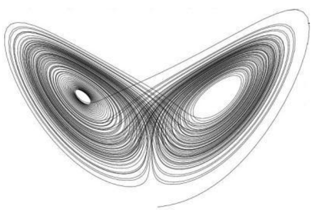
Koopman model predictive control of nonlinear dynamical systems

Milan Korda
(University of California, Santa Barbara)

Linear operator

$$\mathcal{K}g = g \circ f$$

$$\nearrow$$



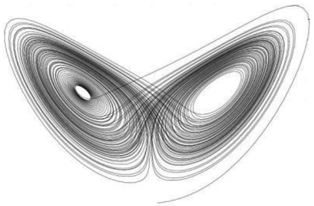
$$x^+ = f(x)$$

Nonlinear system

Linear operator

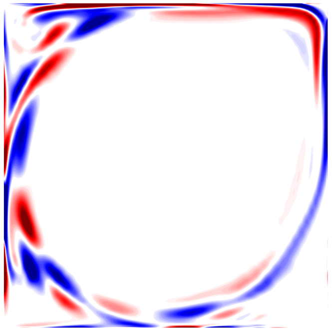
$$\mathcal{K}g = g \circ f$$

$\nearrow$ $\searrow$



$$x^+ = f(x)$$

Nonlinear system

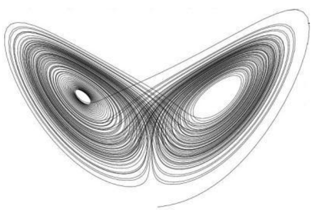


Modal analysis

Linear operator

$$\mathcal{K}g = g \circ f$$

$\nearrow$ $\searrow$



$$x^+ = f(x)$$

Nonlinear system

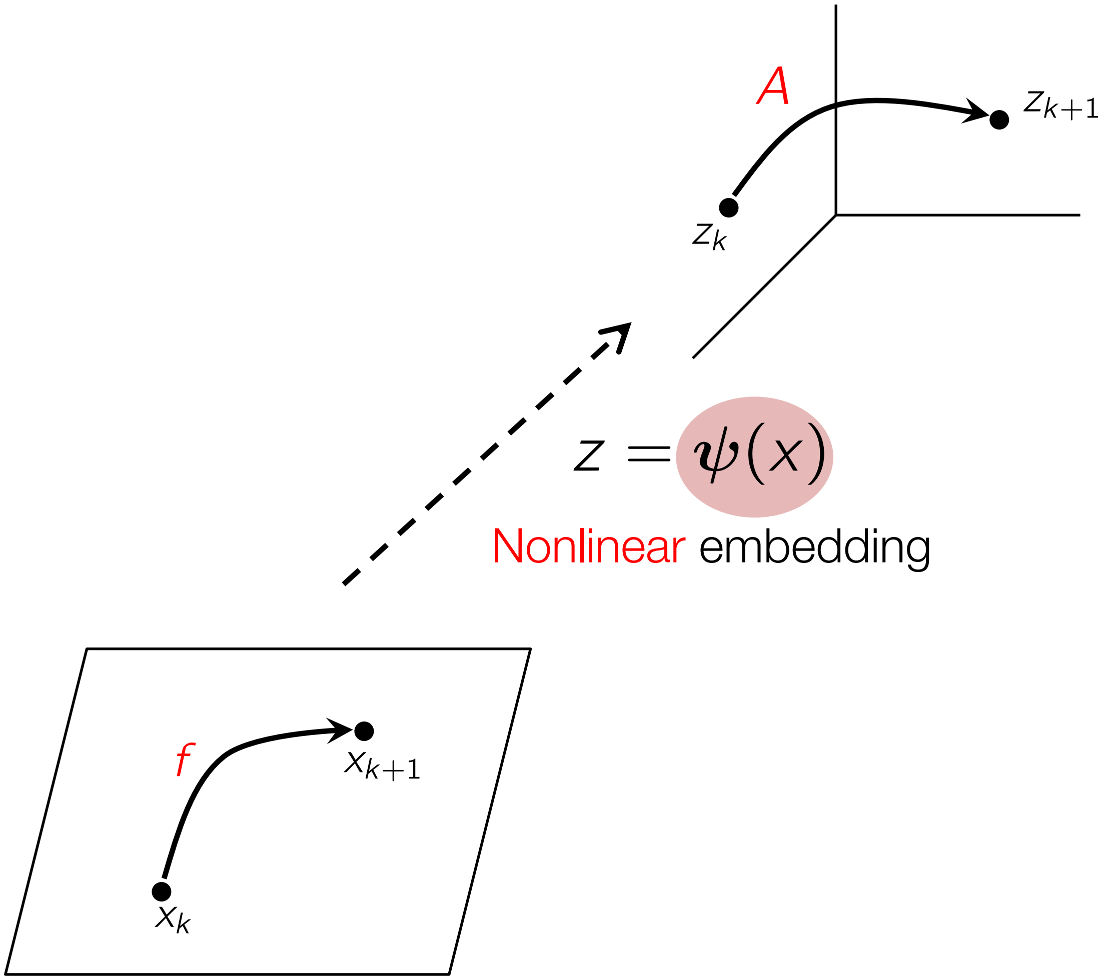
Prediction $\rightarrow$ Control

using **linear** techniques

Nonlinear embedding

Linear dynamics

$$z_{k+1} = Az_k$$

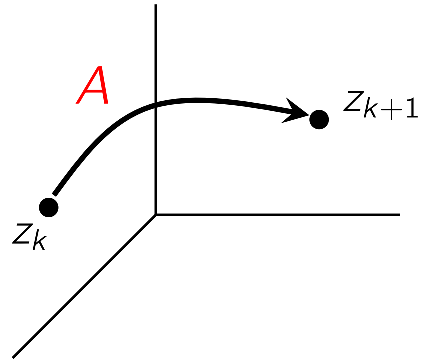


Nonlinear

Nonlinear embedding

Linear dynamics

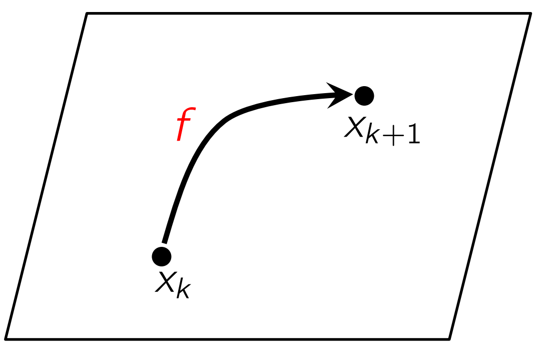
$$z_{k+1} = Az_k$$



$$z = \psi(x)$$

Nonlinear embedding

C



Nonlinear

Linear projection

$$\boldsymbol{\xi}(x_k) \approx Cz_k$$

