Infinite-dimensional embeddings for analysis and control of nonlinear dynamical systems

Milan Korda

(LAAS, CNRS)

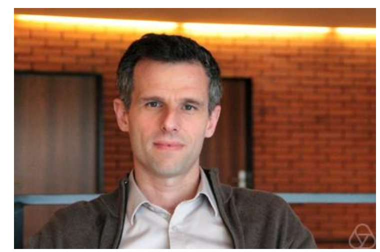
Background

Master *Czech Technical University* (Cybernetics & Robotics)

PhD *École Polytechnique Fédérale de Lausanne*

Colin Jones

Didier Henrion



Postdoc *University of California, Santa Barbara*

Igor Mezić

Mihai Putinar

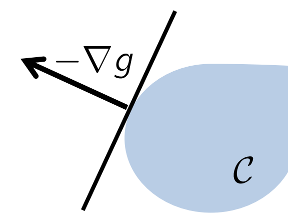
Infinite-dimensional

Linear

Difficult

$$x^+ = f(x, u)$$

Difficult



"Easy"

Convex optimization / Linear algebra

Koopman operator

Linear operator

$$\mathcal{K}g = g \circ f$$

$\nearrow$ $\searrow$

$$x^+ = f(x)$$

Nonlinear system

$$\mathcal{K}_N g = g \circ f$$

$$g \in \mathcal{H}_N$$

Matrix

Linear operator

$$\mathcal{K} g = g \circ f$$

$\nearrow$ $\searrow$

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Nonlinear system

$$\mathcal{K}_N g = g \circ f$$

$$g \in \mathcal{H}_N$$

Matrix

Eigenvectors of $\mathcal{K}_N \Rightarrow$ approximate **eigenfunctions** of $\mathcal{K}$

$\Rightarrow$ Stability, Invariant sets, Ergodic partition, Model reduction …

Linear operator

$$\mathcal{K}g = g \circ f$$

$\Nearrow$ $\Searrow$

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Nonlinear system

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Matrix

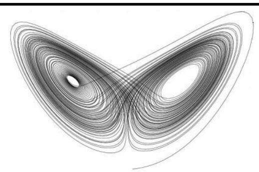
Neuroscience	Fluid mechanics	Power grid	Molecular kinetics
[Brunton et al., 2016]	*[Rowley et al., 2009]*	*[Susuki, Mezić, 2014]*	*[Wu et al., 2017]*

Koopman operator for control

Controlled Koopman operator

$$\mathcal{K}g = g \circ F$$

$$F = (f, \mathcal{S}_{\boldsymbol{u}})$$



$$x^+ = f(x, u)$$

x_0 given

Nonlinear control system

Data

$$z^+ = Az + Bu$$

$$z_0 = \boldsymbol{\psi}(x_0)$$

$$\hat{x} = Cz$$

$$\dim(z) \gg \dim(x)$$

Linear predictor

Koopman operator for control

Controlled Koopman operator

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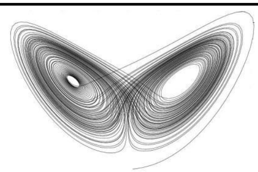

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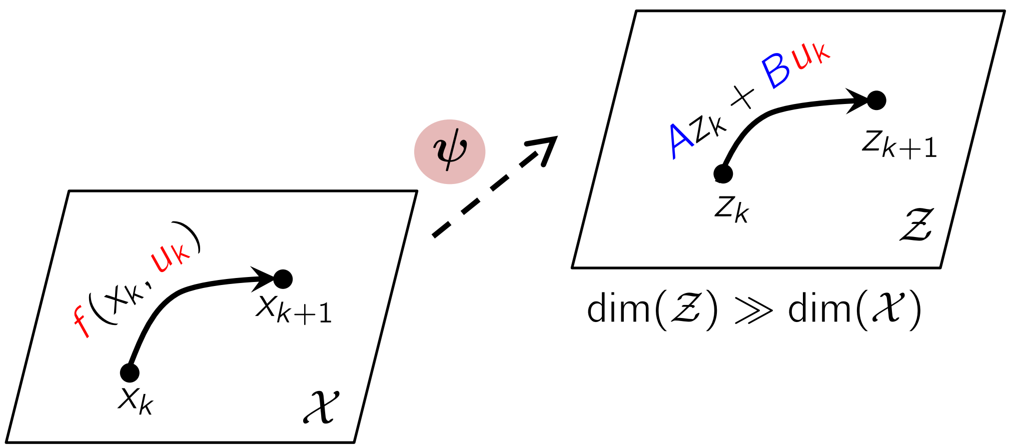
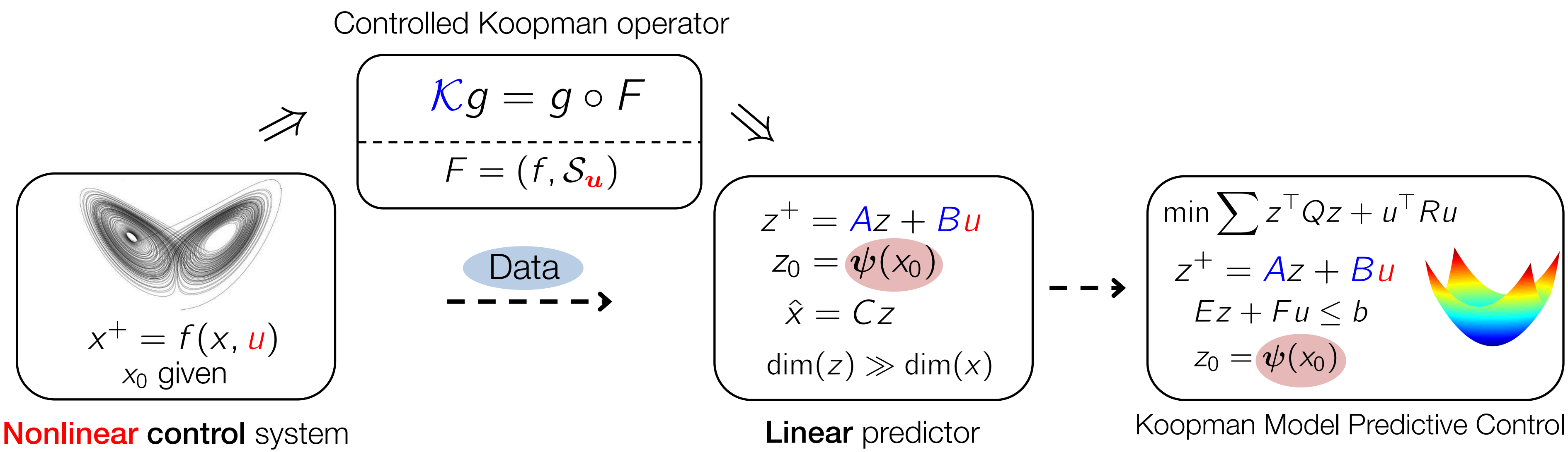
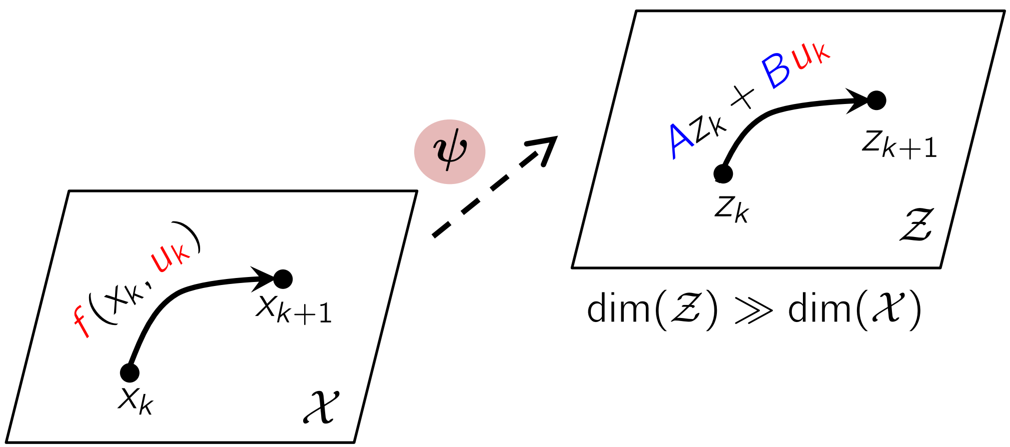


$$\dim(\mathcal{Z}) \gg \dim(\mathcal{X})$$

Koopman operator for control





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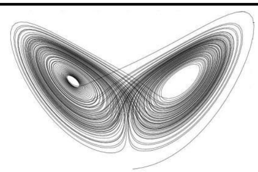

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Nonlinear control system

Data

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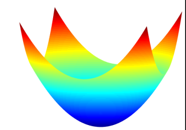
$$z_0 = \psi(x_0)$$
$$\hat{x} = Cz$$
$$\dim(z) \gg \dim(x)$$

Linear predictor

$$\min \sum z^\top Q z + u^\top R u$$
$$z^+ = Az + Bu$$
$$Ez + Fu \leq b$$
$$z_0 = \psi(x_0)$$



Koopman Model Predictive Control

MPC solves **convex** quadratic program
Complexity **independent** of $\dim(z)$

Can solve extremely **fast** for **large** systems

Completely **data-driven**

Contribution
- Koopman operator for controlled systems
- Koopman MPC

[Korda, Mezić, 2018]

Koopman operator for control

Controlled Koopman operator

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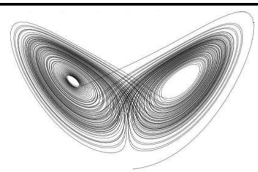

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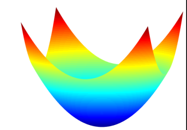


Koopman MPC

Further results:

- **Convergence** analysis of Extended Dynamic Mode Decomposition (with I. Mezić)
  - $\rightarrow$ Convergence $\mathcal{K}_N \rightarrow \mathcal{K}$ in strong operator topology
  - $\rightarrow$ Convergence of finite-horizon predictions
  - $\rightarrow$ Weak spectral convergence
- New **Spectral** approximation algorithm (with M. Putinar & I. Mezić)
  - $\rightarrow$ **Full understanding** of the spectrum from **data**

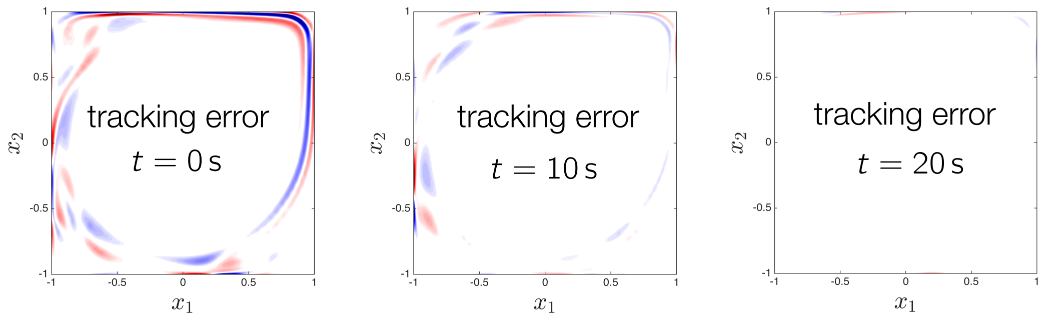
Current work: **Optimal** selection of ψ (with I. Mezić)

Koopman MPC - applications

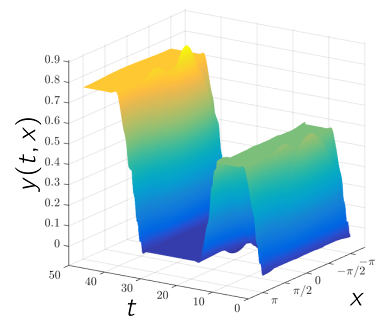
Fluid dynamics

with H. Arbabi & I. Mezić

Cavity flow control (2D Navier-Stokes)

