{f}^{(k)}(x)\right) \begin{cases} = 0 & \text{if} \quad x \in \mathbf{X}^\star \\ > 0 & \text{if} \quad x \notin \mathbf{X}^\star \end{cases}$$

v feasible $\quad \Rightarrow \quad \{x \mid v(x) \leq 0\} \supset \mathbf{X}^\star$

Linear programming formulation

$$\sup_{v \in \mathcal{C}(\mathbf{X})} \int_{\mathbf{X}} v(x)\, dx$$

$$\text{s.t.} \quad v \le \underbrace{\mathrm{dist}_{\mathbf{X}} \circ f}_{\ell} + \alpha v \circ \underbrace{\mathrm{proj}_{\mathbf{X}} \circ f}_{\bar{f}} \quad \text{on} \quad \mathbf{X}$$

Crucial fact

$$v^\star(x) = \sum_{k=0}^{\infty} \alpha^k \ell\big(\bar{f}^{(k)}(x)\big) \begin{cases} = 0 & \text{if} \quad x \in \mathbf{X}^\star \\ > 0 & \text{if} \quad x \notin \mathbf{X}^\star \end{cases}$$

$$\alpha < \frac{1}{\mathrm{Lip}\, f} \quad \Rightarrow \quad v^\star \text{ Lipschitz} \quad \text{with} \quad \mathrm{Lip}\, v^\star \le \frac{1}{1 - \alpha \cdot \mathrm{Lip}\, f}$$

Sampled LP

$$\begin{aligned} \sup_{v \in \mathcal{C}(\mathrm{X})} \quad & \int v(x)\, dx \\ \text{s.t.} \quad & \left. \begin{aligned} & v(x_i) \le \mathrm{dist}_{\mathrm{X}}(x_i^+) + \alpha v(\mathrm{proj}_{\mathrm{X}}(x_i^+)) \\ & -1 \le v(x_i) \le (1-\alpha)^{-1} \end{aligned} \right\} \ \forall (x_i, x_i^+) \in \mathrm{Data} \end{aligned}$$

with the variable $v \in \mathcal{V} \subset \mathcal{C}(\mathbf{X})$, $\dim(\mathcal{V}) < \infty$

Sampled LP

$$
\begin{aligned}
\sup_{v \in \mathcal{C}(\mathrm{X})} \quad & \int v(x)\, dx \\
\text{s.t.} \quad & \left.\begin{aligned} & v(x_i) \le \mathrm{dist}_{\mathrm{X}}(x_i^+) + \alpha v(\mathrm{proj}_{\mathrm{X}}(x_i^+)) \\ & -1 \le v(x_i) \le (1-\alpha)^{-1} \end{aligned}\right\} \quad \forall (x_i, x_i^+) \in \mathrm{Data}
\end{aligned}
$$

with the variable $v \in \mathcal{V} \subset \mathcal{C}(\mathbf{X})$, $\dim(\mathcal{V}) < \infty$

Properties

+ **No assumptions** on f (can be non-polynomial, discontinuous* etc.)
+ **No assumptions** on the subspace $\mathcal{V}$ (can be radial basis functions, wavelets etc.)
+ Boils down to **finite-dimensional linear program**
− No longer guaranteed outer approximation
+ Can analyze **convergence rate** and sample **complexity**

Convergence rate

$$\sup_{v} \int_X v(x)\, dx$$

$$\text{s.t.} \quad v \leq \text{dist}_X \circ f + \alpha v \circ \text{proj}_X \circ f \quad \text{on} \quad \mathbf{X}$$

with the variable $v \in \mathcal{F} \subset \mathcal{C}(\mathbf{X})$, $\dim(\mathcal{F}) < \infty$

$$\text{vol}(\mathbf{X}_{\mathcal{F}} \setminus \mathbf{X}^{\star}) \leq ??$$

Convergence rate

$$\sup_{v} \int_{X} v(x)\, dx$$

$$\text{s.t.} \quad v \le \text{dist}_X \circ f + \alpha v \circ \text{proj}_X \circ f \quad \text{on} \quad \mathbf{X}$$

with the variable $v \in \mathcal{F} \subset \mathcal{C}(\mathbf{X})$, $\dim(\mathcal{F}) < \infty$

$\mathcal{F}$ = multivariate polynomials up to degree d

$$\text{vol}(\mathbf{X}_{\mathcal{F}} \setminus \mathbf{X}^{\star}) \le \frac{c}{(1-\alpha)(1-\alpha\text{Lip}(f))} \frac{1}{\sqrt{d}} + g_{v^{\star}}\Big(\frac{1}{\sqrt{d}}\Big)$$

$$g_{v^{\star}}(\gamma) = \text{vol}(\{x \mid 0 < v^{\star}(x) \le \gamma\})$$

Convergence rate

$$\sup_{v} \int_{X} v(x)\, dx$$

$$\text{s.t.} \quad v \leq \text{dist}_X \circ f + \alpha v \circ \text{proj}_X \circ f \quad \text{on} \quad \mathbf{X}$$

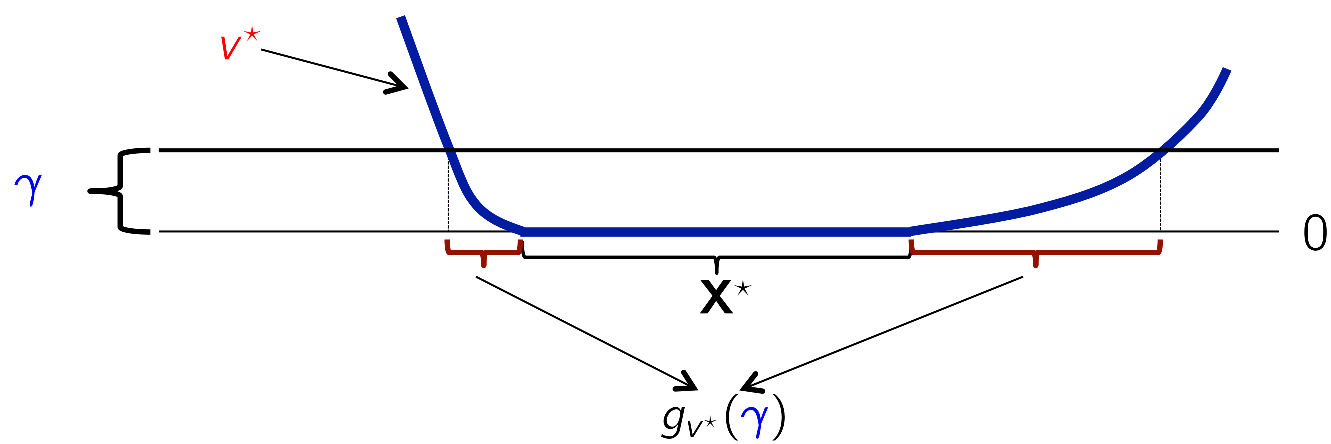
with the variable $v \in \mathcal{F} \subset \mathcal{C}(\mathbf{X})$, $\dim(\mathcal{F}) < \infty$

$\mathcal{F}$ = multivariate polynomials up to degree d

$$\text{vol}(\mathbf{X}_{\mathcal{F}} \setminus \mathbf{X}^\star) \leq \frac{c}{(1-\alpha)(1-\alpha \text{Lip}(f))} \frac{1}{\sqrt{d}} + g_{v^\star}\Big(\frac{1}{\sqrt{d}}\Big)$$

[K., 2019]

$g_{v^\star}(\gamma) = \text{vol}(\{x \mid 0 < v^\star(x) \leq \gamma\})$



Sample complexity

$$\sup_{v \in \mathcal{C}(X)} \int v(x)\,dx$$

$$\text{s.t.} \quad \left. \begin{array}{l} v(x_i) \leq \mathrm{dist}_X(x_i^+) + \alpha v(\mathrm{proj}_X(x_i^+)) \\ -1 \leq v(x_i) \leq (1-\alpha)^{-1} \end{array} \right\} \quad \begin{array}{c} \forall (x_i, x_i^+) \in \mathrm{Data} \\ |\mathrm{Data}| = K \end{array}$$

with the variable $v \in \mathcal{F} \subset \mathcal{C}(\mathbf{X})$, $\dim(\mathcal{F}) < \infty$

$$\left| \int_X v_{\mathcal{F},K} - \int_X v_{\mathcal{F}} \right| < \epsilon$$

with probability at least $1 - \delta$ if

$$K \geq \frac{\log(\frac{1}{\delta}) + n \log(\frac{L_{X,\mathcal{F}}}{\epsilon(1-\alpha)})}{\log\left(\frac{1}{1-\left[\frac{\epsilon(1-\alpha)}{L_{X,\mathcal{F}}}\right]^n}\right)}$$

[K., 2019]