$\boldsymbol{\xi}$ = vector of “observables”

(e.g. point measurements of velocity)

Linear predictor

$$
\begin{aligned}
z_{k+1} &= A z_k \\
z_0 &= \boldsymbol{\psi}(x_0) \\
\hat{y}_k &= C z_k
\end{aligned}
$$

$$
\hat{y}_k \approx \boldsymbol{\xi}(x_k)
$$

Why linear predictors?

$$
\begin{aligned}
z_{k+1} &= A z_k \\
z_0 &= \psi(x_0) \\
\hat{y}_k &= C z_k
\end{aligned}
$$

$$\hat{y}_k \approx \xi(x_k)$$

Nonlinear feedback control & estimation using linear techniques

$\Rightarrow$ Model predictive control $\rightarrow$ this talk

$\Rightarrow$ State estimation *[Surana & Banaszuk, 2016]*

Mature & well understood

Fast computation (linear algebra / convex optimization)

Rapid deployment in applications

Choosing the embedding

$$\begin{aligned} z_{k+1} &= A z_k \\ z_0 &= \boldsymbol{\psi}(x_0) \\ \hat{y}_k &= C z_k \end{aligned}$$

When can we predict exactly?

$$\hat{y}_k = \boldsymbol{\xi}(x_k)$$

Choosing the embedding

$$\begin{aligned} z_{k+1} &= A z_k \\ z_0 &= \boldsymbol{\psi}(x_0) \\ \hat{y}_k &= C z_k \end{aligned}$$