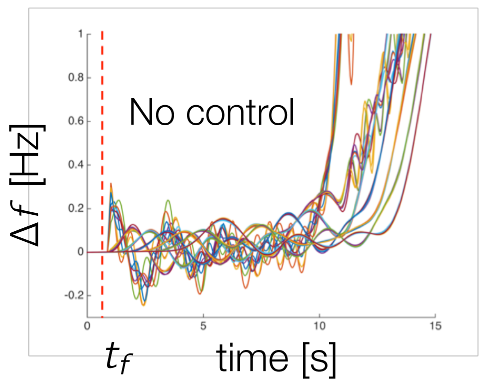


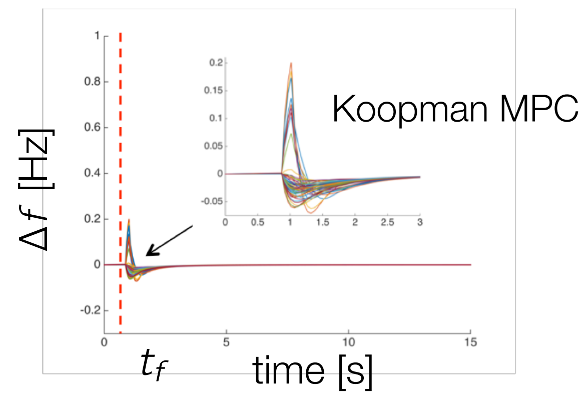
Kortweg-de Vries



Powergrid

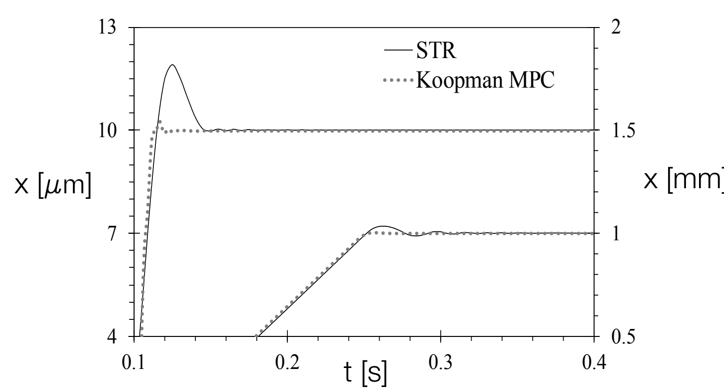
with Y. Susuki & I. Mezić





High-precision positioning

with E. Kamenar et al.



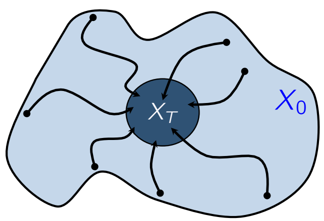
Moment-sum-of-squares

$$\min_{\mu} \langle g, \mu \rangle$$
$$\mathcal{A}\mu = b$$
$$\mu \in \mathcal{M}^+$$

$\nearrow$ $\searrow$

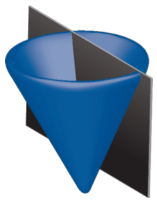
Linear program
(**Infinite**-dimensional)

$$\dot{x} = f(x, u)$$



Highly **nonconvex**

$$\min_{\boldsymbol{y}} \langle g_N, \boldsymbol{y} \rangle$$
$$\mathcal{A}_N \boldsymbol{y} = b_N$$
$$\boldsymbol{y} \in \mathcal{M}_N^+$$

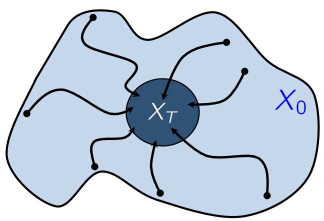


Convex semidefinite program (SDP)

$$\min_{\mu} \langle g, \mu \rangle$$
$$\mathcal{A}\mu = b$$
$$\mu \in \mathcal{M}^+$$

$\Nearrow$ $\Searrow$

$$\dot{x} = f(x, u)$$



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Linear program
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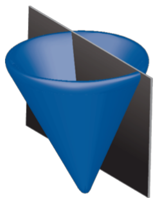


Convex semidefinite program (SDP)

(with D. Henrion and C. Jones)

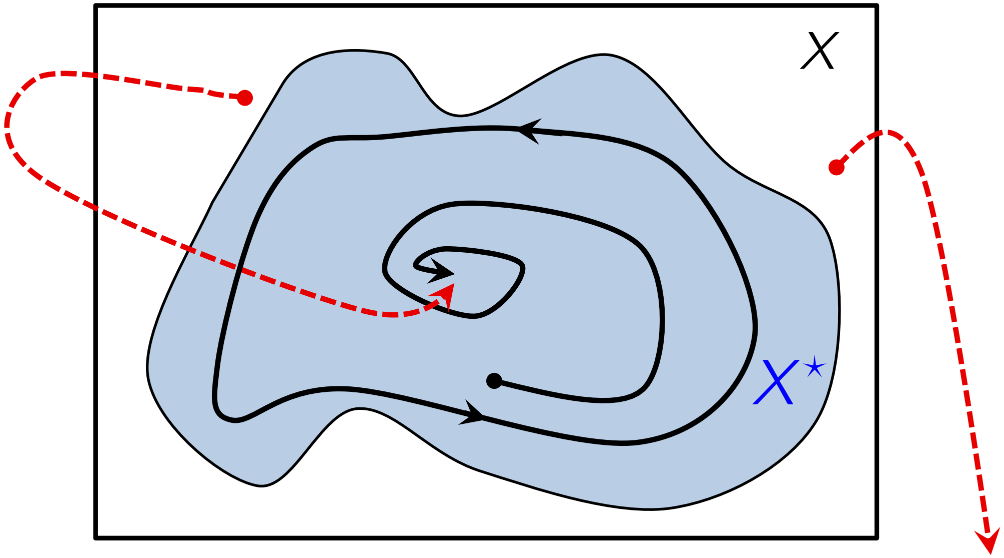
- Region of attraction
- Maximum controlled invariant set
- Value function of OCPs
- Verification of optimization-based controllers

Maximum controlled invariant set

$$x^+ = f(x, u)\,, \quad x \in X\,, \quad u \in U$$

MCI set

Set of all initial states that can be kept in the state constraint set using admissible controls



Primal LP

The MCI set is characterized by the optimization problem

Primal LP

$$\begin{aligned} \sup_{\mu,\mu_0} \quad & \int_X 1 \, d\mu_0 \\ \text{s.t.} \quad & \mu_0 + \alpha\, f_\# \mu - \mu = 0 \\ & \mu_0 \leq \lambda \\ & \mu \in \mathcal{M}(X \times U)_+, \; \mu_0 \in \mathcal{M}(X)_+ \end{aligned}$$

Infinite dimensional **linear program** in the cone of nonnegative measures

Dual LP on continuous functions

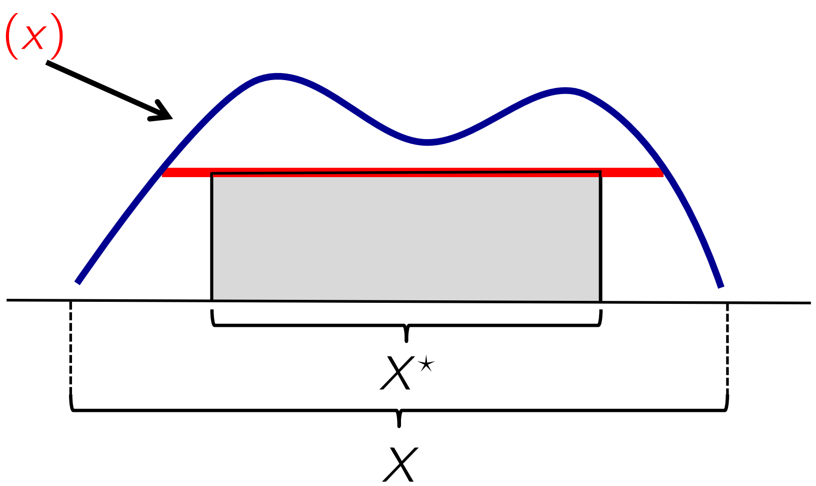
Dual LP

$$
\begin{aligned}
\inf_{v,w} \quad & \int_X w(x)\,dx \\
\text{s.t.} \quad & \alpha v(f(x,u)) \le v(x), && \forall (x,u) \in X \times U \\
& w(x) \ge v(x) + 1, && \forall x \in X \\
& w(x) \ge 0 && \forall x \in X
\end{aligned}
$$

where the infimum is over $v \in C(X)$ and $w \in C(X)$

key obseravtion:

$$w \ge \mathbb{I}_{X^\star} \quad \Rightarrow \quad \{x \mid w(x) \ge 1\} \supset X^\star$$



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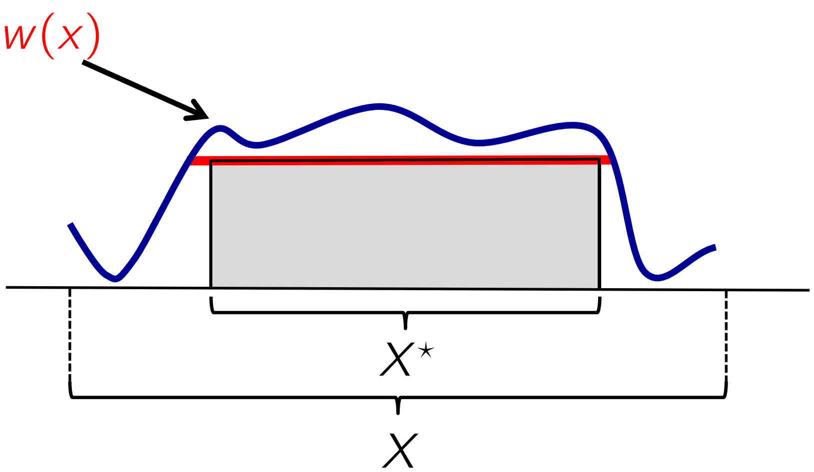

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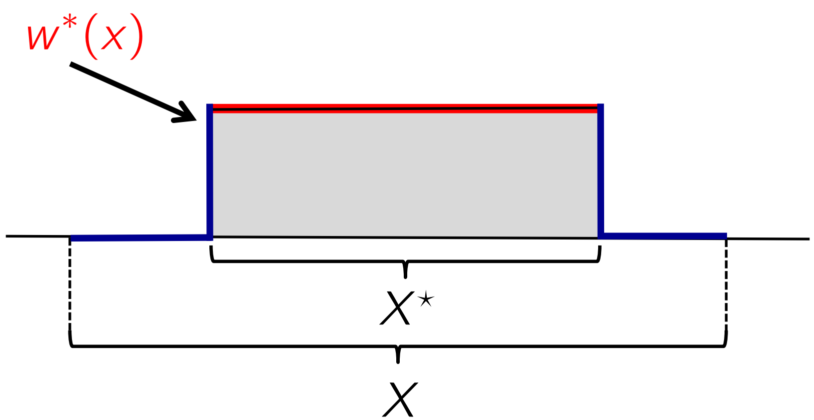
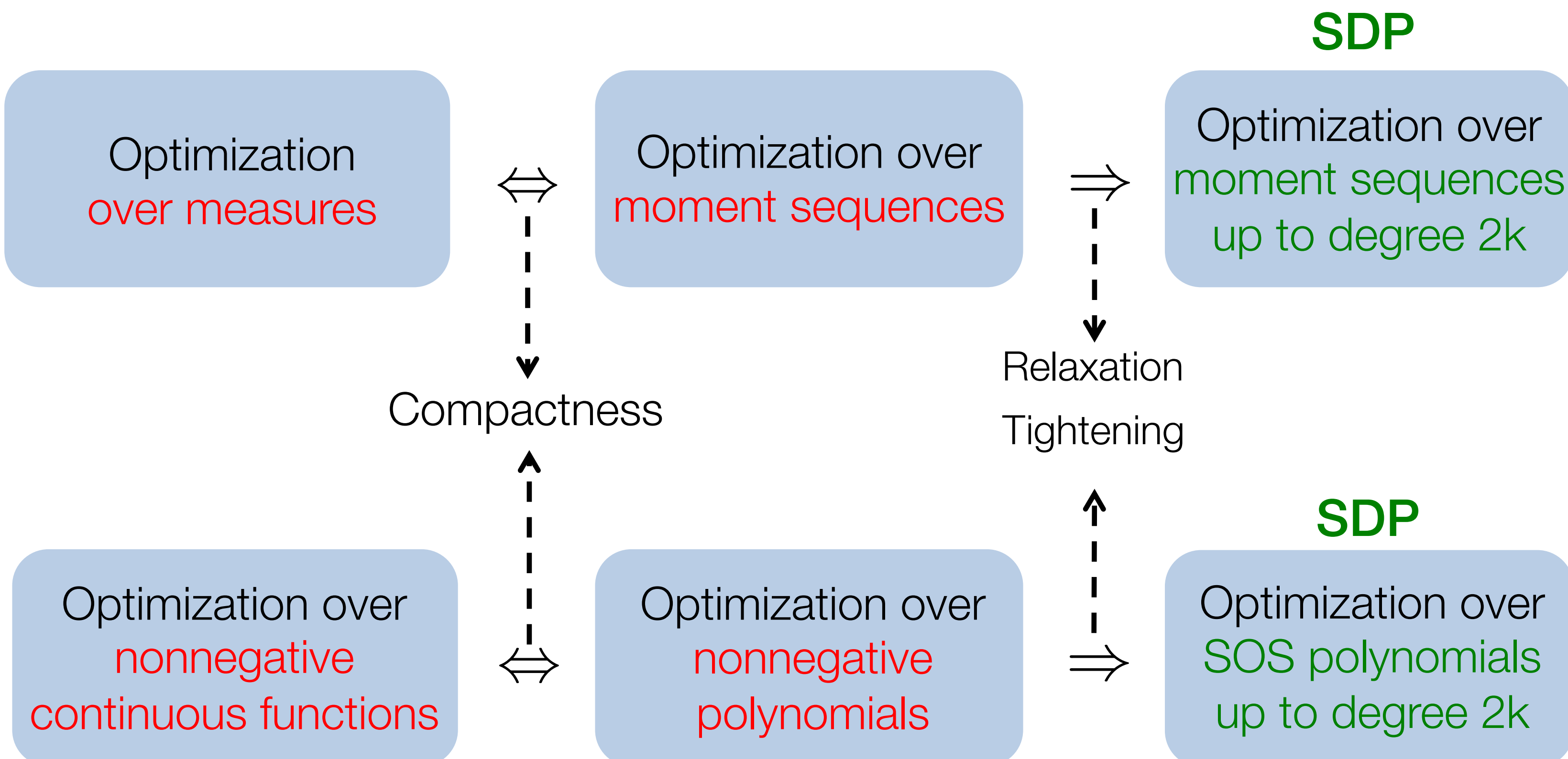