Sample complexity

$$
\begin{aligned}
\sup_{v \in \mathcal{C}(\mathbf{X})} \quad & \int v(x)\, dx \\
\text{s.t.} \quad & \left.\begin{aligned} & v(x_i) \le \mathrm{dist}_{\mathbf{X}}(x_i^+) + \alpha v(\mathrm{proj}_{\mathbf{X}}(x_i^+)) \\ & -1 \le v(x_i) \le (1-\alpha)^{-1} \end{aligned}\right\} \quad \begin{aligned} & \forall (x_i, x_i^+) \in \mathrm{Data} \\ & |\mathrm{Data}| = K \end{aligned}
\end{aligned}
$$

with the variable $v \in \mathcal{F} \subset \mathcal{C}(\mathbf{X})$, $\dim(\mathcal{F}) < \infty$

$$
\left| \int_X v_{\mathcal{F},K} - \int_X v_{\mathcal{F}} \right| < \epsilon
$$

with probability at least $1 - \delta$ if

$$
K \ge \frac{\log(\frac{1}{\delta}) + n \log(\frac{L_{\mathbf{X},\mathcal{F}}}{\epsilon(1-\alpha)})}{\log\left(\frac{1}{1-\left[\frac{\epsilon(1-\alpha)}{L_{\mathbf{X},\mathcal{F}}}\right]^n}\right)} \approx \frac{\log(\frac{1}{\delta}) + n \log(\frac{L_{\mathbf{X},\mathcal{F}}}{\epsilon(1-\alpha)})}{\left[\frac{\epsilon(1-\alpha)}{L_{\mathbf{X},\mathcal{F}}}\right]^n}
$$

[K., 2019]

Sample complexity

$$\sup_{v \in \mathcal{C}(\mathrm{X})} \int v(x)\, dx$$

$$\text{s.t.} \quad \left.\begin{aligned} & v(x_i) \le \mathrm{dist}_{\mathrm{X}}(x_i^+) + \alpha v(\mathrm{proj}_{\mathrm{X}}(x_i^+)) \\ & -1 \le v(x_i) \le (1-\alpha)^{-1} \end{aligned}\right\} \quad \begin{aligned} & \forall (x_i, x_i^+) \in \mathrm{Data} \\ & |\mathrm{Data}| = K \end{aligned}$$

with the variable $v \in \mathcal{F} \subset \mathcal{C}(\mathbf{X})$, $\dim(\mathcal{F}) < \infty$

$$\mathrm{vol}(\mathbf{X}_{\mathcal{F},K} \setminus \mathbf{X}^\star) \le ??$$

Sample complexity

$$\begin{array}{ll} \sup\limits_{v\in\mathcal{C}(\mathrm{X})} & \int v(x)\,dx \\ \text{s.t.} & \left.\begin{array}{l} v(x_i) \le \mathrm{dist}_{\mathrm{X}}(x_i^+) + \alpha v(\mathrm{proj}_{\mathrm{X}}(x_i^+)) \\ -1 \le v(x_i) \le (1-\alpha)^{-1} \end{array}\right\} \begin{array}{c} \forall (x_i, x_i^+) \in \mathrm{Data} \\ |\mathrm{Data}| = K \end{array} \end{array}$$

with the variable $v \in \mathcal{F} \subset \mathcal{C}(\mathbf{X})$, $\dim(\mathcal{F}) < \infty$

$\mathcal{F}$ = multivariate polynomials up to degree d

$$\mathrm{vol}(\mathbf{X}_{\mathcal{F},K} \setminus \mathbf{X}^\star) \le \frac{C}{K^{1/(2n+1)}} + g_{v^\star}\left(\frac{1}{K^{1/(2n+1)}}\right)$$

with probability at least $1-\delta$ if

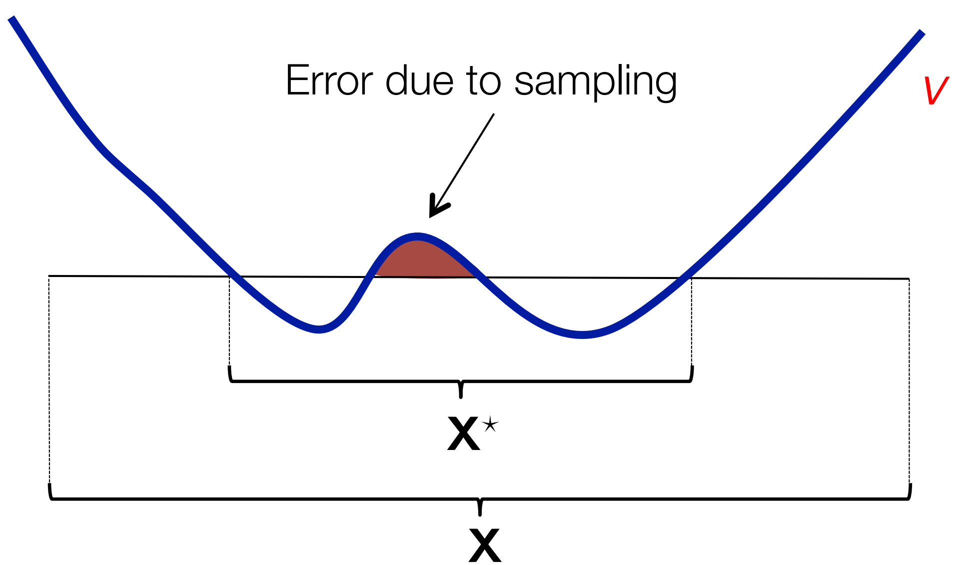
[K., 2019]

Guaranteed outer approximation

$$\sup_{v \in \mathcal{C}(\mathbf{X})} \int v(x)\, dx$$

$$\text{s.t.} \quad \left.\begin{aligned} & v(x_i) \le \mathrm{dist}_{\mathbf{X}}(x_i^+) + \alpha v(\mathrm{proj}_{\mathbf{X}}(x_i^+)) \\ & -1 \le v(x_i) \le (1-\alpha)^{-1} \end{aligned}\right\} \quad \begin{aligned} & \forall (x_i, x_i^+) \in \mathrm{Data} \\ & |\mathrm{Data}| = K \end{aligned}$$

Problem: $\{x \mid v \le 0\}$ may not be an outer approximation of **X**

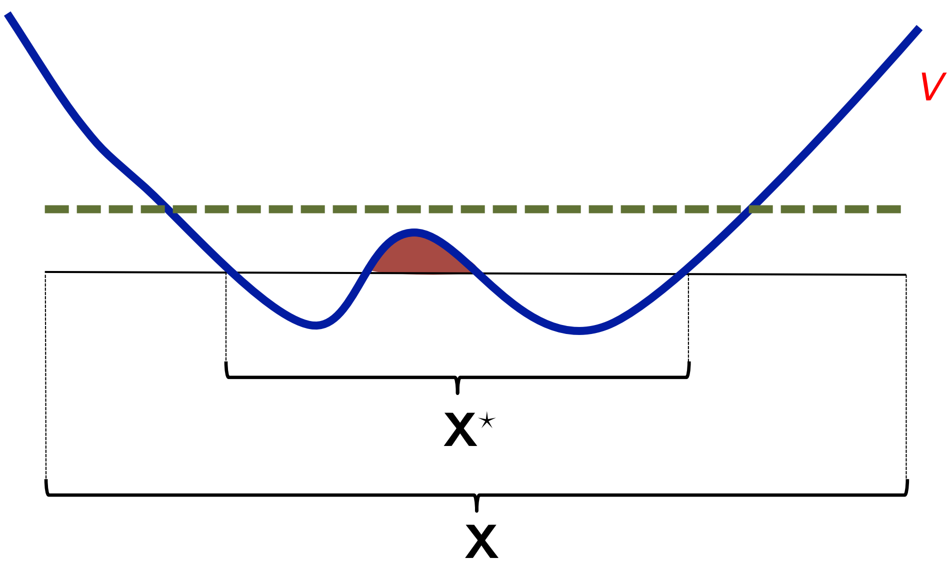


Guaranteed outer approximation

$$\sup_{v \in \mathcal{C}(X)} \int v(x)\,dx$$

$$\text{s.t.} \quad \left.\begin{aligned} & v(x_i) \le \mathrm{dist}_X(x_i^+) + \alpha v(\mathrm{proj}_X(x_i^+)) \\ & -1 \le v(x_i) \le (1-\alpha)^{-1} \end{aligned}\right\} \begin{aligned} & \forall (x_i, x_i^+) \in \mathrm{Data} \\ & |\mathrm{Data}| = K \end{aligned}$$

Solution: look at a different sublevel set



Guaranteed outer approximation

$$
\begin{aligned}
\sup_{v\in\mathcal{C}(\mathrm{X})} \quad & \int v(x)\,dx \\
\text{s.t.} \quad & \left.\begin{aligned} & v(x_i) \le \mathrm{dist}_{\mathrm{X}}(x_i^+) + \alpha v(\mathrm{proj}_{\mathrm{X}}(x_i^+)) \\ & -1 \le v(x_i) \le (1-\alpha)^{-1} \end{aligned}\right\} \quad \begin{aligned} & \forall (x_i, x_i^+) \in \mathrm{Data} \\ & |\mathrm{Data}| = K \end{aligned}
\end{aligned}
$$