When can we predict exactly? $\hat{y}_k = \boldsymbol{\xi}(x_k)$

equality if and only if

$\mathrm{span}\{\psi_1, \ldots, \psi_N\}$ is Koopman invariant & $\boldsymbol{\xi} \in \mathrm{span}\{\psi_1, \ldots, \psi_N\}$

This talk: Assume $\boldsymbol{\psi}$ given

Constructing good $\boldsymbol{\psi}$: *[Korda, Mezić, in preparation]*

Extended dynamic mode decomposition

Data $(x_i)_{i=1}^K$ $(x_i^+)_{i=1}^K$ $x_i^+ = f(x_i)$

Basis functions $\psi = [\psi_1, \ldots, \psi_N]^\top$

LS problem

$$\min_{A \in \mathbb{R}^{N\times N}} \sum_{i=1}^{K} \|\boldsymbol{\psi}(x_i^+) - A\boldsymbol{\psi}(x_i)\|_2^2$$

LS problem

$$\min_{C \in \mathbb{R}^{N\times N}} \sum_{i=1}^{K} \|\boldsymbol{\xi}(x_i) - C\boldsymbol{\psi}(x_i)\|_2^2$$

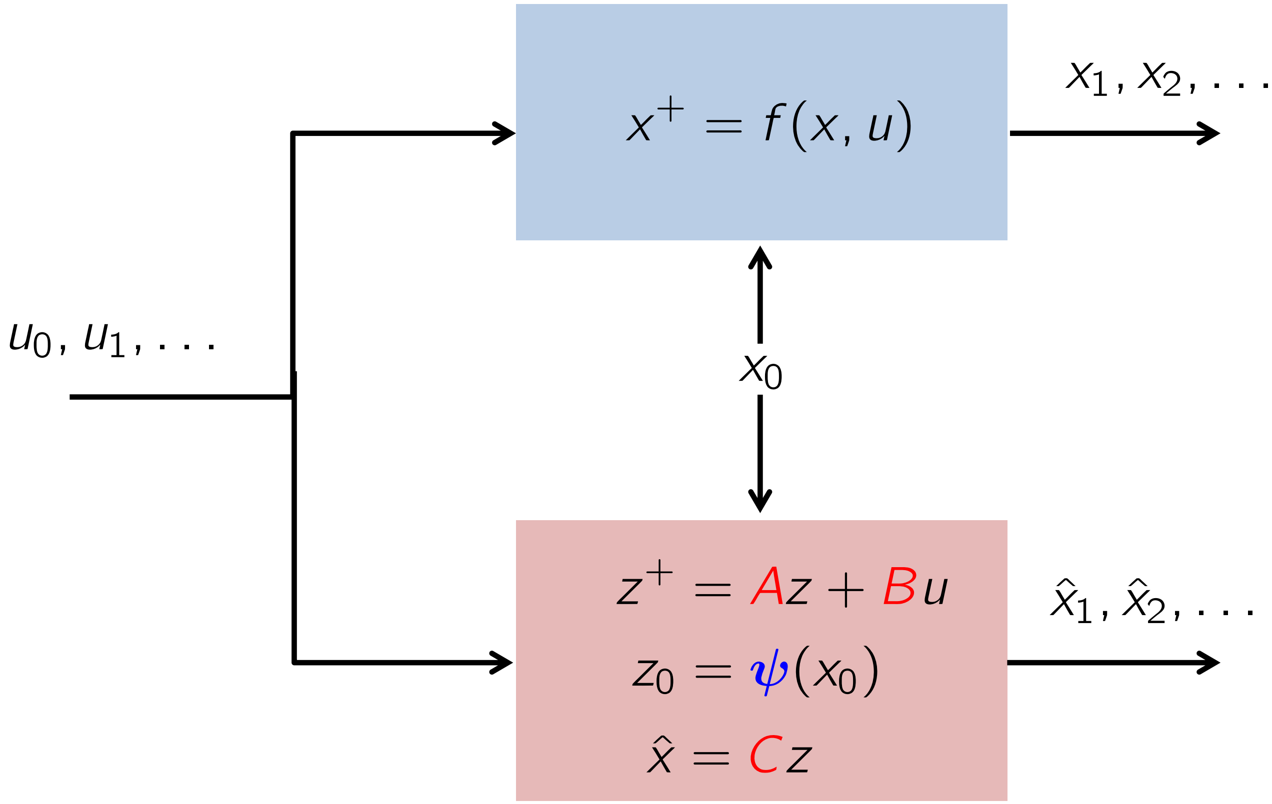
[Williams et al., 2015]