SDP hierarchy

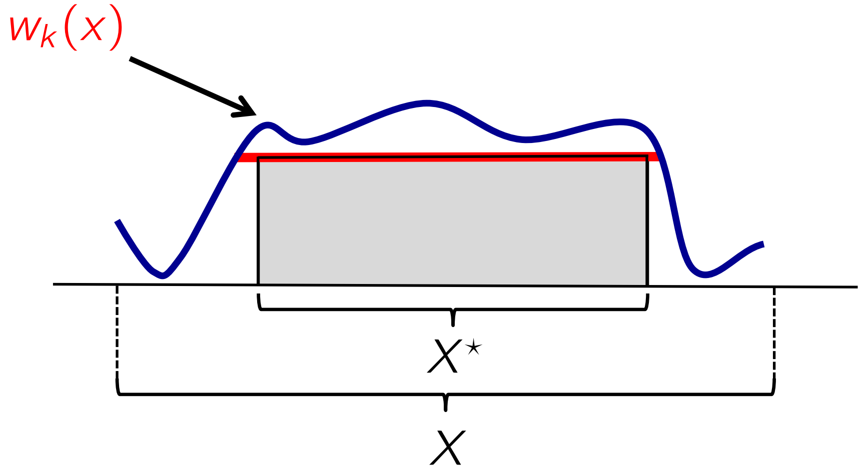


Convergence

Let $w_k(x)$ be the optimal solution to the dual SDP relaxation of order k

$$X_k^\star := \{x \mid w_k(x) \geq 1\}$$

$$\boxed{\mathrm{vol}(X_k^\star \setminus X^\star) \to 0}$$



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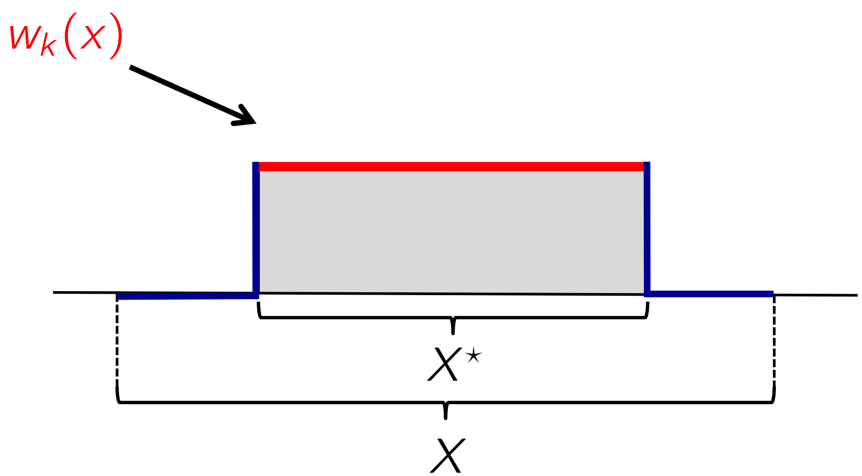
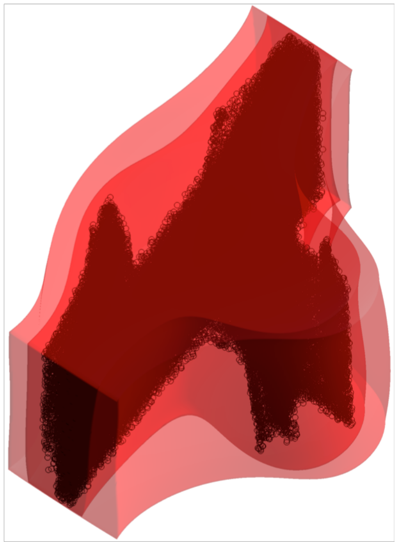
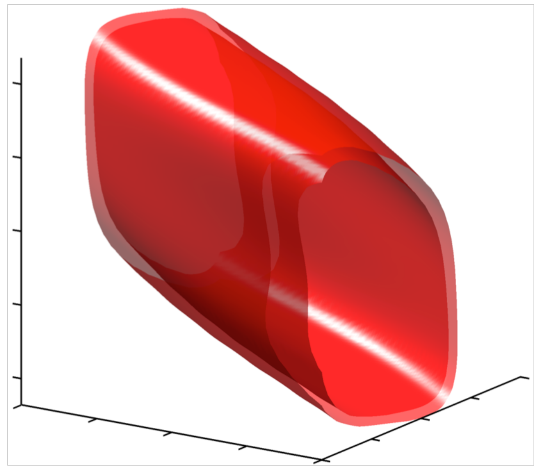


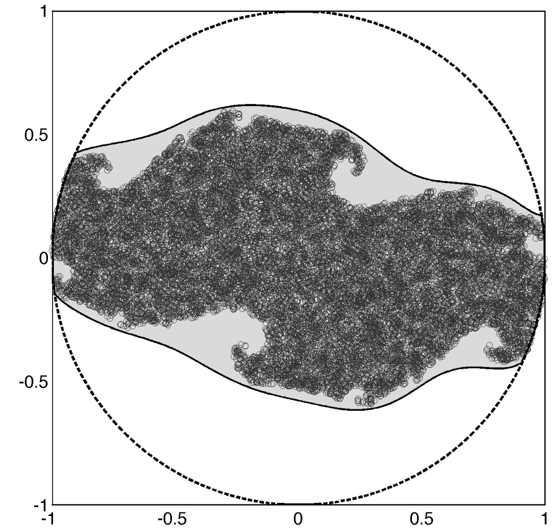
3D Hénon



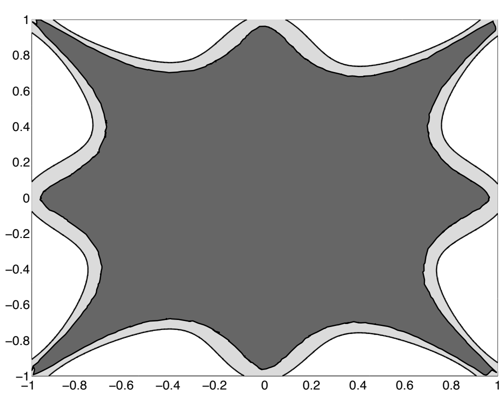
Double pendulum on cart



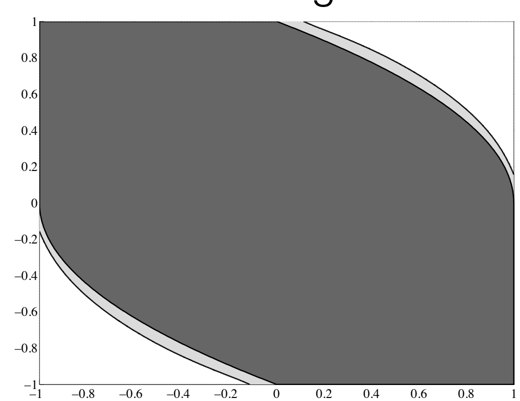
Julia



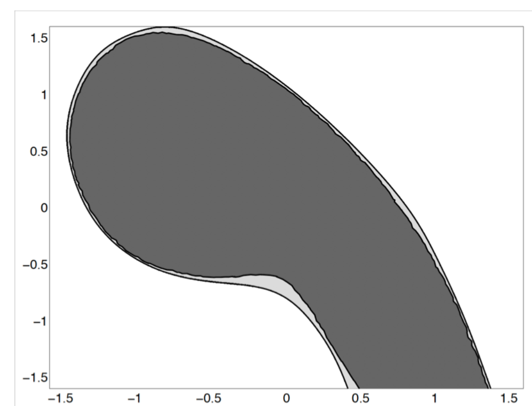
Spider web



Double integrator



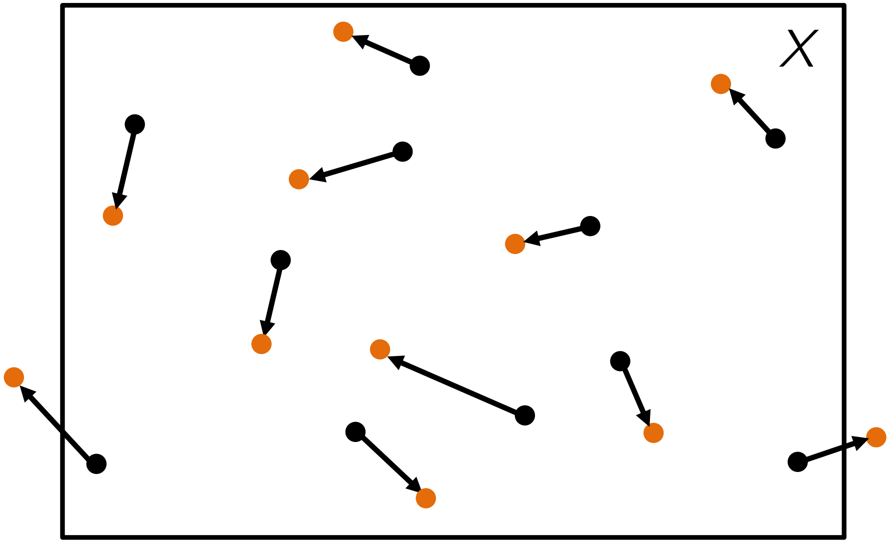
Cathala



Data-driven invariant set estimation

Maximum controlled invariant set from data

f **not given**, only **data** $\{x_i^+, x_i, u_i\}_{i=1}^{M}$ available



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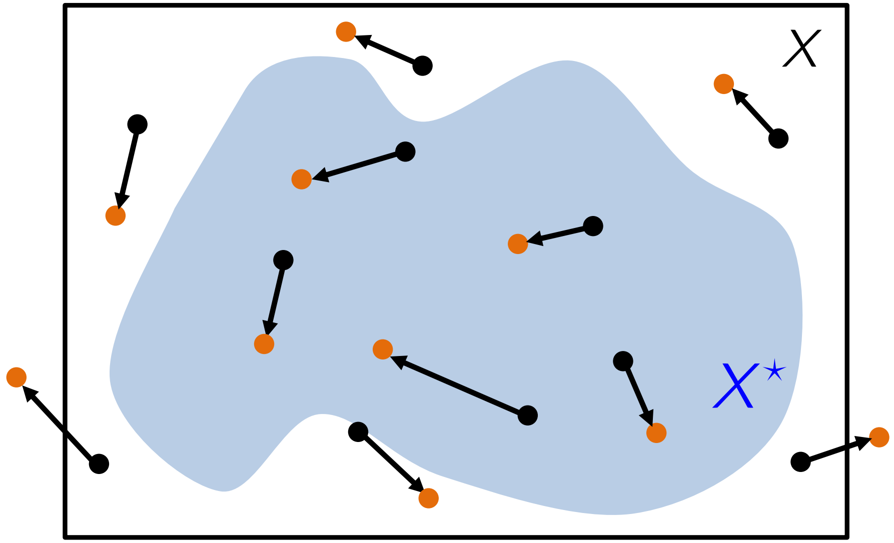