Proposition:

$$E(x) := v(x) - \mathrm{dist}_{\mathrm{X}}(f(x)) - \alpha v(\mathrm{proj}_{\mathrm{X}}(f(x)))$$

$$\Rightarrow \qquad \mathbf{X}_{\mathrm{G}} := \left\{ x \mid v(x) \le \frac{1}{1-\alpha} \sup_{y\in\mathrm{X}} E(y) \right\} \supset \mathbf{X}^\star$$

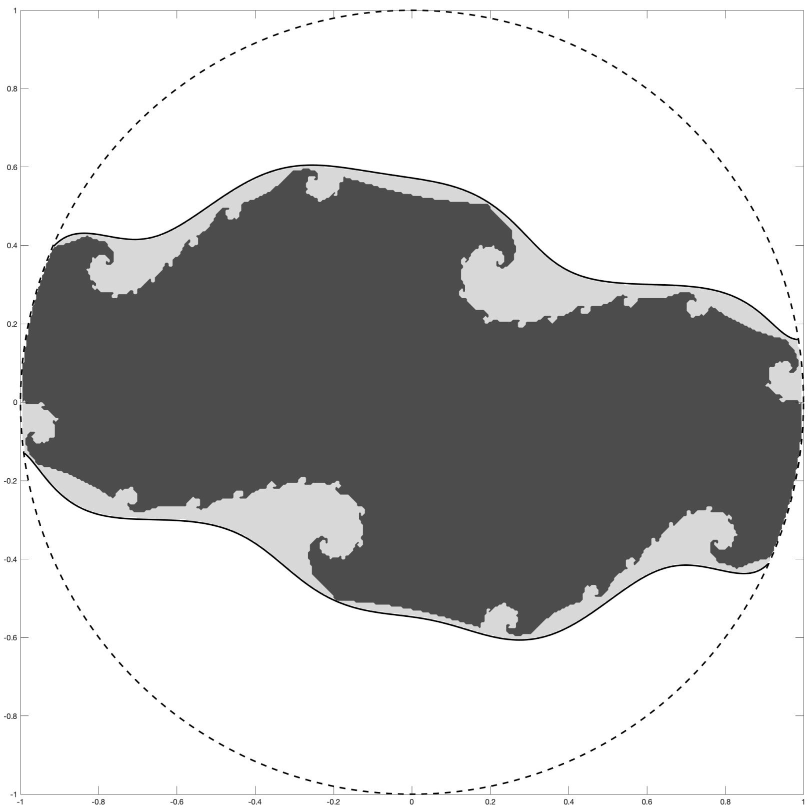
[K., 2019]