Control

(Joint work with Igor Mezić)

M. Korda, I. Mezić. Linear predictors for nonlinear dynamical systems: Koopman operator meets model predictive control. Automatica, 2018

Linear predictor



$$\psi : \mathbb{R}^n \to \mathbb{R}^N, \quad N \gg n$$

Koopman operator for controlled systems

$$x^+ = f(x, u),\; x \in \mathbb{R}^n,\; u \in \mathbb{R}^m$$

Koopman operator for controlled systems

$$x^+ = f(x,u),\ x \in \mathbb{R}^n,\ u \in \mathbb{R}^m$$

$$\Downarrow$$

$$\chi^+ = F(\chi) := \begin{bmatrix} f(x, \boldsymbol{u}(0)) \\ \mathcal{S}\boldsymbol{u} \end{bmatrix}$$

- Extended state $\chi := (x, \boldsymbol{u}) \in \mathcal{X} := \mathbb{R}^n \times \ell(\mathbb{R}^m)$
- Shift operator $(\mathcal{S}\boldsymbol{u})(i) = \boldsymbol{u}(i+1)$

Space of all control sequences $(u_i)_{i=0}^{\infty} =: \boldsymbol{u}$

Koopman operator for controlled systems

$$\boxed{x^+ = f(x,u),\ x \in \mathbb{R}^n,\ u \in \mathbb{R}^m}$$

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Space of all control sequences $(u_i)_{i=0}^{\infty} =: \boldsymbol{u}$

Koopman operator $\boxed{\mathcal{K}\phi = \phi \circ F}$ $\phi : \mathcal{X} \to \mathbb{R}$

Linear predictors from Koopman - EDMD

Data $\quad (x_i)_{i=1}^{K} \quad (x_i^+)_{i=1}^{K} \quad x_i^+ = F(x_i)$

LS problem

$$\min_{\mathcal{A}\in\mathbb{R}^{N_\phi\times N_\phi}} \sum_{i=1}^{K} \|\phi(x_i^+) - \mathcal{A}\phi(x_i)\|_2^2$$

$$\phi(x) = [\phi_1(x), \ldots, \phi_{N_\phi}(x)]^\top$$

Linear predictors from Koopman - EDMD

Data $(x_i)_{i=1}^{K}$ $(x_i^{+})_{i=1}^{K}$ $x_i^{+} = F(x_i)$

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$$\phi(x) = [\phi_1(x), \ldots, \phi_{N_\phi}(x)]^\top$$

linear operator

Predictor linear in u $\Rightarrow$ $\phi_i(x, \boldsymbol{u}) = \psi_i(x) + \mathcal{L}_i(\boldsymbol{u})$

Linear predictors from Koopman - EDMD

Data $(x_i)_{i=1}^{K}$ $(x_i^+)_{i=1}^{K}$ $x_i^+ = F(x_i)$

LS problem

$$\min_{\mathcal{A}\in\mathbb{R}^{N_\phi \times N_\phi}} \sum_{i=1}^{K} \|\phi(x_i^+) - \mathcal{A}\phi(x_i)\|_2^2$$

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linear operator

Predictor linear in u $\Rightarrow$ $\phi_i(x, \boldsymbol{u}) = \psi_i(x) + \mathcal{L}_i(\boldsymbol{u})$

Without loss of generality

$$\phi(x, \boldsymbol{u}) = [\psi_1(x), \ldots, \psi_N(x), \boldsymbol{u}(0)^\top]^\top$$

Linear predictors from Koopman - EDMD

Data $\quad (\chi_i)_{i=1}^K \quad (\chi_i^+)_{i=1}^K \quad \chi_i^+ = F(\chi_i)$

LS problem

$$\min_{\mathcal{A}\in\mathbb{R}^{N_\phi\times N_\phi}} \sum_{i=1}^{K} \|\phi(\chi_i^+) - \mathcal{A}\phi(\chi_i)\|_2^2$$

$$\phi(\chi) = [\phi_1(\chi), \ldots, \phi_{N_\phi}(\chi)]^\top$$

linear operator

Predictor linear in u $\quad \Rightarrow \quad \phi_i(x, \boldsymbol{u}) = \psi_i(x) + \mathcal{L}_i(\boldsymbol{u})$