Idea

Sampled dual LP

$$
\begin{aligned}
\inf \quad & \frac{1}{M}\sum_{i=1}^{M} w(x_i) \\
\text{s.t.} \quad & \left.\begin{aligned}
&\alpha v(x_i^+) \le v(x_i) \\
&w(x_i) \ge v(x_i) + 1 \\
&w(x_i) \ge 0
\end{aligned}\right\} \ \forall (x_i, x_i^+) \in \text{Data}
\end{aligned}
$$

with variables $v, w \in \mathcal{F} \subset \mathcal{C}(X)$, $\dim(\mathcal{F}) < \infty$

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with variables $v, w \in \mathcal{F} \subset \mathcal{C}(X)$, $\dim(\mathcal{F}) < \infty$

Properties

+ **No assumptions** on f (can be non-polynomial, discontinuous etc.)
+ **No assumptions** on the subspace $\mathcal{F}$ (can be radial basis functions, wavelets etc.)
+ Boils down to **finite-dimensional LP**

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Sampled dual LP

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+ **No assumptions** on the subspace $\mathcal{F}$ (can be radial basis functions, wavelets etc.)
+ Boils down to **finite-dimensional LP**

– No longer guaranteed outer approximation

Idea

Sampled dual LP

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\begin{aligned}
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\end{aligned}
$$

with variables $v, w \in \mathcal{F} \subset \mathcal{C}(X)$, $\dim(\mathcal{F}) < \infty$

Heuristic?

Robust convex optimization

Dual LP

$$\begin{array}{lll} \inf\limits_{v,w} & \int_X w(x)\,dx & \\ \text{s.t.} & \alpha v(f(x,u)) \le v(x), & \forall\,(x,u) \in X \times U \\ & w(x) \ge v(x) + 1, & \forall\, x \in X \\ & w(x) \ge 0, & \forall\, x \in X, \end{array}$$

$$\Updownarrow$$

$$\begin{array}{ll} \inf\limits_{v,w} & \mathbb{E}_{\lambda_X} w \\ \text{s.t.} & L(w,v)(x,u) \le 0 \quad \forall (x,u) \in \underbrace{X \times U}_{\text{“'uncertainty set'}} \end{array}$$

Sample / scenario approximation: **well-studied** (Shapiro, Nemirovski, Campi, Calafiore,...)

Convergence

$N :=$ number of basis functions

$M :=$ number of samples

$$X_{N,M} = \{x \mid w_{N,M}(x) \geq 1\}$$

Easy statement: $$\lim_{N\to\infty} \mathrm{vol}(X_{N,M} \setminus X^\star) = 0$$

More tricky statement: $$\lim_{M\to\infty} \mathrm{vol}(X^\star \setminus X_{N,M}) = 0$$

Open questions

(1) $$\mathbb{P}\{x \in X \;\; \& \;\; x \notin X_{N,M}\} \leq \mathrm{function}(N, M)$$

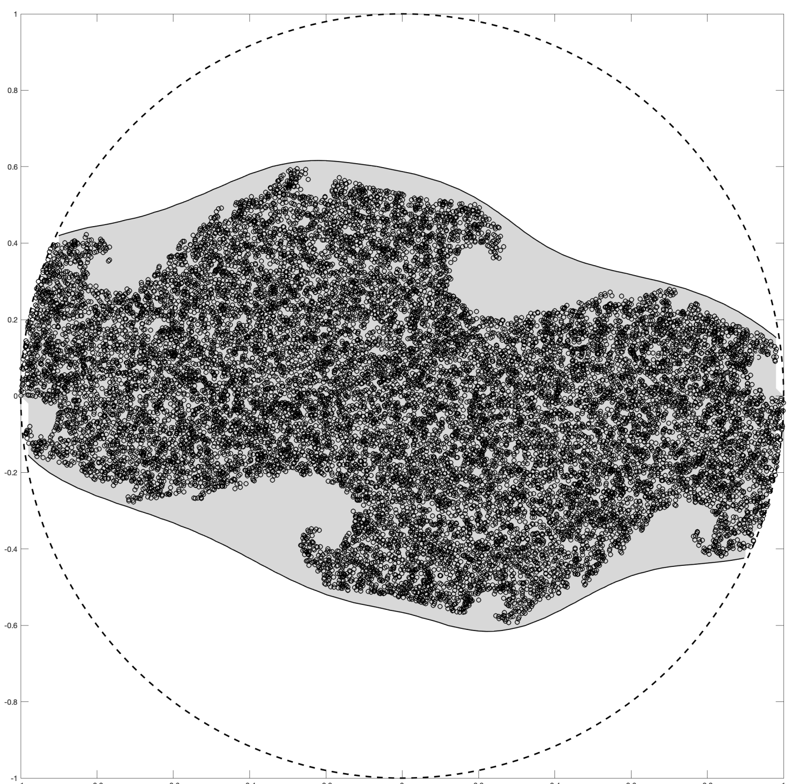
(2) Given ϵ, δ, N, how many samples are needed such that