Numerical examples

Julia set – sampling vs SDP

Basis: polynomials up to degree 10

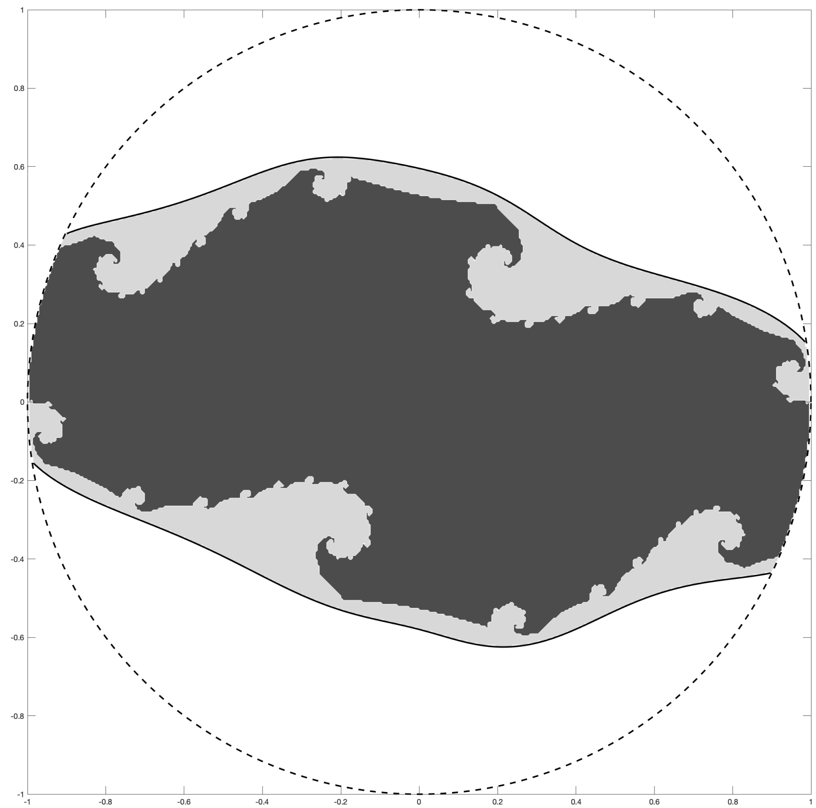
Sampling



Volume error 20.31 %

Misclassification 0 %

SDP



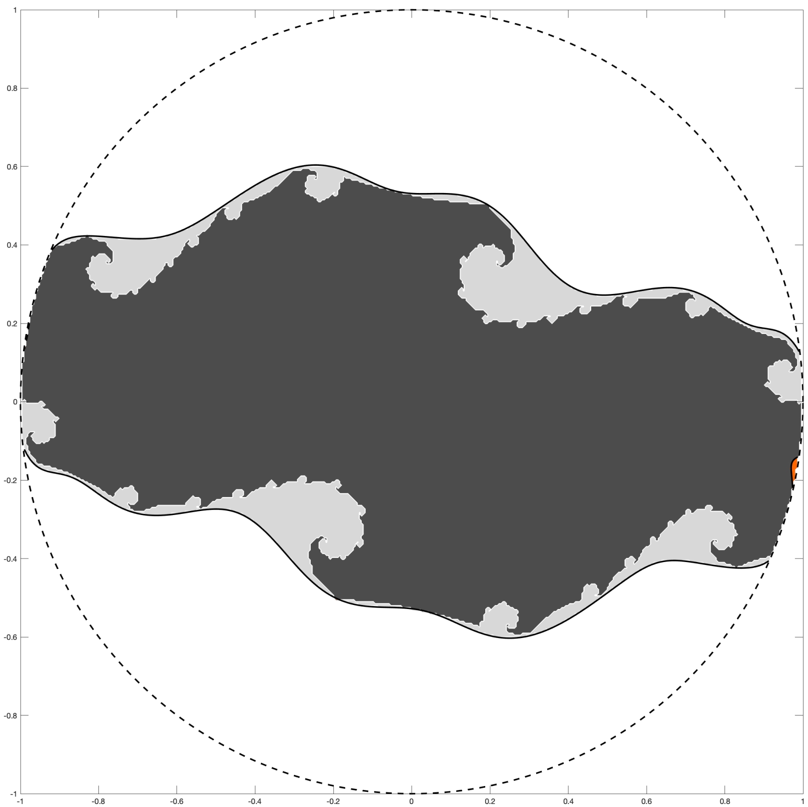
Volume error 28.7 %

Misclassification 0 %

Julia set – sampling vs SDP

Basis: polynomials up to degree 14

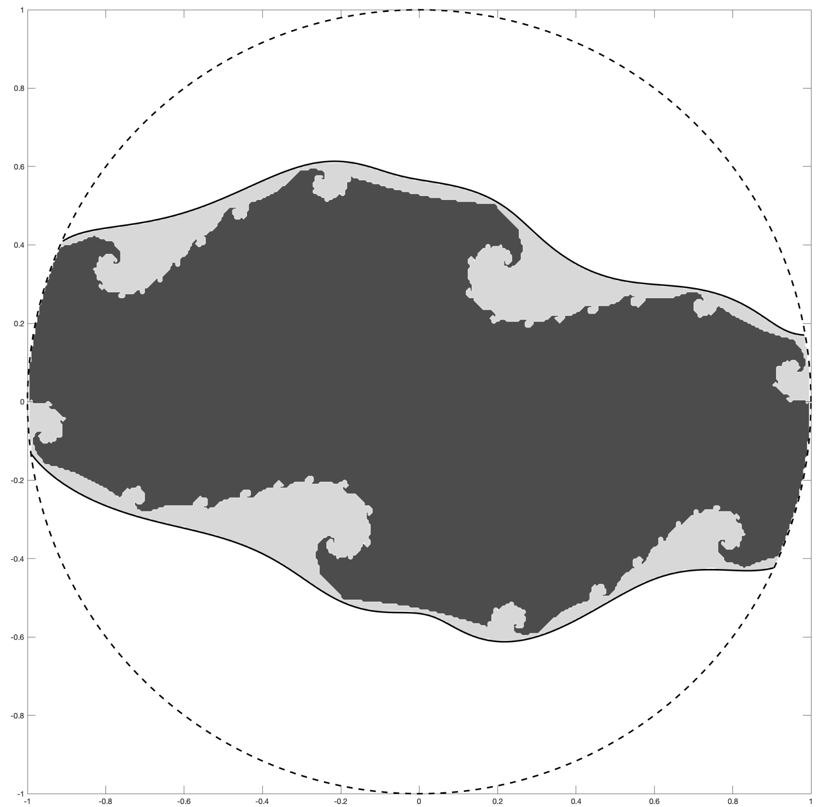
Sampling



Volume error 14.98 %

Misclassification 0.086 %

SDP



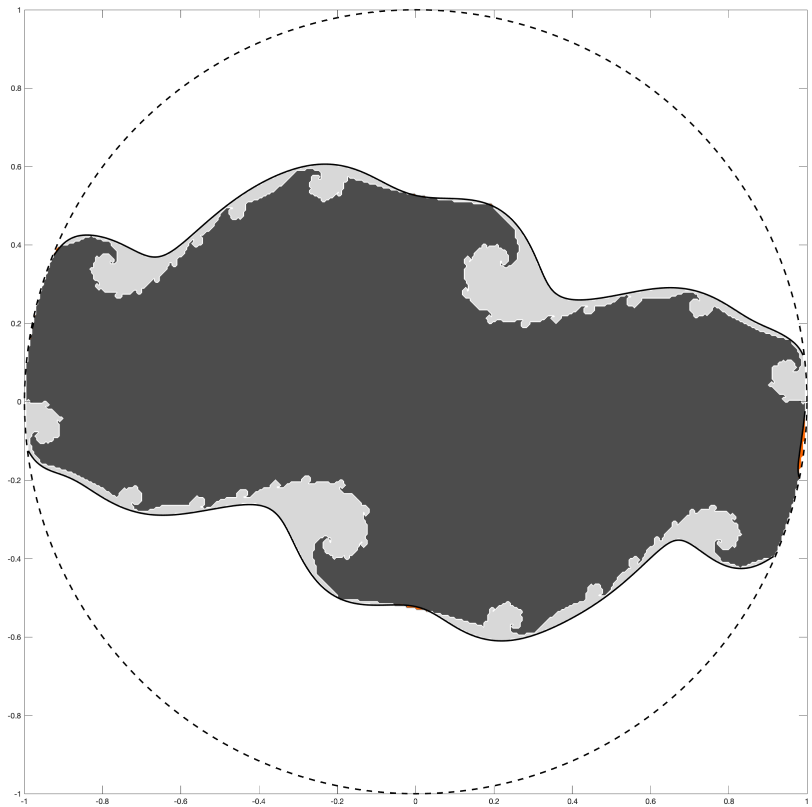
Volume error 21.9 %

Misclassification 0 %

Julia set – sampling vs SDP

Basis: polynomials up to degree 18

Sampling



Volume error 13.24 %