Without loss of generality

$$\phi(x, \boldsymbol{u}) = [\psi_1(x), \ldots, \psi_N(x), \boldsymbol{u}(0)^\top]^\top$$

$$\min_{A\in\mathbb{R}^{N\times N},\, B\in\mathbb{R}^{N\times m}} \sum_{i=1}^{K} \|\boldsymbol{\psi}(x_i^+) - A\boldsymbol{\psi}(x_i) - B\boldsymbol{u}_i(0)\|_2^2$$

Algorithm summary

Data $\mathbf{X} = \left[x_1, \ \ldots, x_K\right], \ \mathbf{Y} = \left[x_1^+, \ldots, x_K^+\right], \mathbf{U} = \left[u_1, \ldots, u_K\right]$

Embedding $\mathbf{X}_{\text{lift}} = \left[\boldsymbol{\psi}(x_1), \ \ldots, \boldsymbol{\psi}(x_K)\right], \ \mathbf{Y}_{\text{lift}} = \left[\boldsymbol{\psi}(x_1^+), \ldots, \boldsymbol{\psi}(x_K^+)\right]$

LS problem $\min_{A,B} \|\mathbf{Y}_{\text{lift}} - A\mathbf{X}_{\text{lift}} - B\mathbf{U}\|_F \ , \quad \min_{C} \|\mathbf{X} - C\mathbf{X}_{\text{lift}}\|_F$

Solution $[A, B] = \mathbf{Y}_{\text{lift}}[\mathbf{X}_{\text{lift}}, \mathbf{U}]^\dagger, \quad C = \mathbf{X}\mathbf{X}_{\text{lift}}^\dagger$

$$
\begin{aligned}
z^+ &= Az + Bu \\
\hat{x} &= Cz \\
z_0 &= \boldsymbol{\psi}(x_0)
\end{aligned}
$$

MPC design

Koopman MPC

Nonlinear MPC

$$
\begin{array}{lll}
\underset{u_i, x_i}{\text{minimize}} & \sum_{i=0}^{N_p-1} l_x(x_i) + u_i^\top R u_i + r^\top u_i & \\
\text{subject to} & x_{i+1} = f(x_i, u_i), & i = 0, \ldots, N_p - 1 \\
 & c_x(x_i) + C_u u_i \leq b, & i = 0, \ldots, N_p - 1 \\
\text{parameter} & x_0 = x &
\end{array}
$$

Nonconvex

x

$$\kappa(x) = \{u_0^\star, u_1^\star, \ldots, u_{N_p-1}^\star\} \longrightarrow \boxed{x^+ = f(x, u)}$$

Koopman MPC

Koopman MPC

$$\begin{array}{lll} \underset{u_i, z_i}{\text{minimize}} & \sum_{i=0}^{N_p-1} z_i^\top Q z_i + u_i^\top R u_i + q^\top z_i + r^\top u_i & \\ \text{subject to} & z_{i+1} = {\color{red}A} z_i + {\color{red}B} u_i, & i = 0, \ldots, N_p - 1 \\ & E z_i + F u_i \le b, & i = 0, \ldots, N_p - 1 \\ \text{parameter} & {\color{blue} z_0 = \psi(x)} & \end{array}$$

Convex

$$x$$

$$\kappa(x) = \{u_0^\star, u_1^\star, \ldots, u_{N_p-1}^\star\} \longrightarrow \quad x^+ = f(x, u)$$

Can handle **nonlinear constraints** and **costs** in a linear fashion

Koopman MPC

Dense-form Koopman MPC

$$
\begin{array}{ll}
\underset{\mathbf{u}\in\mathbb{R}^{mN_p}}{\text{minimize}} & \mathbf{u}^\top H\mathbf{u}^\top + h^\top\mathbf{u} + z_0^\top G\mathbf{u} \\
\text{subject to} & L\mathbf{u} + Mz_0 \le c \\
\text{parameter} & z_0 = \boldsymbol{\psi}(x)
\end{array}
$$

x

$$
\kappa(x) = \begin{bmatrix} u_0^\star \\ \vdots \\ u_{N_p-1}^\star \end{bmatrix} \longrightarrow \boxed{x^+ = f(x,u)}
$$

Computation cost independent of the size of the embedding!