$$\mathbb{P}\{\mathrm{vol}(X^\star \setminus X_{N,M}) > \epsilon\} < \delta$$

Julia set – sampling vs SDP

Basis: polynomials up to degree 10

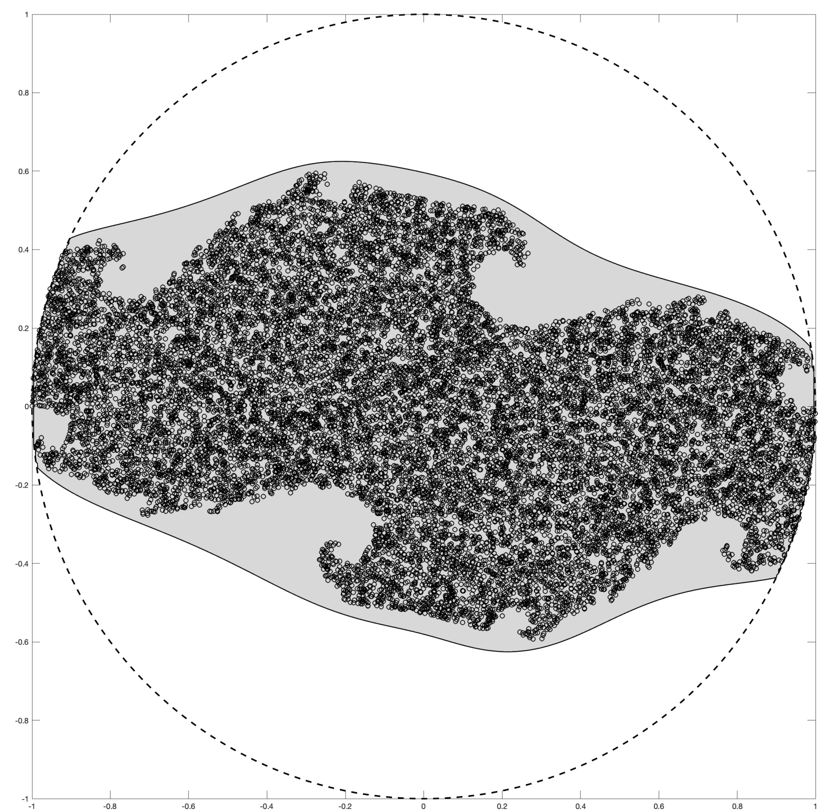
Sampling



Volume error 25.1 %

Misclassification 0 %

SDP



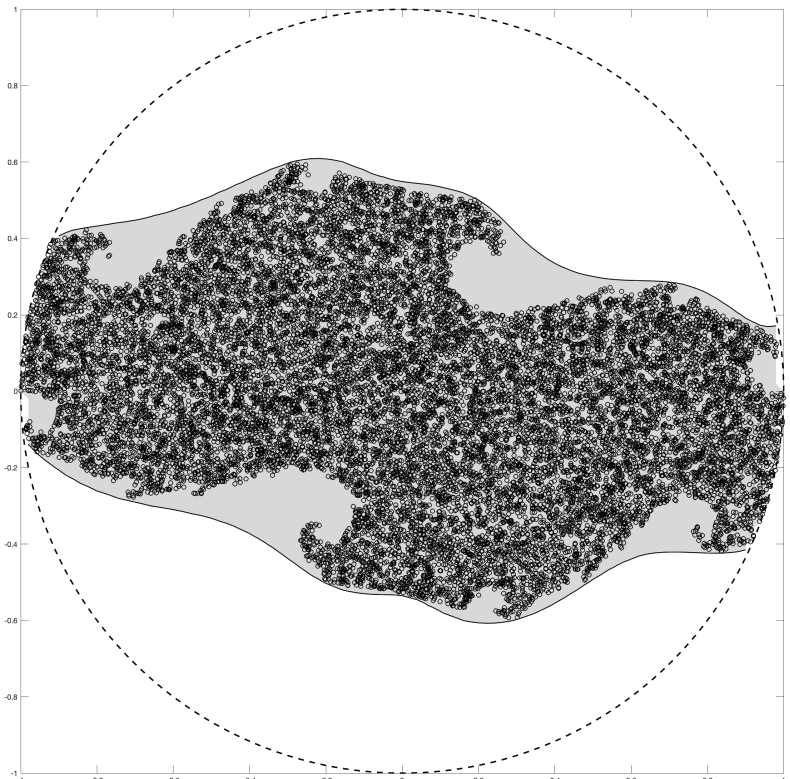
Volume error 28.7 %

Misclassification 0 %

Julia set – sampling vs SDP

Basis: polynomials up to degree 14

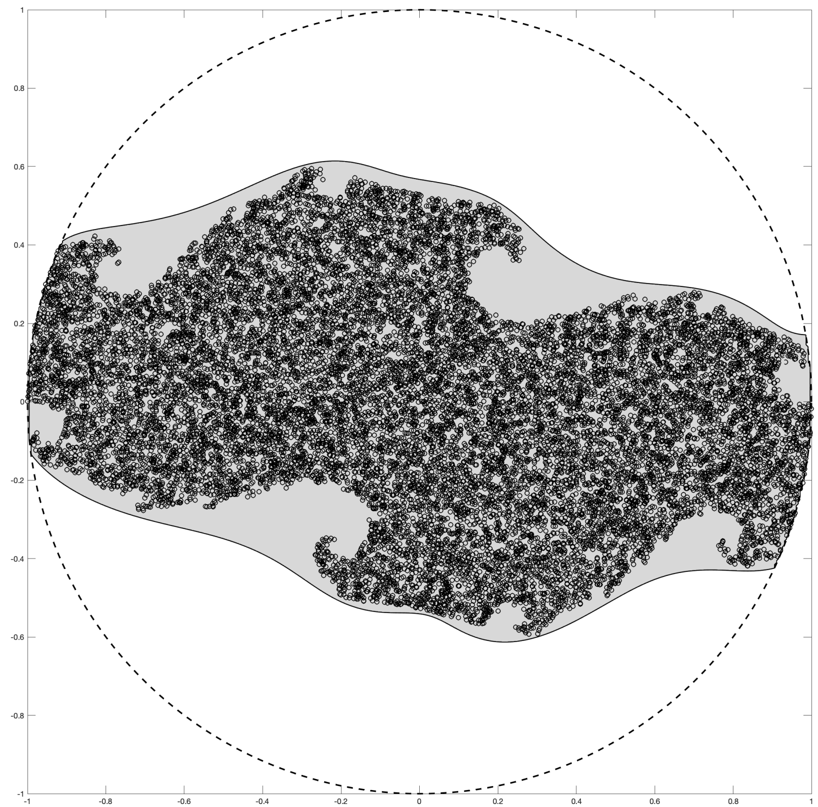
Sampling



Volume error 19.7 %

Misclassification 0 %

SDP



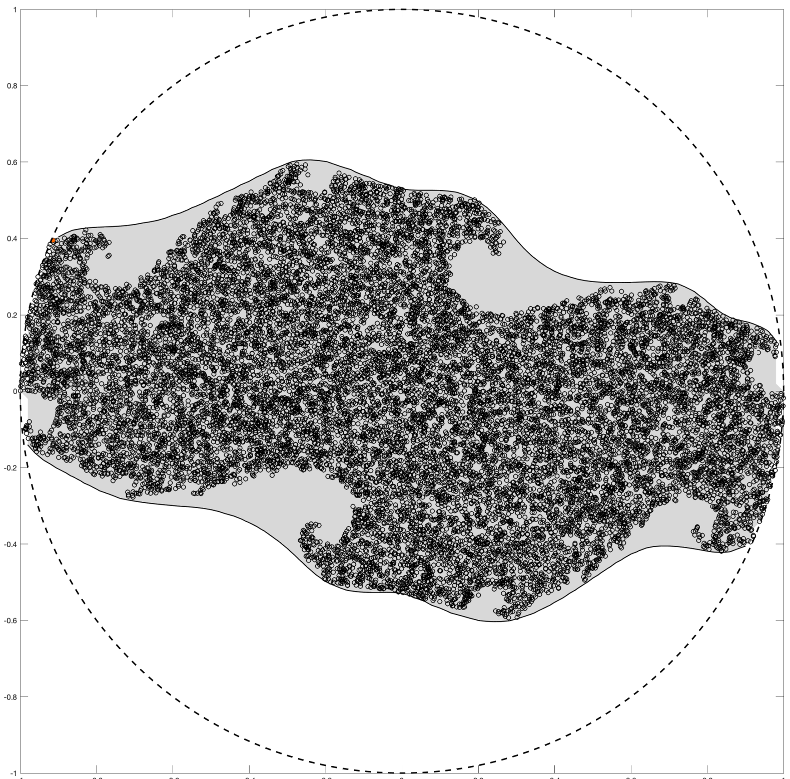
Volume error 21.9 %

Misclassification 0 %

Julia set – sampling vs SDP

Basis: polynomials up to degree 18

Sampling



Volume error 17.1 %

Misclassification 0.0025 %

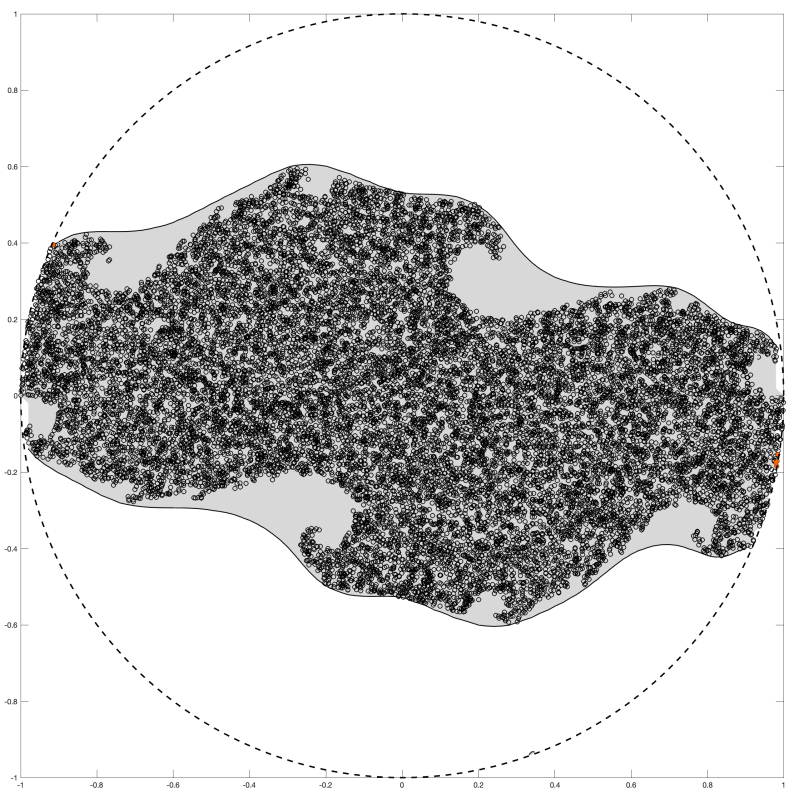
SDP

Numerical problems

Julia set – sampling vs SDP

Basis: polynomials up to degree 22

Sampling



Volume error 16.3 %

Misclassification 0.0125 %