Misclassification 0.157 %

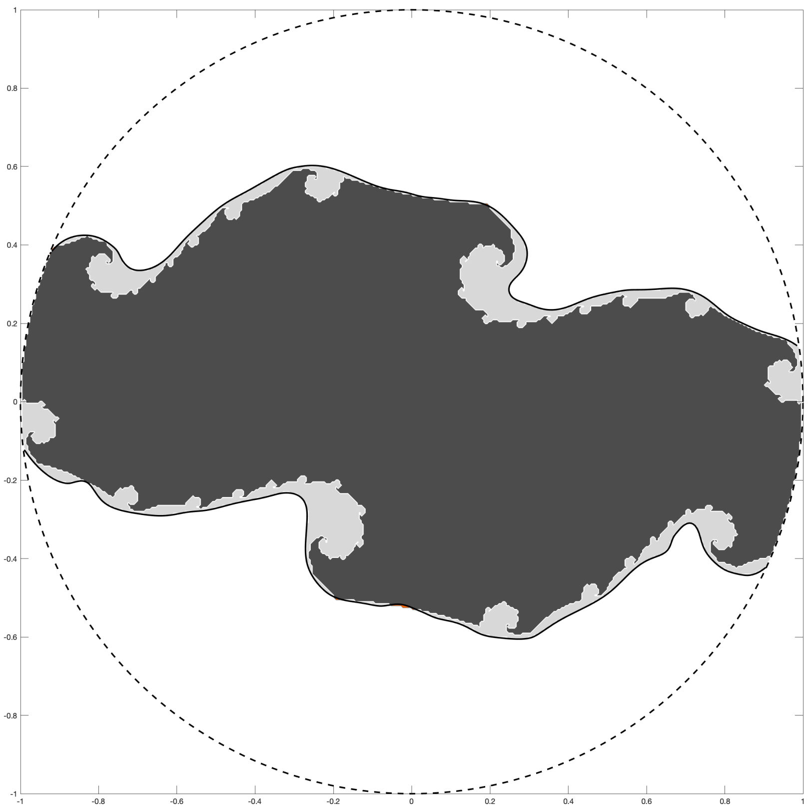
SDP

Numerical problems

Julia set – different bases

Basis: 400 thin-plate spline RBFs

Sampling



Volume error 10.78 %

Misclassification 0.041 %

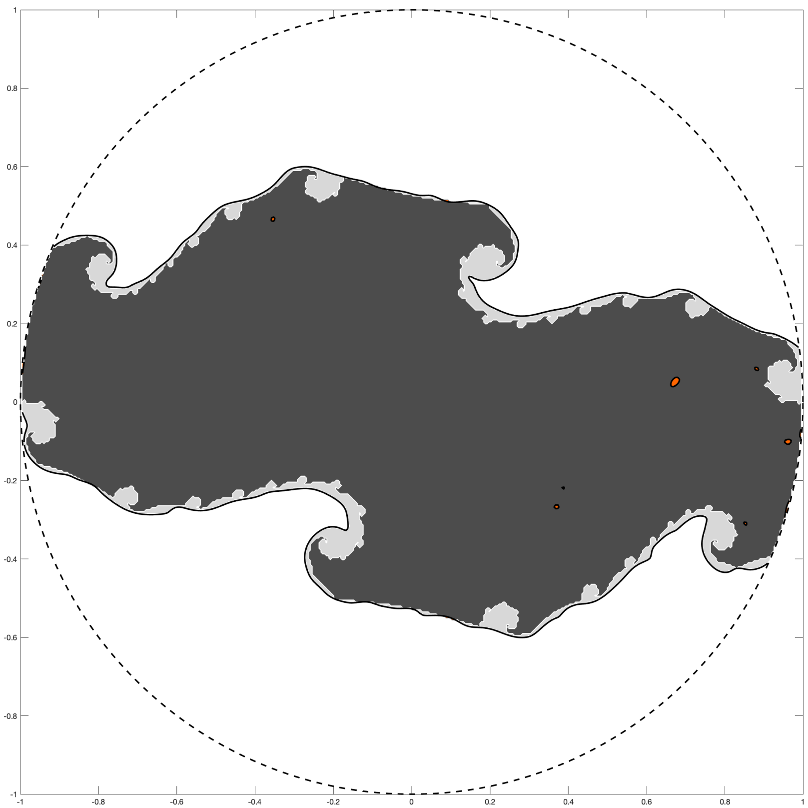
SDP

NA

Julia set – different bases

Basis: 1000 thin-plate spline RBFs

Sampling



Volume error 7.35 %

Misclassification 0.014 %

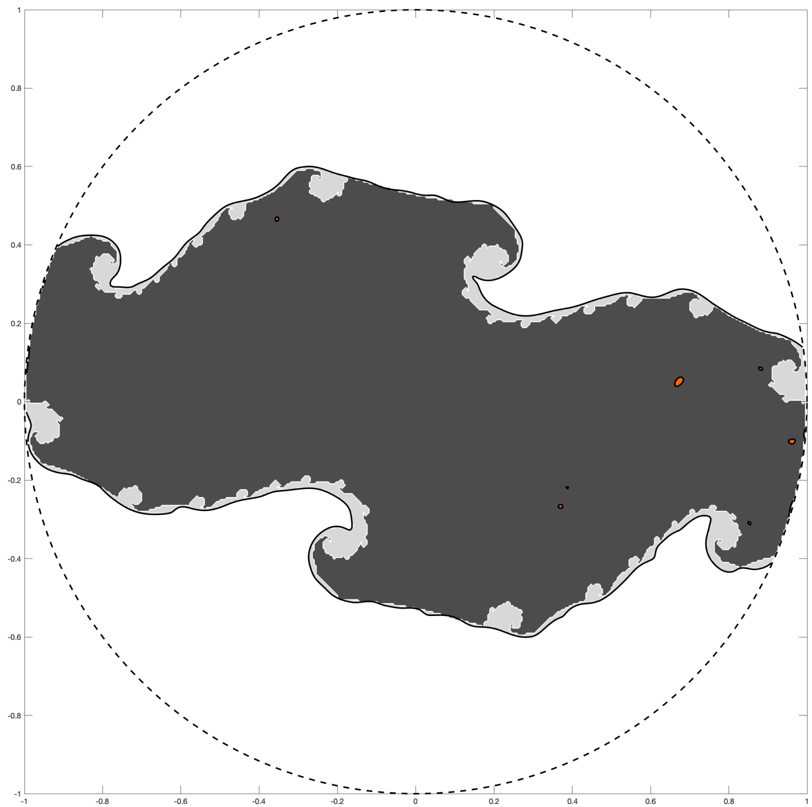
SDP

NA

Julia set – postprocessing

Basis: 1000 thin-plate spline RBFs

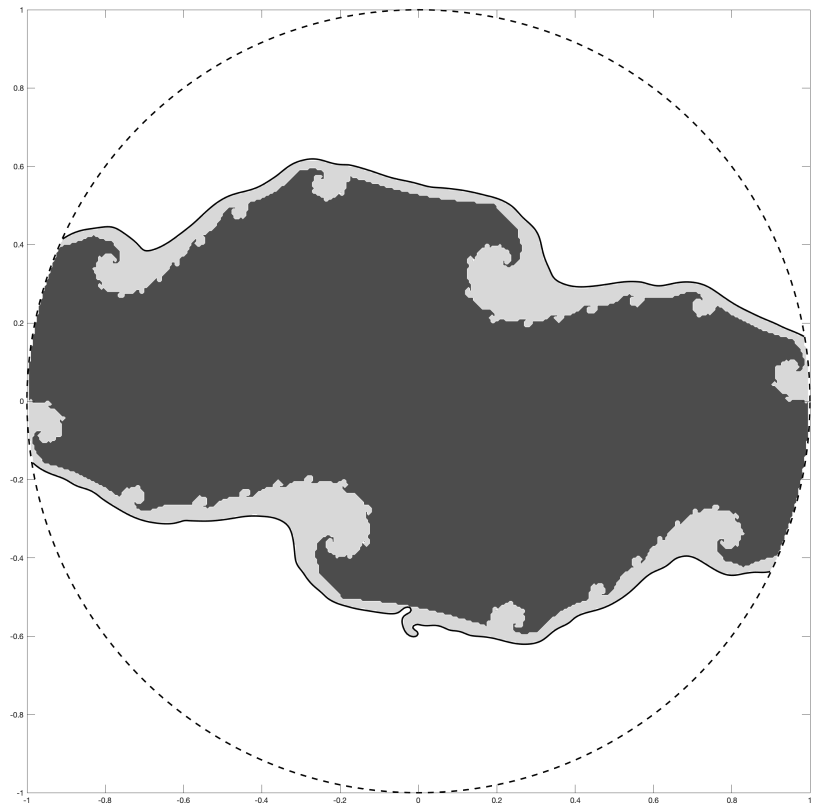
Sampling



Volume error 7.35 %

Misclassification 0.014 %

Sampling (with **postprocessing**)