Computation cost **independent** of the size of the embedding!

Koopman MPC summary

At each step k of closed-loop operation

- Set $z_0 = \boldsymbol{\psi}(x_k)$

- Solve

$$\begin{array}{ll} \underset{\mathbf{u}\in\mathbb{R}^{mN_p}}{\text{minimize}} & \mathbf{u}^\top H \mathbf{u}^\top + h^\top \mathbf{u} + z_0^\top G \mathbf{u} \\ \text{subject to} & L\mathbf{u} + M z_0 \leq c \end{array} \quad \Rightarrow \quad \mathbf{u}^\star = \begin{bmatrix} u_0^\star \\ \vdots \\ u_{N_p-1}^\star \end{bmatrix}$$

- Apply $u_0^\star$ to the system

Koopman MPC summary

At each step k of closed-loop operation

- Set $z_0 = \boldsymbol{\psi}(x_k)$

- Solve

$$\begin{array}{ll} \underset{\mathbf{u}\in\mathbb{R}^{mN_p}}{\text{minimize}} & \mathbf{u}^\top H \mathbf{u}^\top + h^\top \mathbf{u} + z_0^\top G \mathbf{u} \\ \text{subject to} & L\mathbf{u} + M z_0 \leq c \end{array} \quad \Rightarrow \quad \mathbf{u}^\star = \begin{bmatrix} u_0^\star \\ \vdots \\ u_{N_p-1}^\star \end{bmatrix}$$

- Apply $u_0^\star$ to the system

Main benefits

Data-driven: No model required

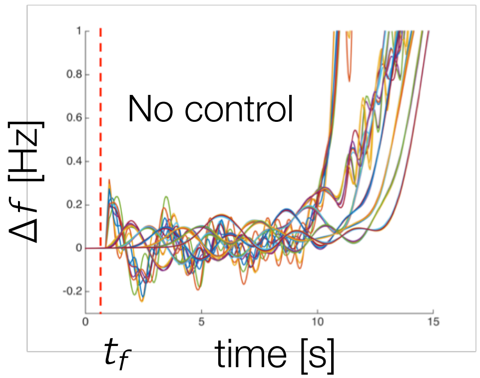
Fast & **simple**: only small **convex quadratic program** solved online

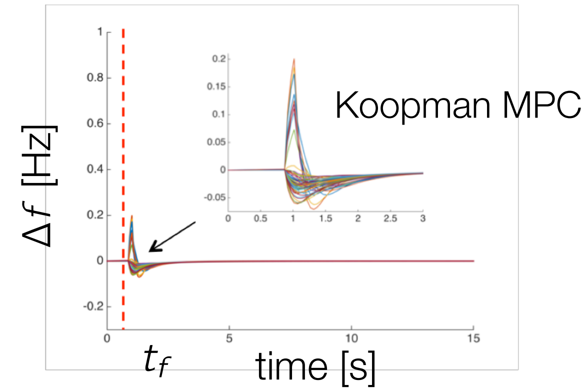
Nonlinear constraints and **costs** handled in a linear fashion

Koopman MPC - applications

Powergrid

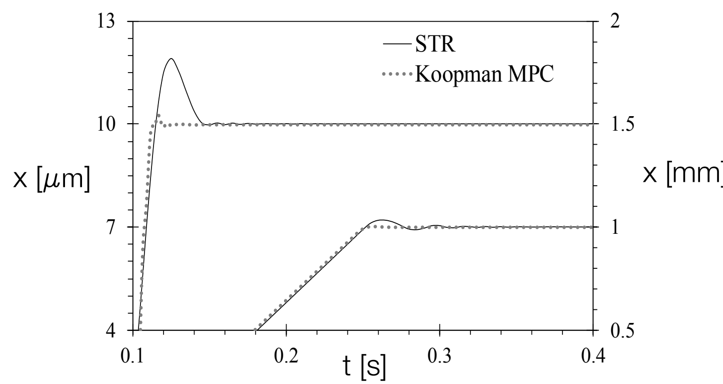
[Korda et al. 2017]