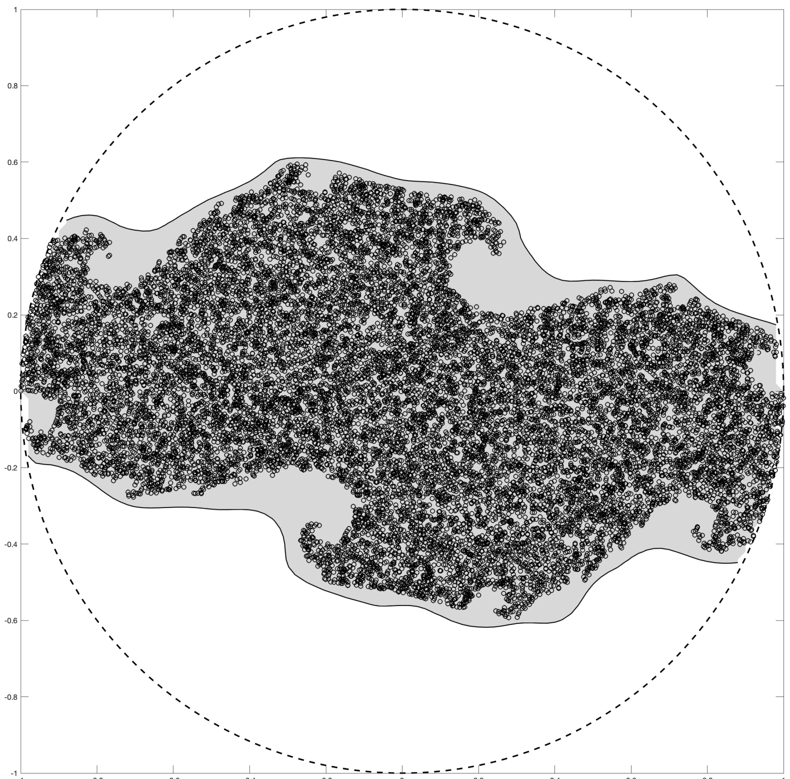
SDP

Numerical problems

Julia set – different bases

Basis: 200 thin-plate spline RBFs

Sampling



Volume error 20.6 %

Misclassification 0 %

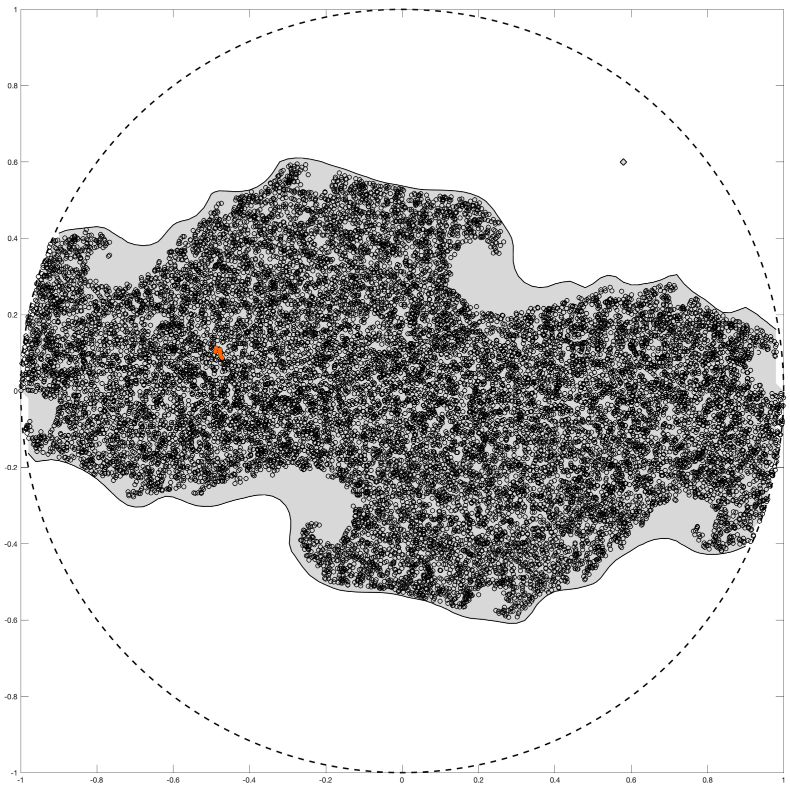
SDP

NA

Julia set – different bases

Basis: 400 thin-plate spline RBFs

Sampling



Volume error 14.7 %

Misclassification 0.0175 %

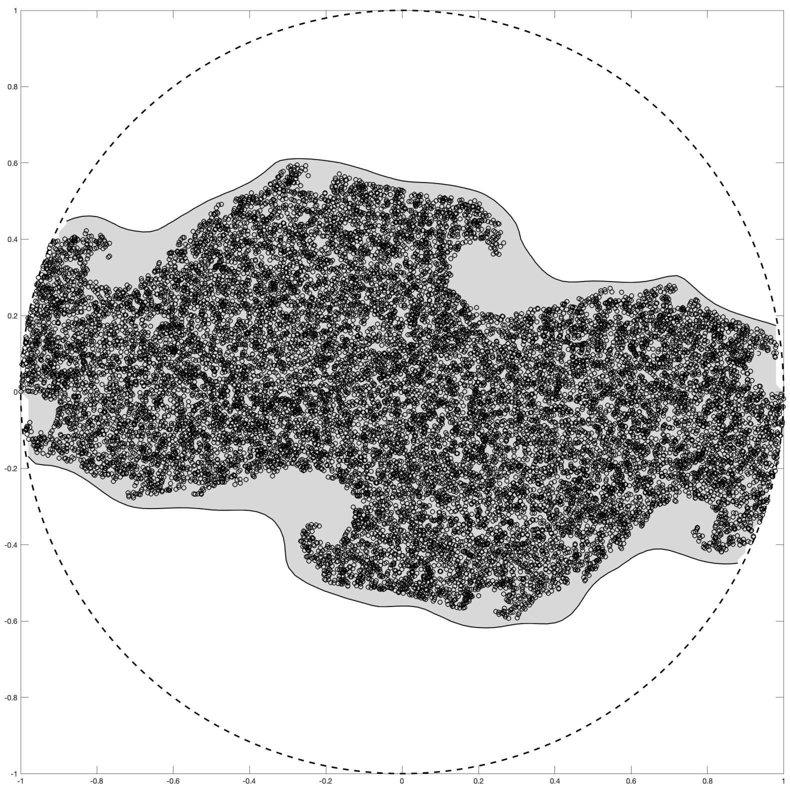
SDP

NA

Julia set – # samples

Basis: 200 thin-plate spline RBFs

 # Samples: 10000



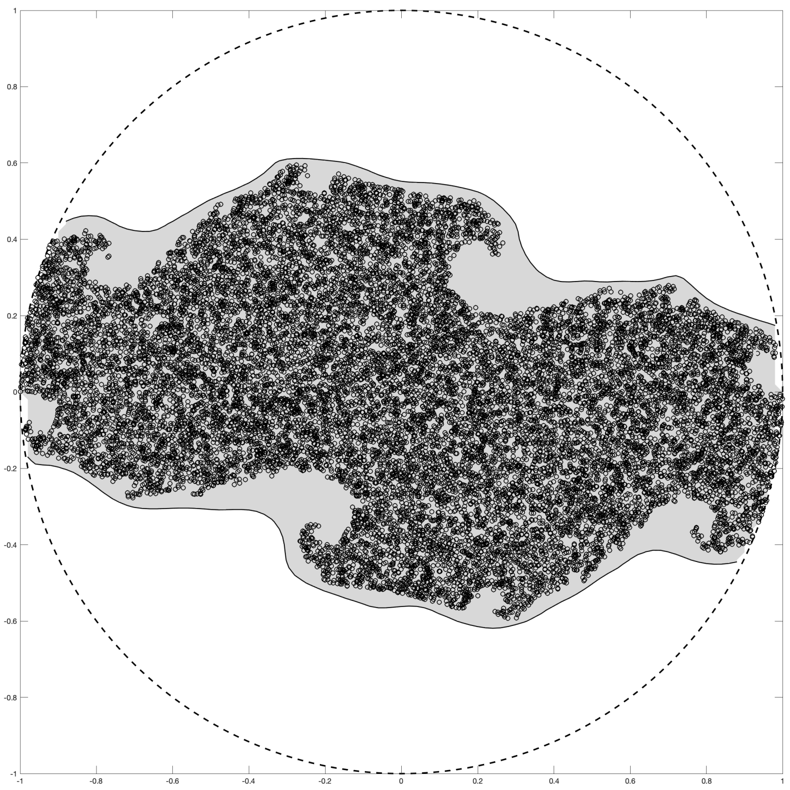
Volume error 20.6 %

Misclassification 0 %

Julia set – # samples

Basis: 200 thin-plate spline RBFs

 # Samples: 5000



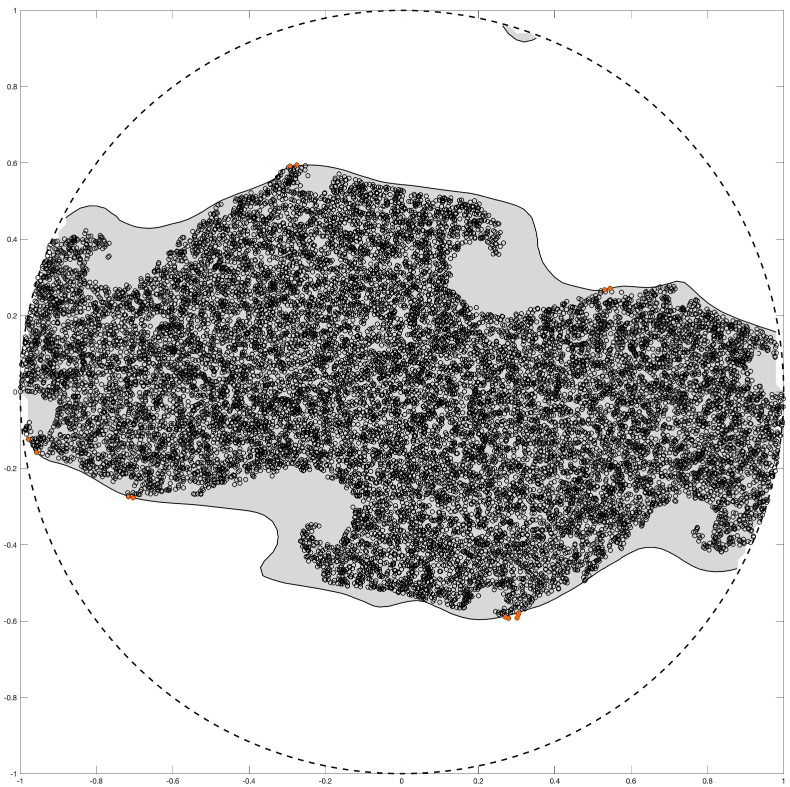
Volume error 20.1 %

Misclassification 0 %

Julia set – # samples

Basis: 200 thin-plate spline RBFs

 # Samples: 2000



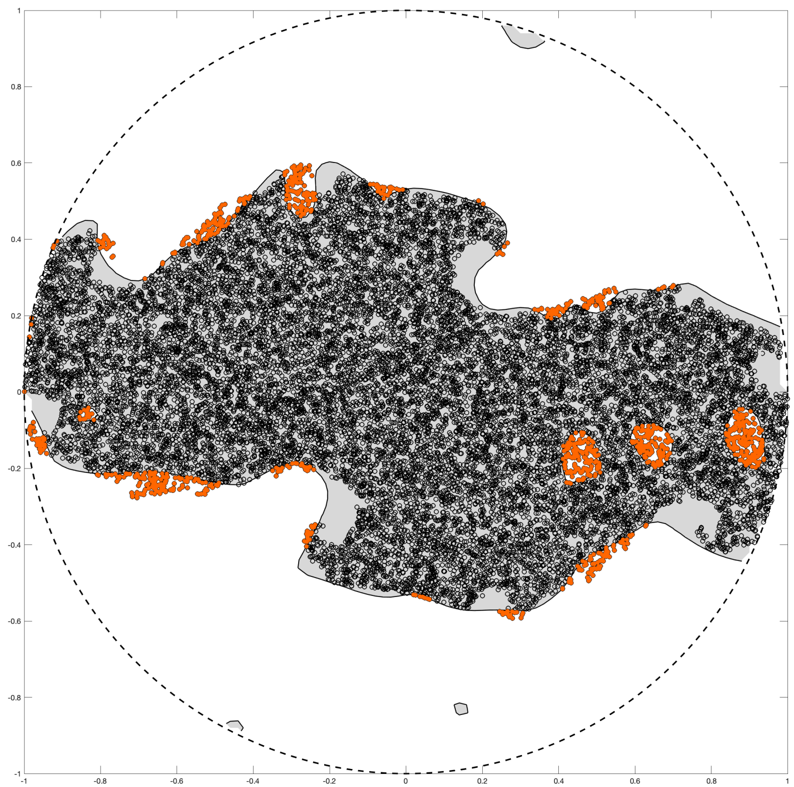
Volume error 18.4 %

Misclassification 0 %

Julia set – # samples