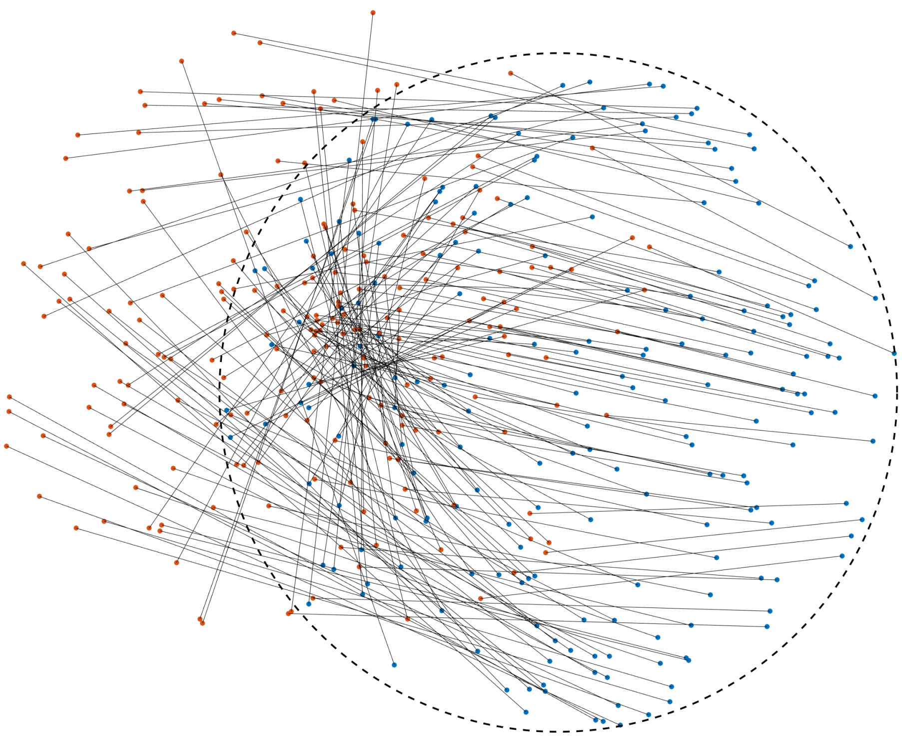
Volume error 18.86 %

Misclassification 0 %

Julia set – low data limit

 # Samples: 200

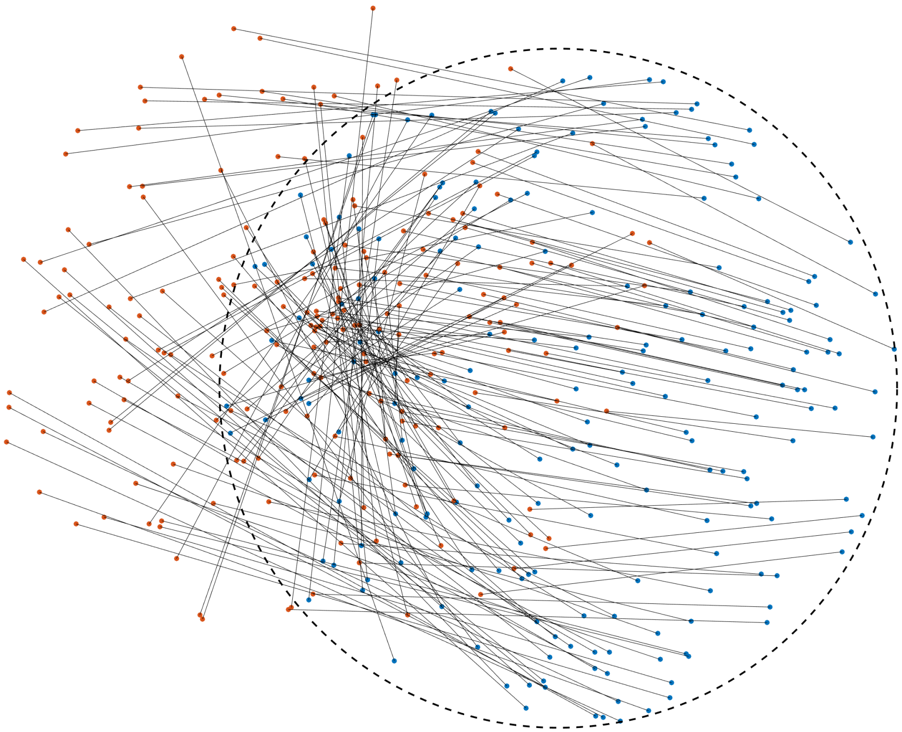
Data



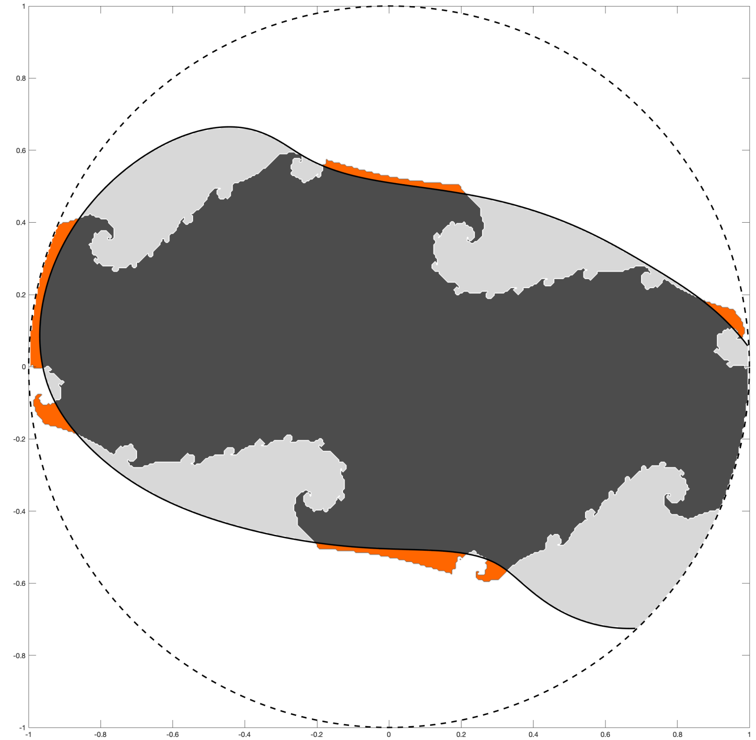
Julia set – low data limit

 # Samples: 200

Data



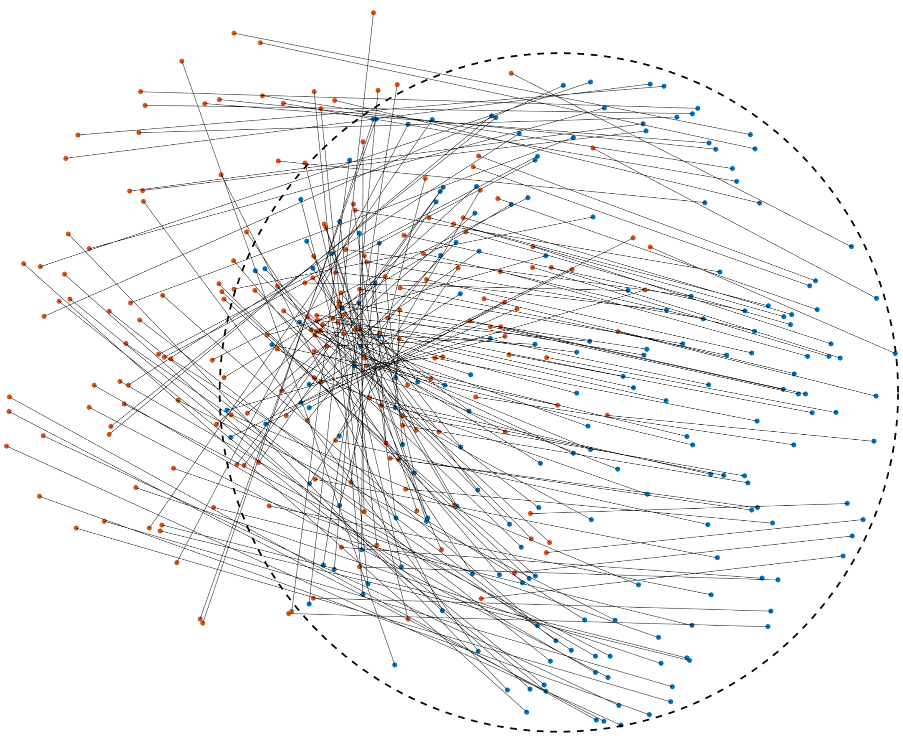
Approximation using 15 RBFs



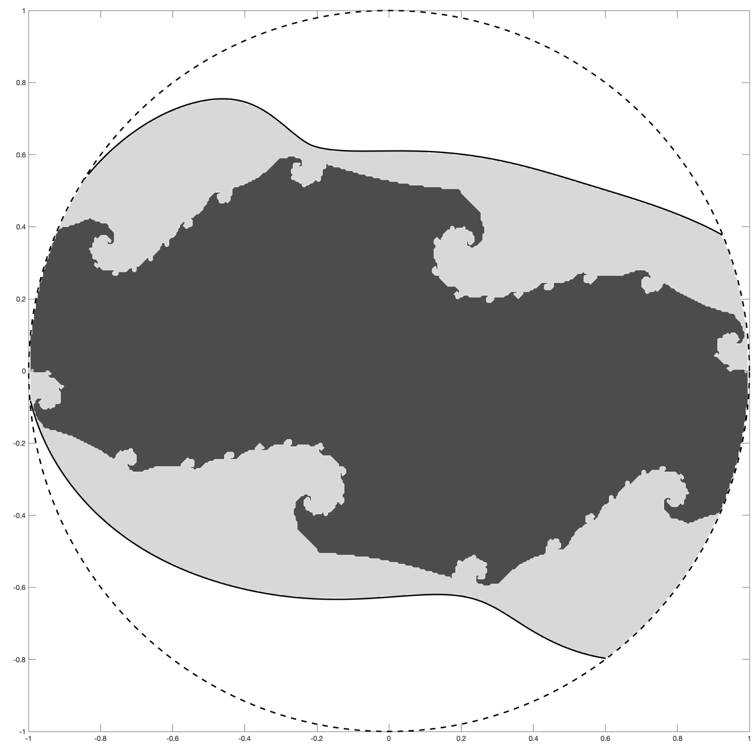
Julia set – low data limit

 # Samples: 200

Data



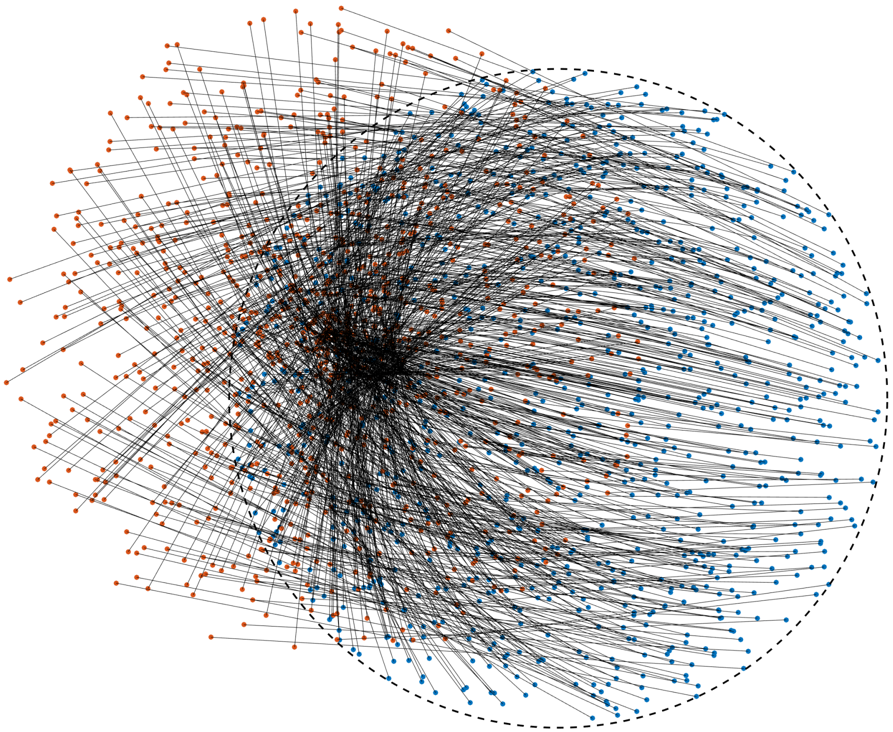
Approximation using 10 RBFs



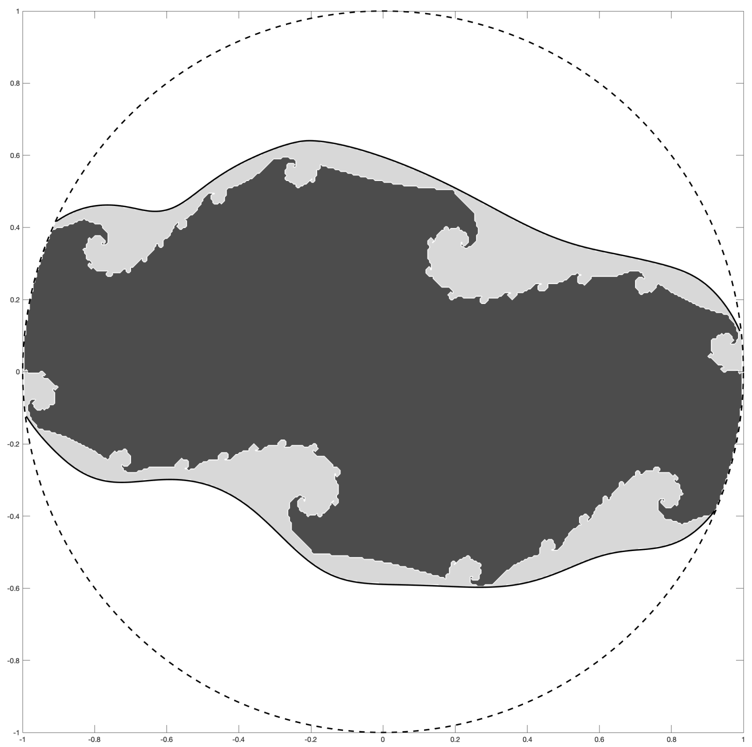
Julia set – low data limit

 # Samples: 1000

Data

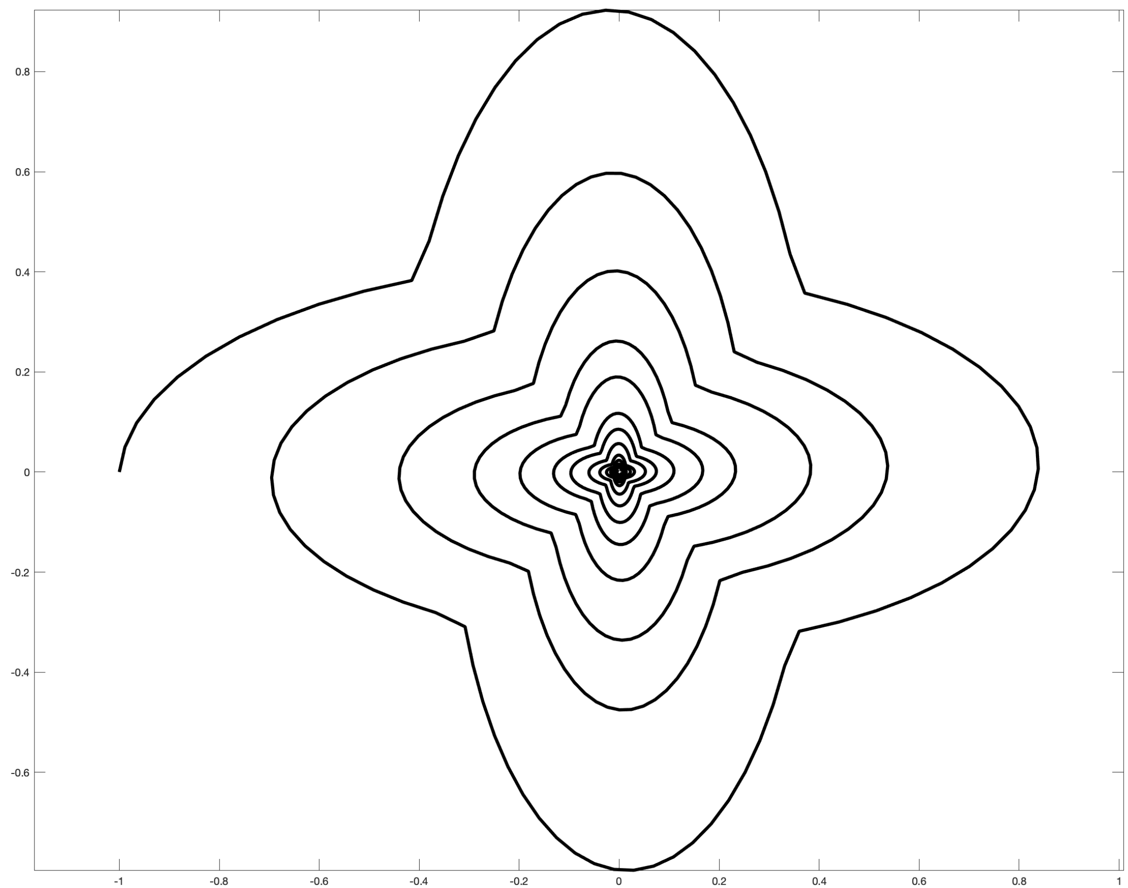


Approximation using 30 RBFs



Switched system