High-precision positioning

[Kamenar et al. 2018]



Fluids control →

Fluids control

(Joint work with Hassan Arbabi and Igor Mezić)

Burgers' equation

$$\frac{\partial y}{\partial t} - \nu \frac{\partial^2 y}{\partial x^2} + y \frac{\partial y}{\partial x} = u(t, x)$$

$y(x, 0) = y_0(x)$, periodic boundary

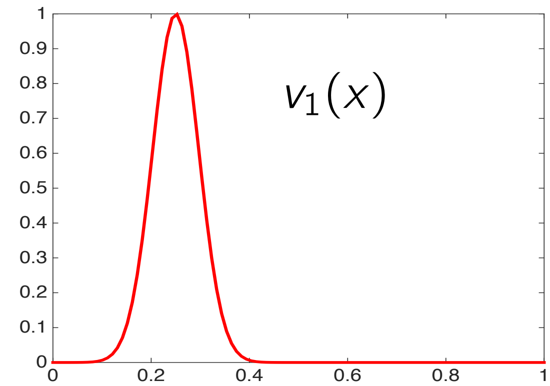
Burgers' equation

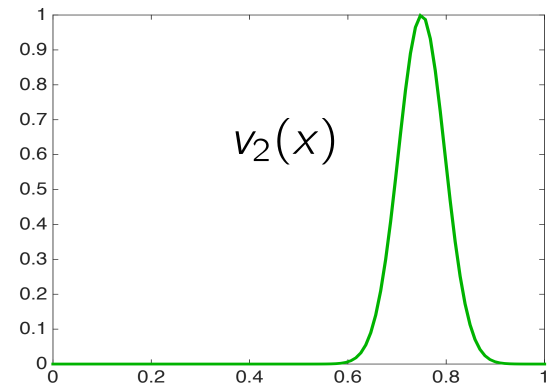
$$\frac{\partial y}{\partial t} - \nu \frac{\partial^2 y}{\partial x^2} + y \frac{\partial y}{\partial x} = u(t,x)$$

$y(x,0) = y_0(x)$, periodic boundary

Setup from *[Peitz, Klus 2017]*

$$u(t,x) = u_1(t) v_1(x) + u_2(t) v_2(x)$$





Burgers' equation

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$y(x, 0) = y_0(x)$, periodic boundary

$$u(t, x) = u_1(t) v_1(x) + u_2(t) v_2(x)$$

$$|u_1(t)| \leq 0.1 \ , \quad |u_2(t)| \leq 0.1$$

Tracking piecewise-constant reference

Burgers' equation

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$y(x,0) = y_0(x)$, periodic boundary