Basis: 200 thin-plate spline RBFs

 # Samples: 1000



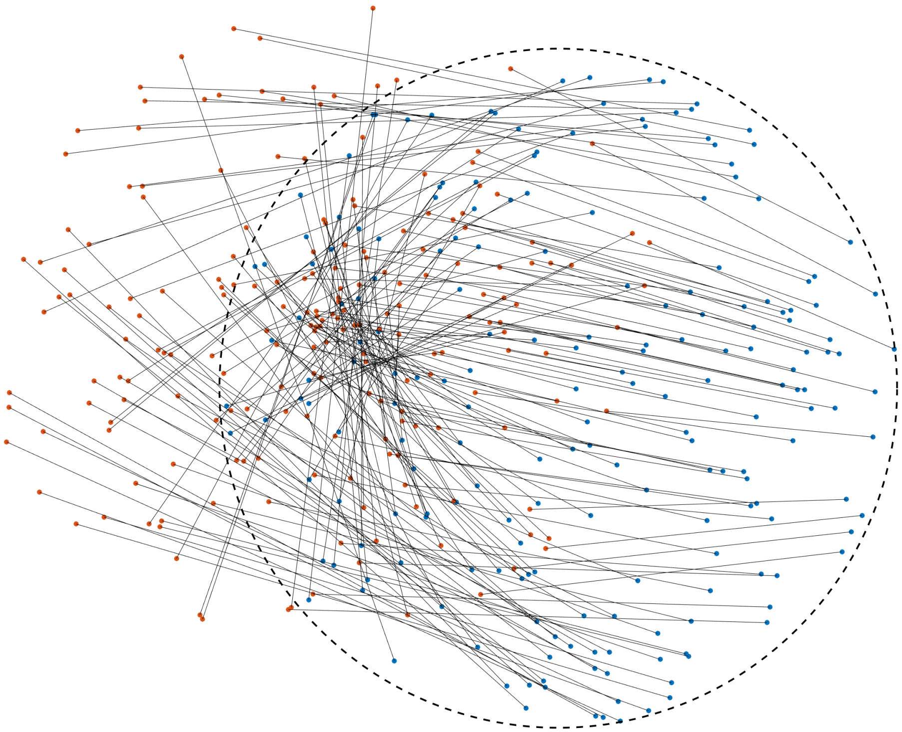
Volume error 7.35 %

Misclassification 5.95 %

Julia set – low data limit

 # Samples: 200

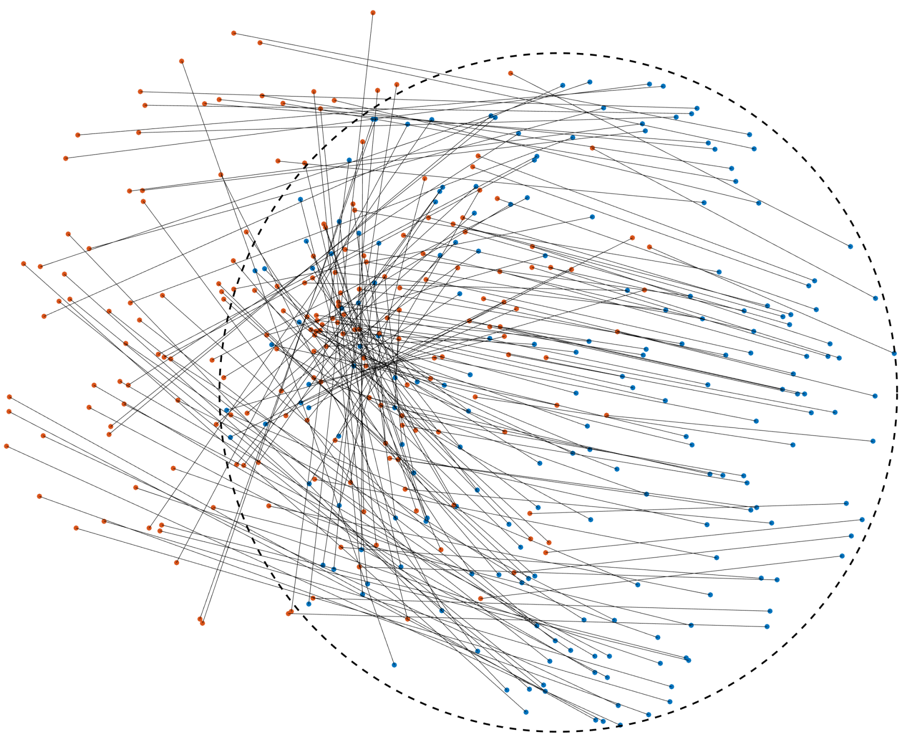
Data



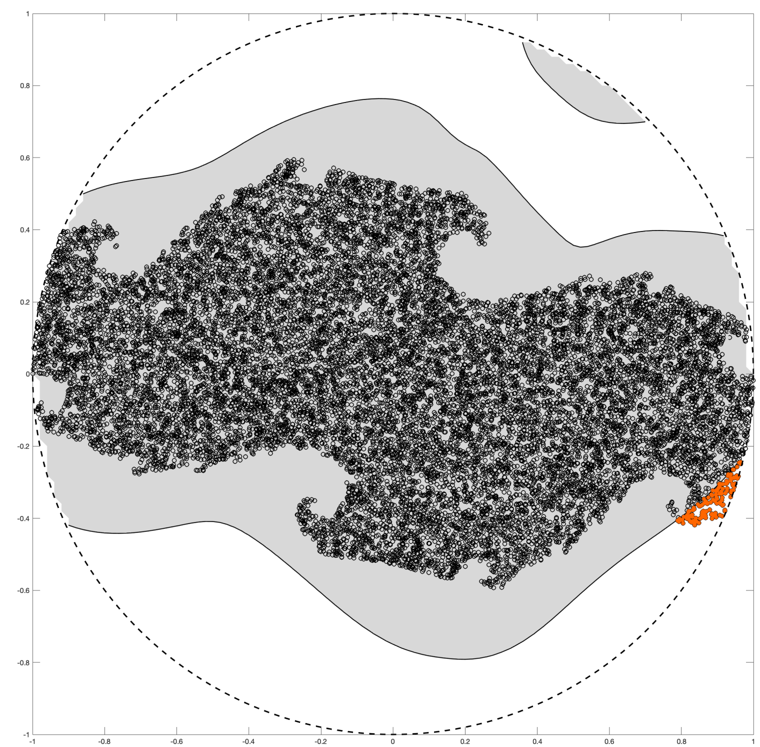
Julia set – low data limit

 # Samples: 200

Data



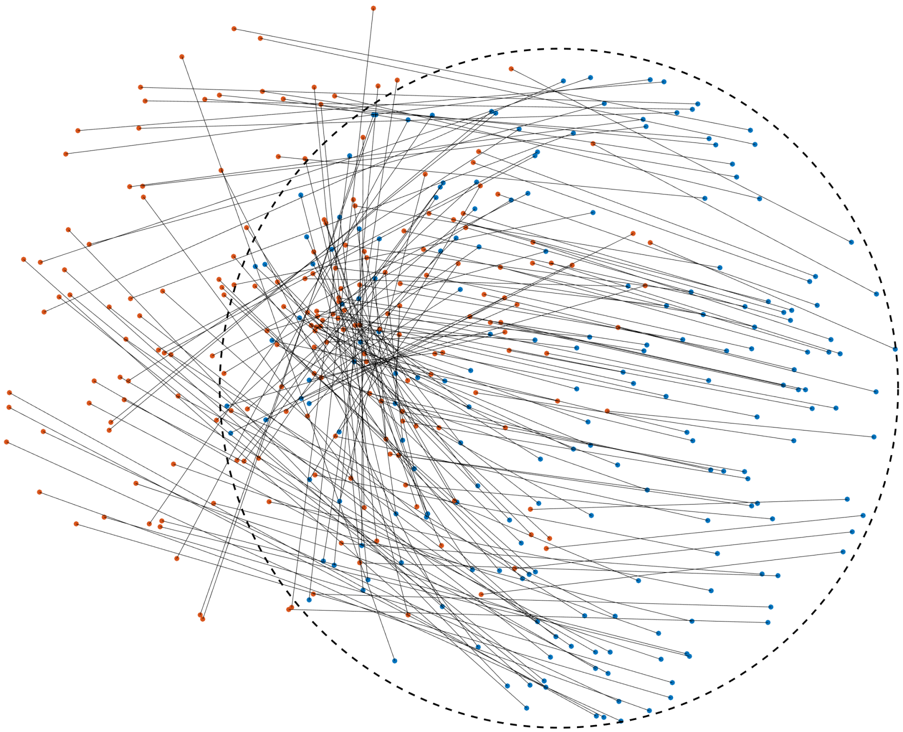
Approximation using 30 RBFs



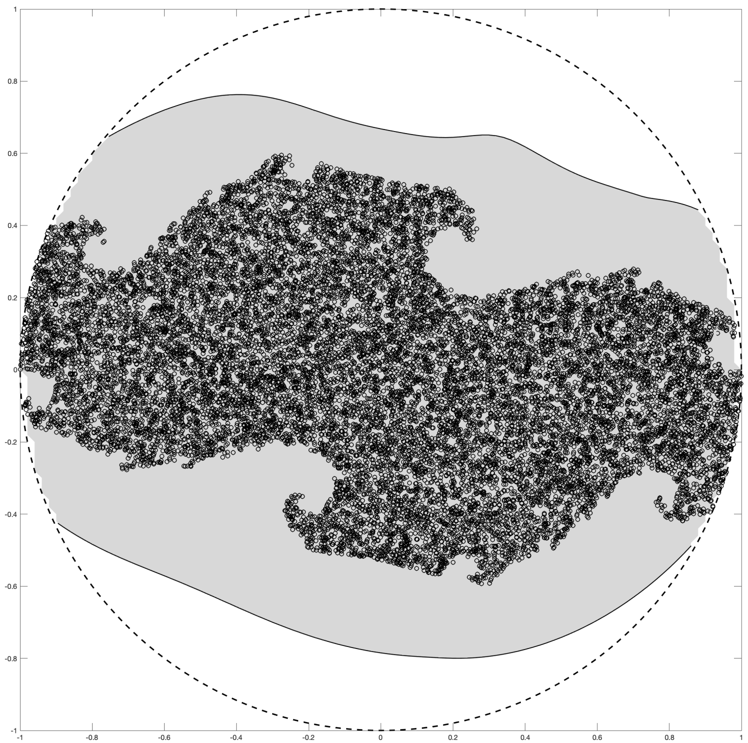
Numerics – Julia set – low data limit

 # Samples: 200

Data



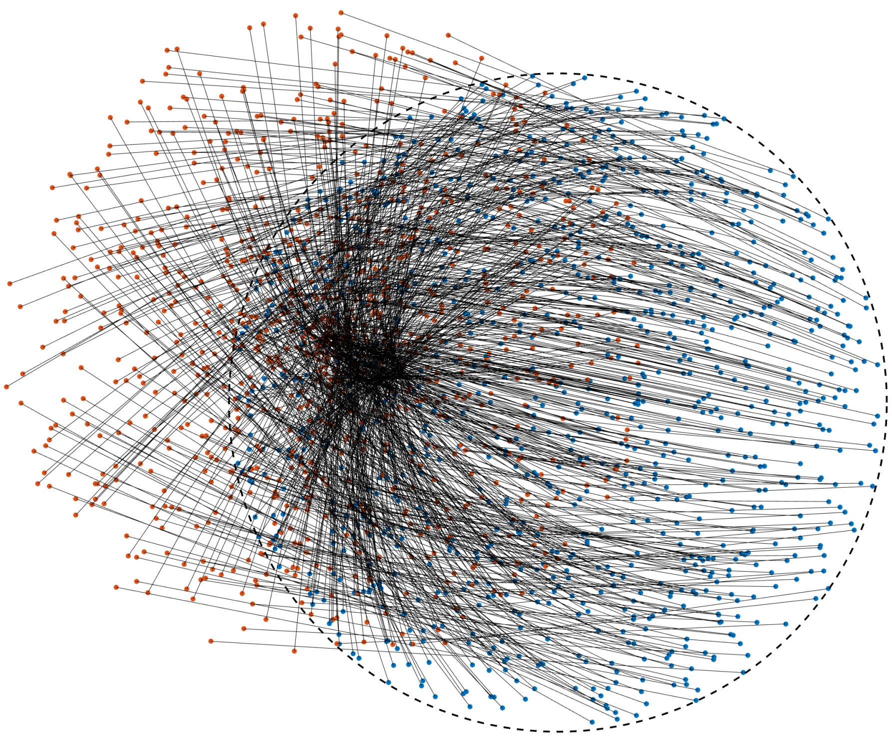
Approximation using 25 RBFs



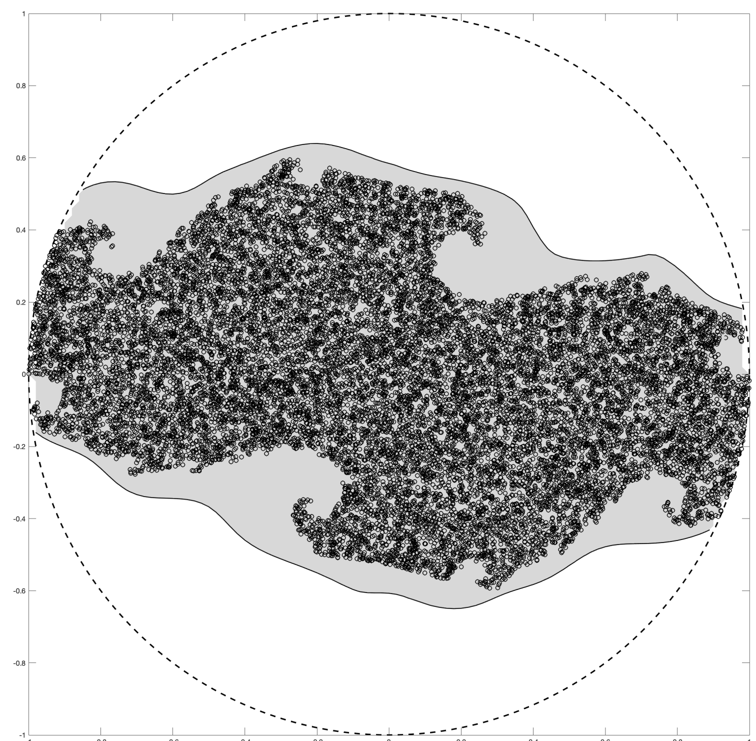
Numerics – Julia set – low data limit

 # Samples: 1000

Data

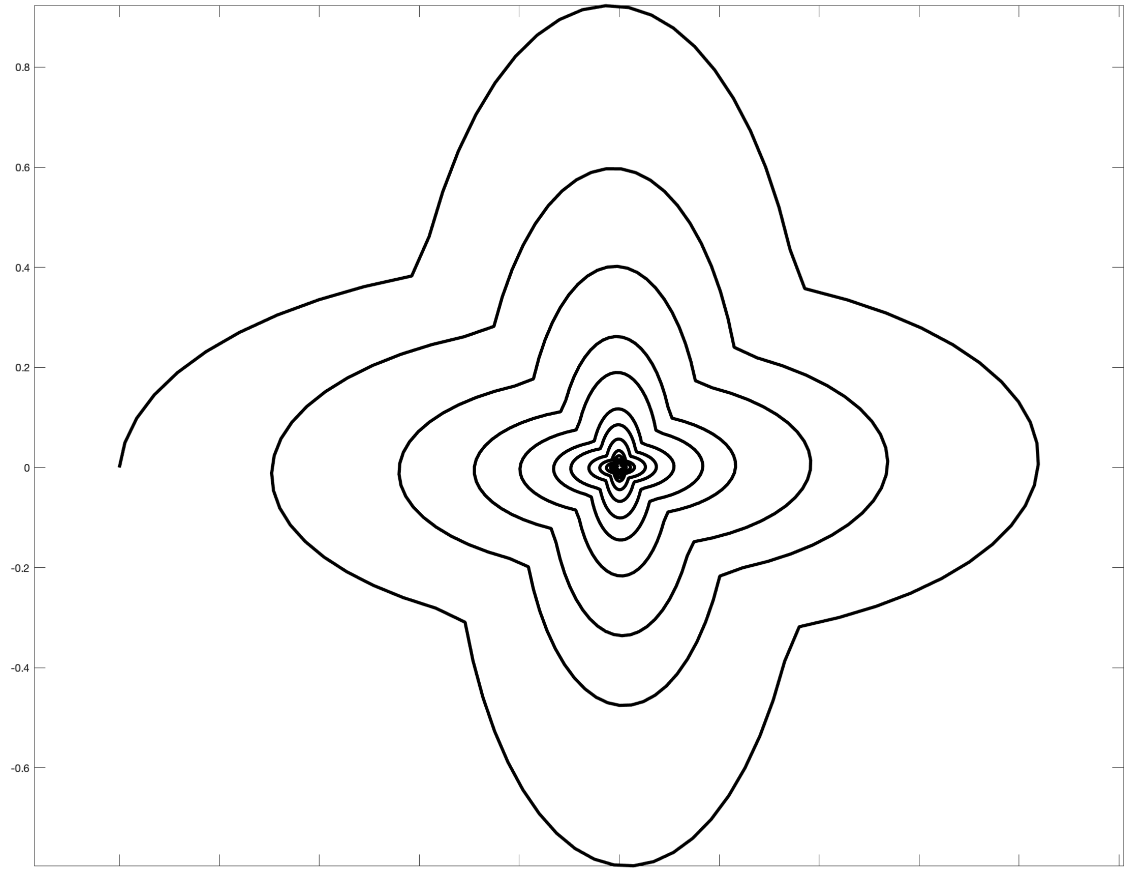


Approximation using 100 RBFs



Switched system

Flower system $\begin{cases} \dot{x} = A_1 x, & x \in \mathcal{X}_1 \\ \dot{x} = A_2 x, & x \in \mathcal{X}_2 \end{cases}$