Flower system $$\begin{cases} \dot{x} = A_1 x, & x \in \mathcal{X}_1 \\ \dot{x} = A_2 x, & x \in \mathcal{X}_2 \end{cases}$$

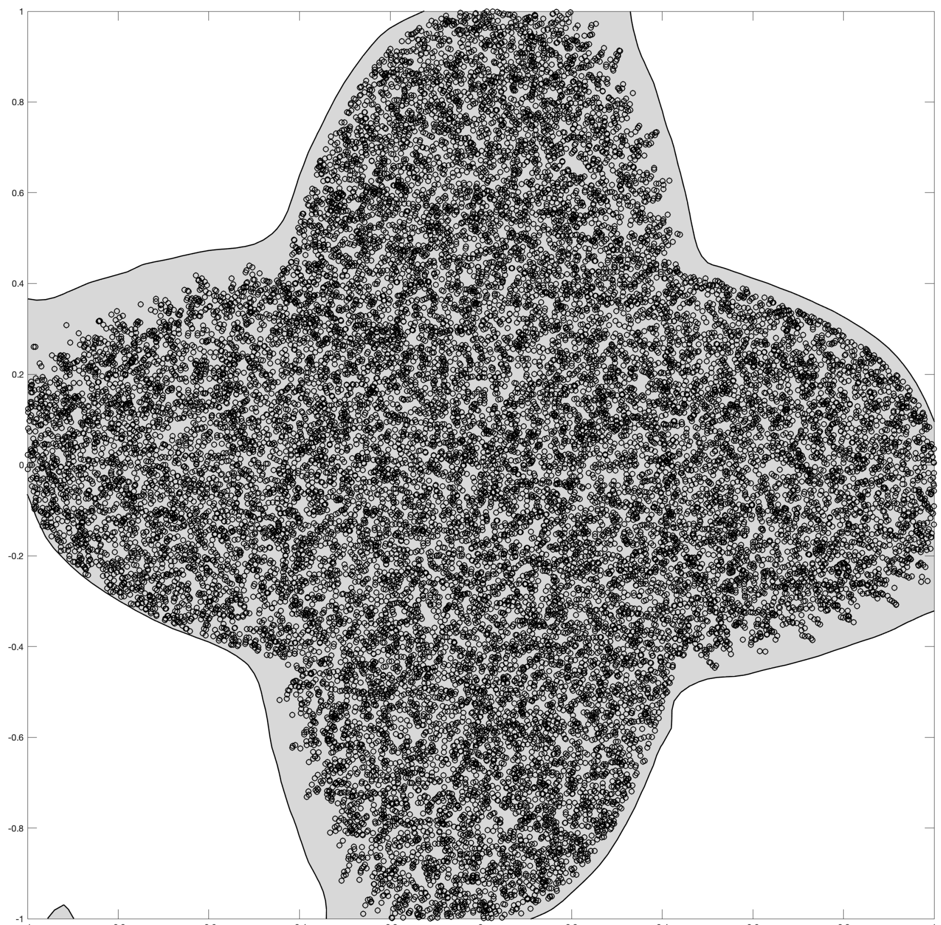


Switched system

Flower system
$$\begin{cases} \dot{x} = A_1 x, & x \in \mathcal{X}_1 \\ \dot{x} = A_2 x, & x \in \mathcal{X}_2 \end{cases}$$

Basis: 400 RBFs

 # Samples: 10000



Switched system

Modified flower system

$$\begin{cases} \dot{x} = A_1 \sin(x^3), & x \in \mathcal{X}_1 \\ \dot{x} = A_2 \sin(x^3), & x \in \mathcal{X}_2 \end{cases}$$

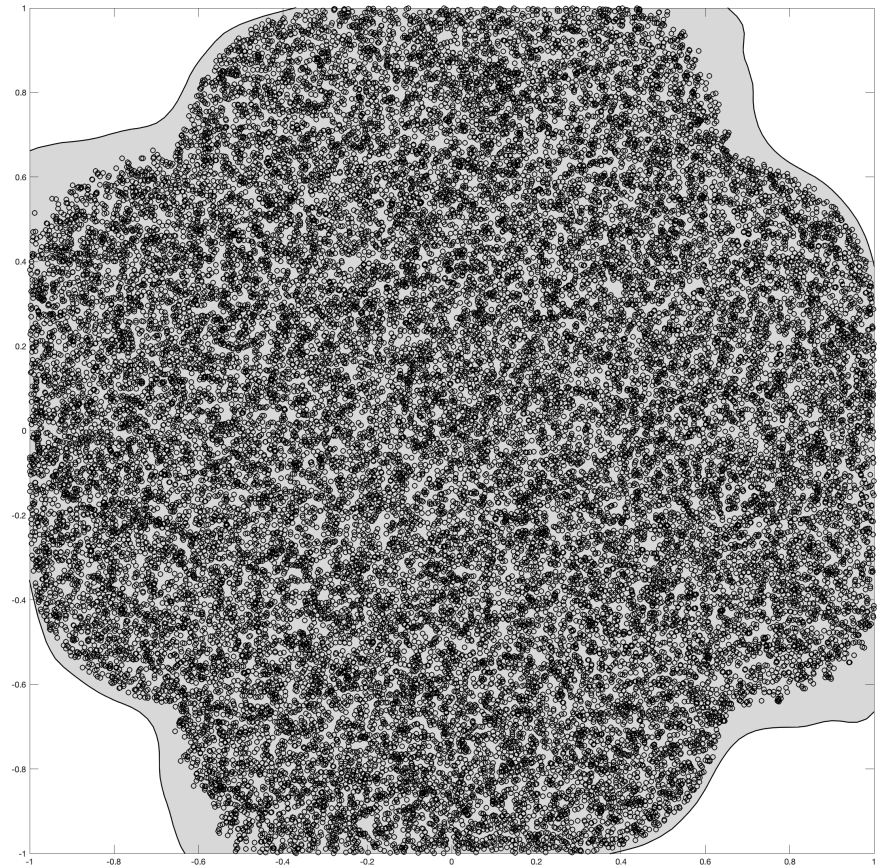
Switched system

Modified flower system

$$\begin{cases} \dot{x} = A_1 \sin(x^3), & x \in \mathcal{X}_1 \\ \dot{x} = A_2 \sin(x^3), & x \in \mathcal{X}_2 \end{cases}$$