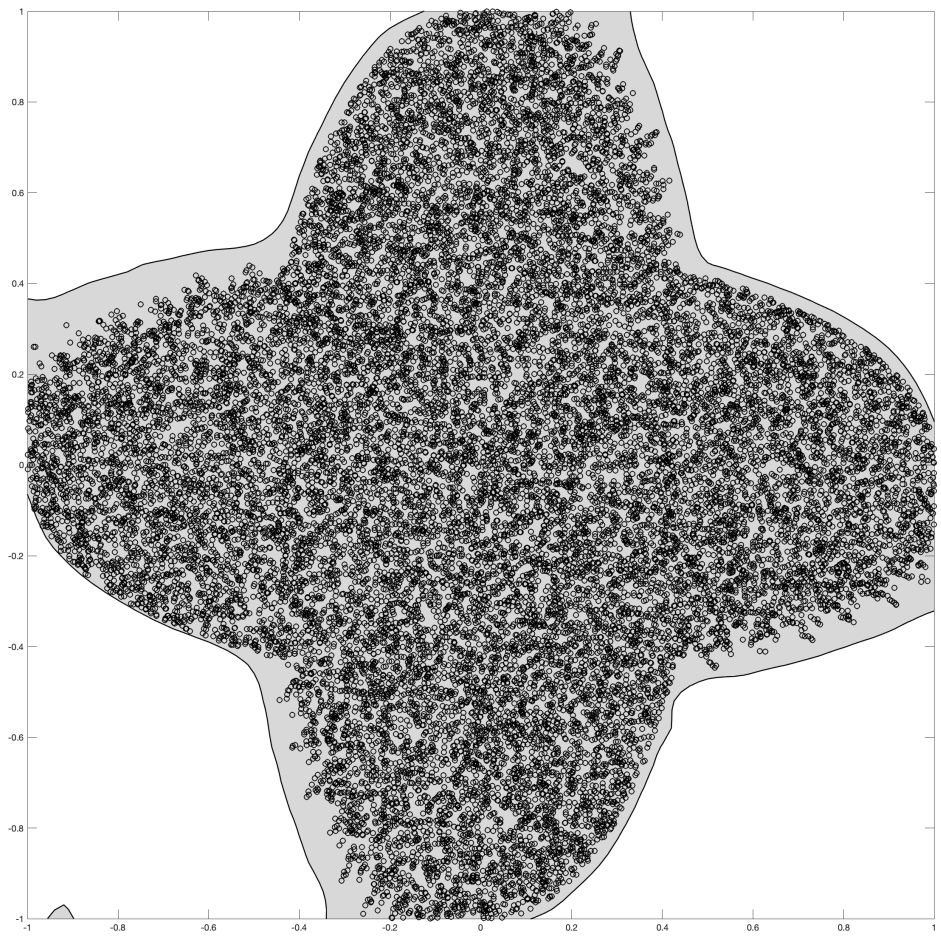


Switched system

Flower system $\begin{cases} \dot{x} = A_1 x, & x \in \mathcal{X}_1 \\ \dot{x} = A_2 x, & x \in \mathcal{X}_2 \end{cases}$

Basis: 400 RBFs

 # Samples: 10000



Switched system

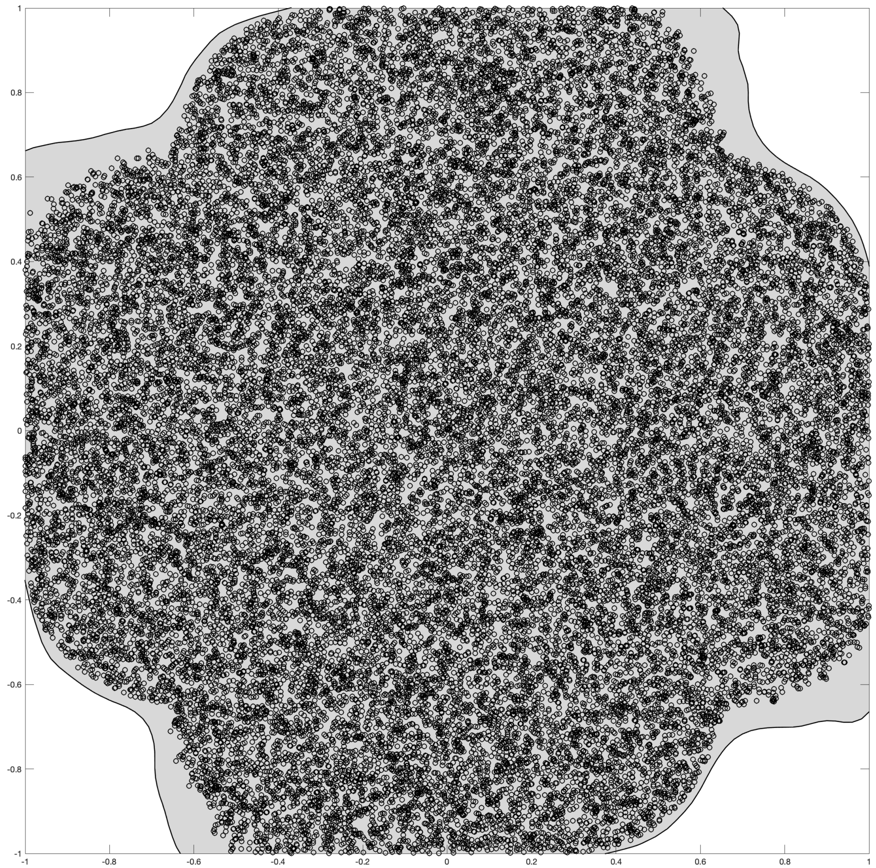
Modified flower system
$$\begin{cases} \dot{x} = A_1 \sin(x^3), & x \in \mathcal{X}_1 \\ \dot{x} = A_2 \sin(x^3), & x \in \mathcal{X}_2 \end{cases}$$

Switched system

Modified flower system $\begin{cases} \dot{x} = A_1 \sin(x^3), & x \in \mathcal{X}_1 \\ \dot{x} = A_2 \sin(x^3), & x \in \mathcal{X}_2 \end{cases}$

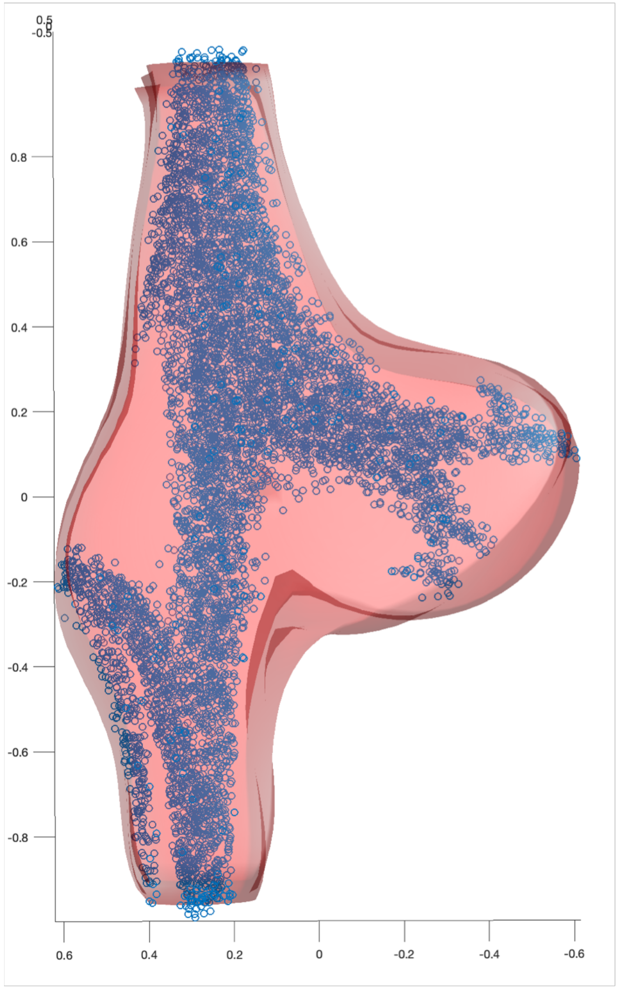
Basis: 400 RBFs

 # Samples: 10000

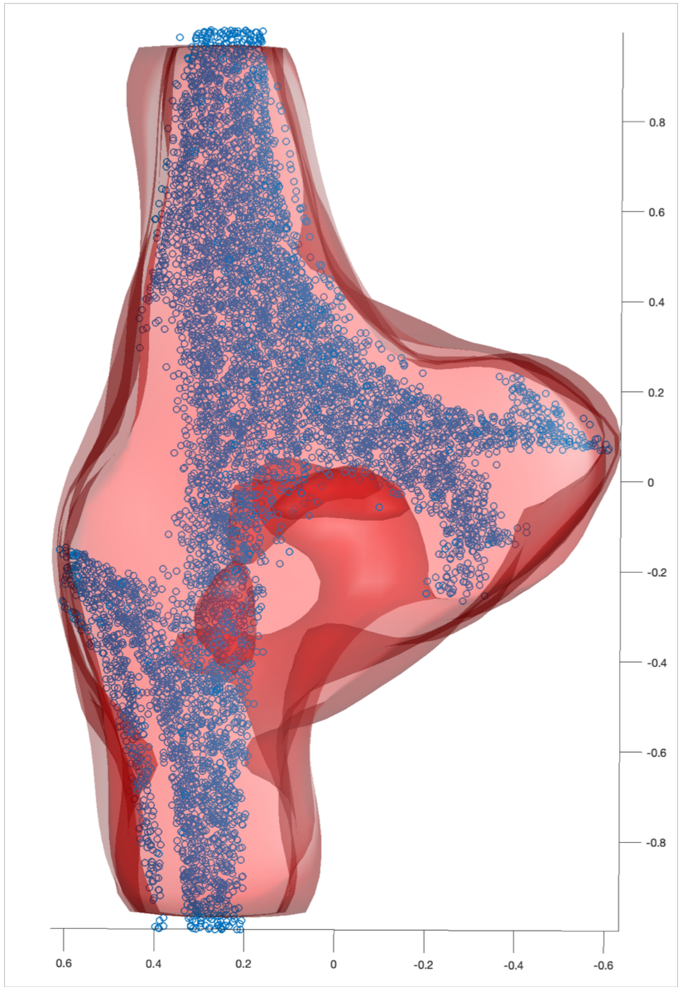


3D Hénon map

Basis: Monomials up to degree 10



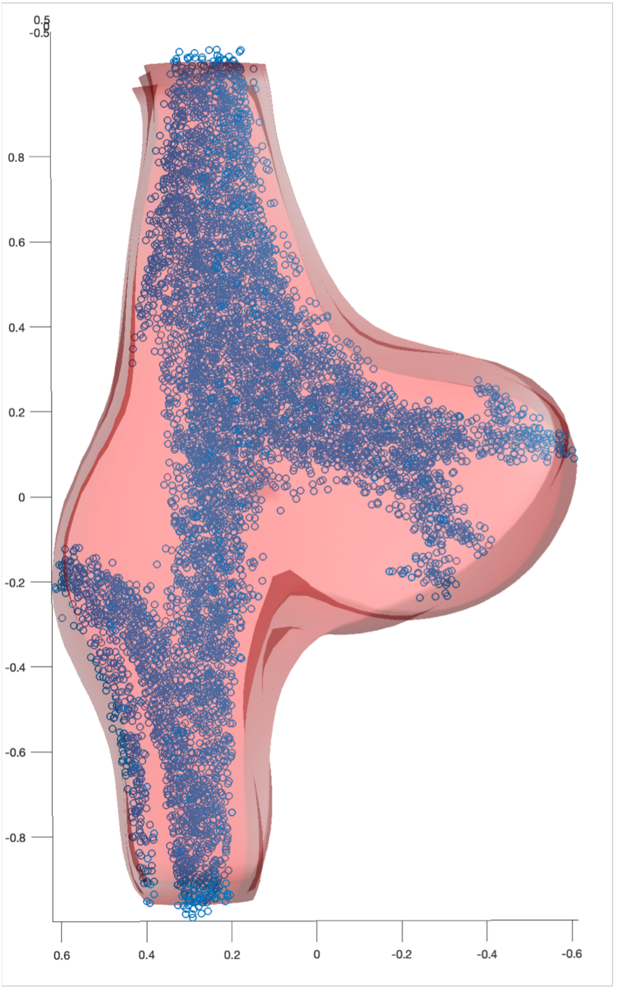
Basis: 286 RBFs



Thank you

3D Hénon map

Basis: Monomials up to degree 10



Basis: 200 RBFs