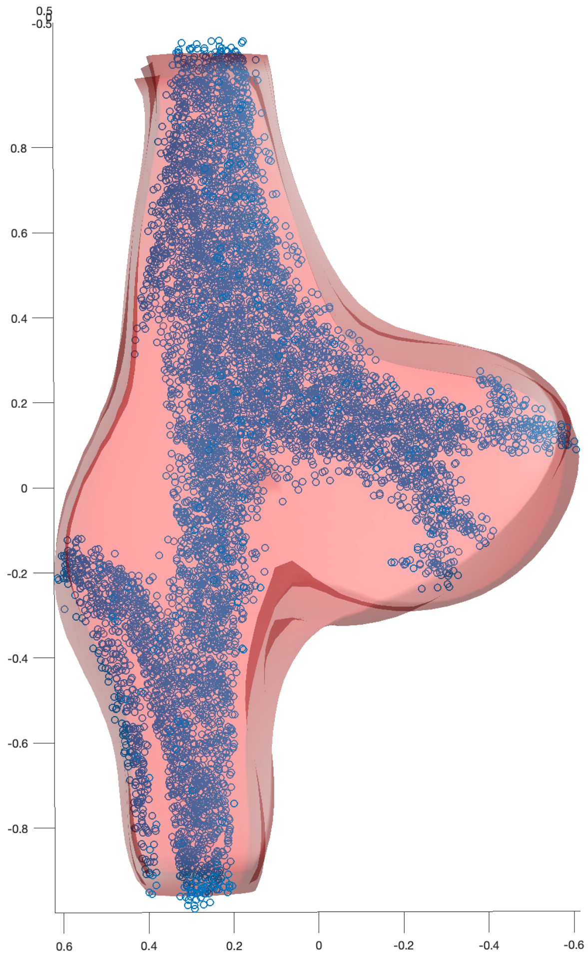
Basis: 400 RBFs

 # Samples: 10000

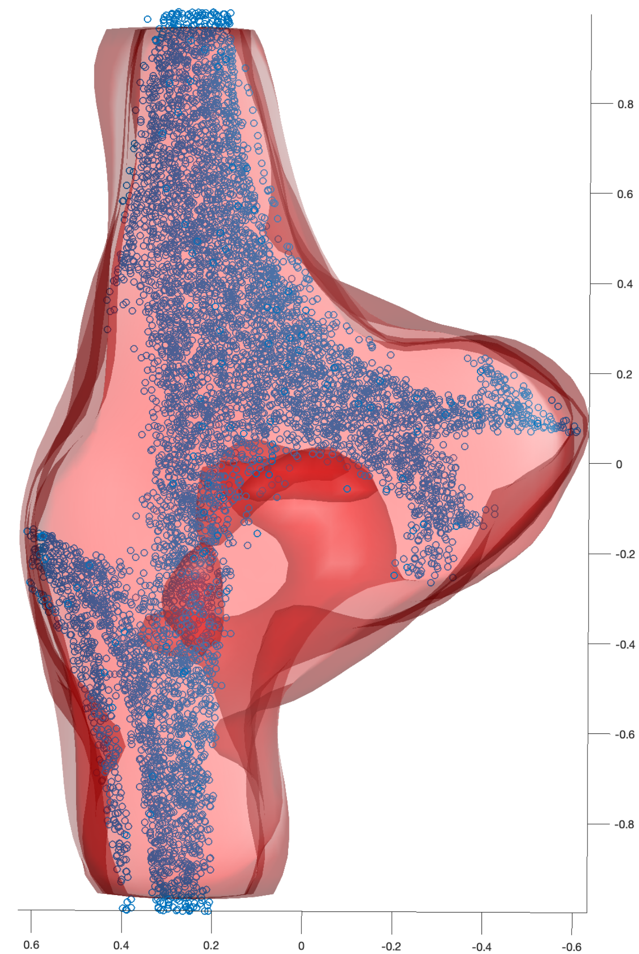


3D Hénon map

Basis: Monomials up to degree 10



Basis: 286 RBFs



Dimensionality dependence

$f = [\underbrace{f_{\text{Julia}}, \dots, f_{\text{Julia}}}_{n/2 \text{ times}}]^\top$ $\Rightarrow$ state-space of dimension n

Box constraints: $-1 \le x_i \le 1$, $i = 1, \dots, n$

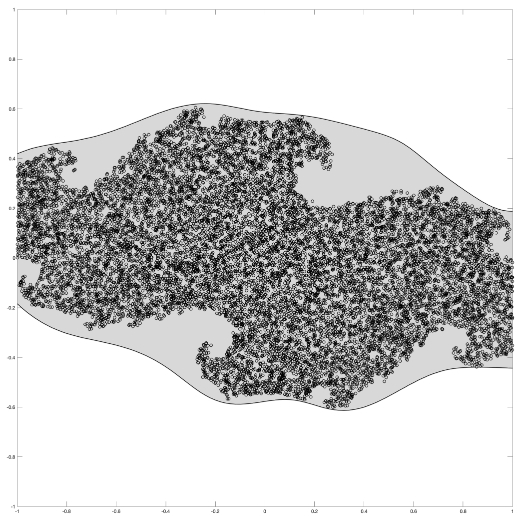
Random n-dimensional unitary state-space transformation

1600 thin-plate spline RBFs

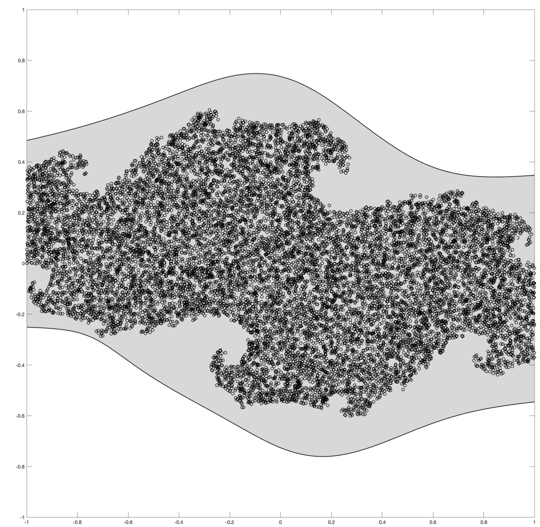
$3 \cdot 10^4$ samples

Dimensionality dependence

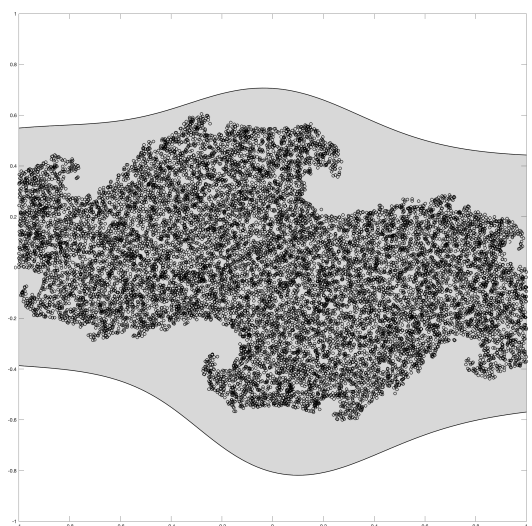
State-space dim = 4



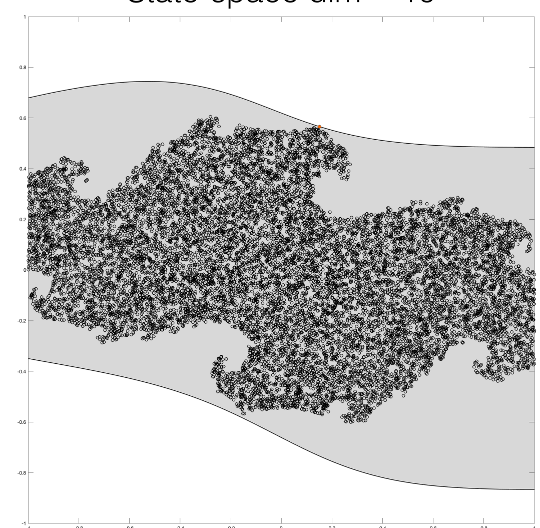
State-space dim = 6



State-space dim = 8



State-space dim = 10



Summary

- **Measure optimization** can solve dynamical systems and control problems
- The harder the problem, the more useful it seems to be
- Extends to a data-driven setting
- No free lunch – curse of dimensionality

Topics not covered and perspectives

Nonlinear PDE analysis and control

Complexity reduction (sparsity, symmetries, redundant constraints)

Invariant measures, reachability …

Postprocessing in data-driven approach: cross-validation etc.