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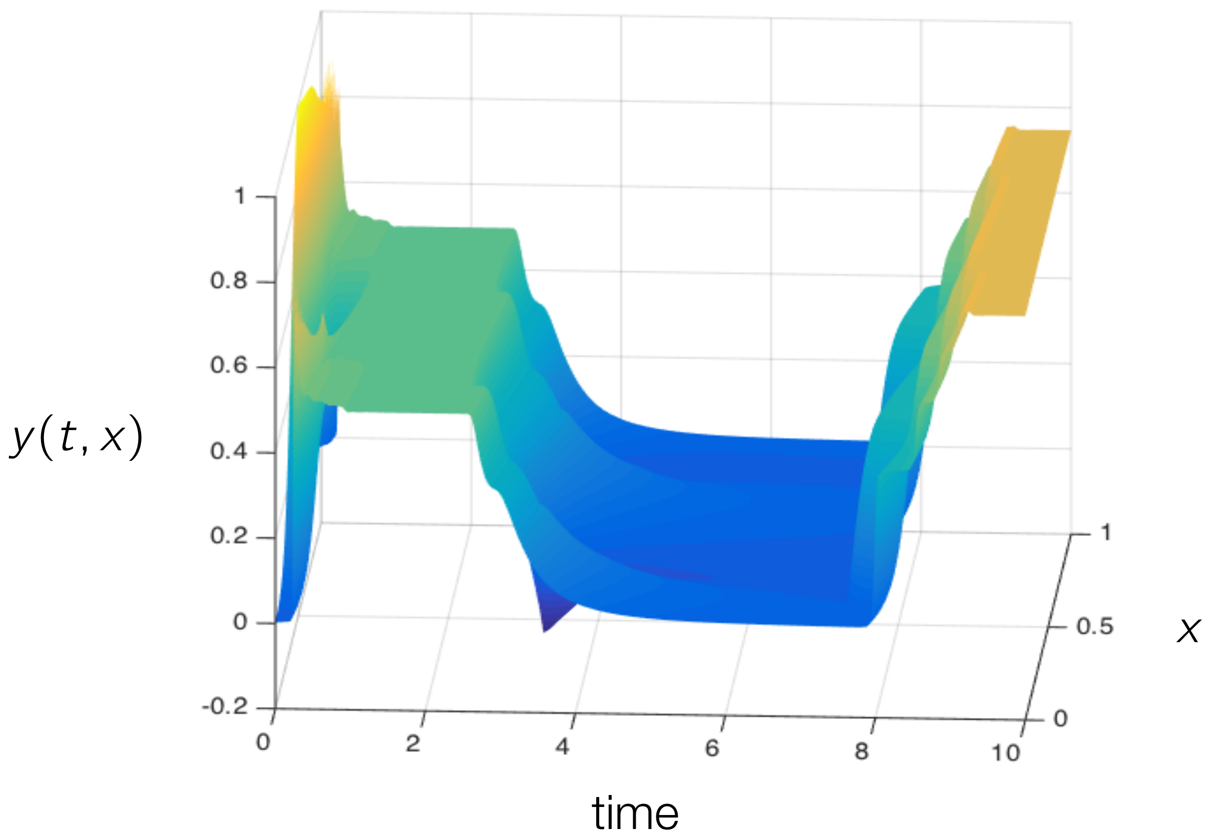

Burgers' equation

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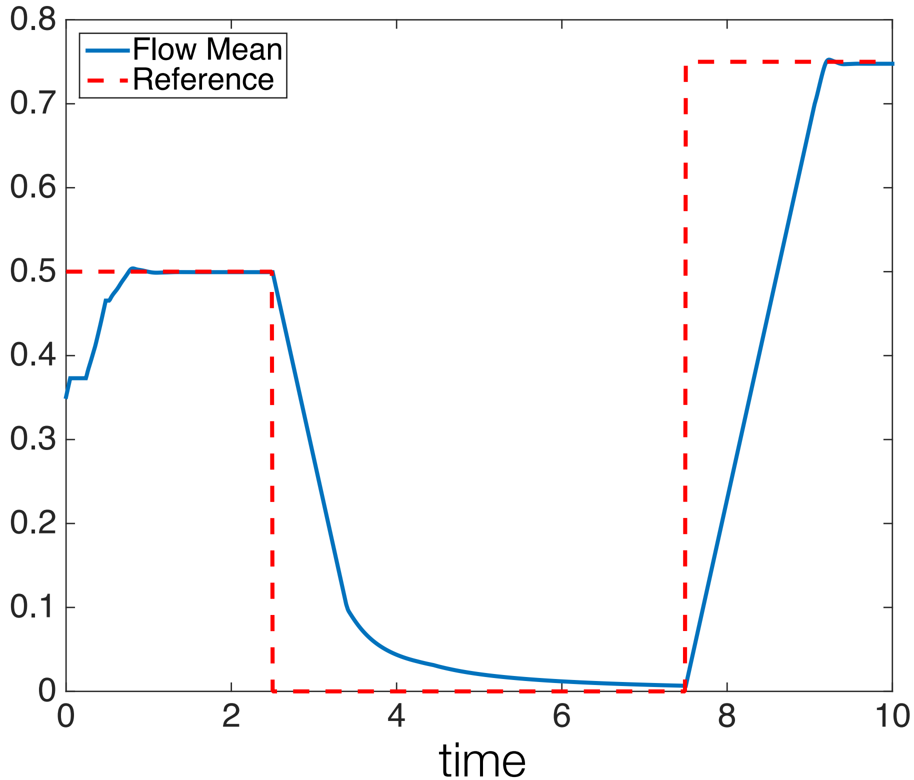
$$|u_1(t)| \leq 0.1 \,, \quad |u_2(t)| \leq 0.1$$



Burgers' equation

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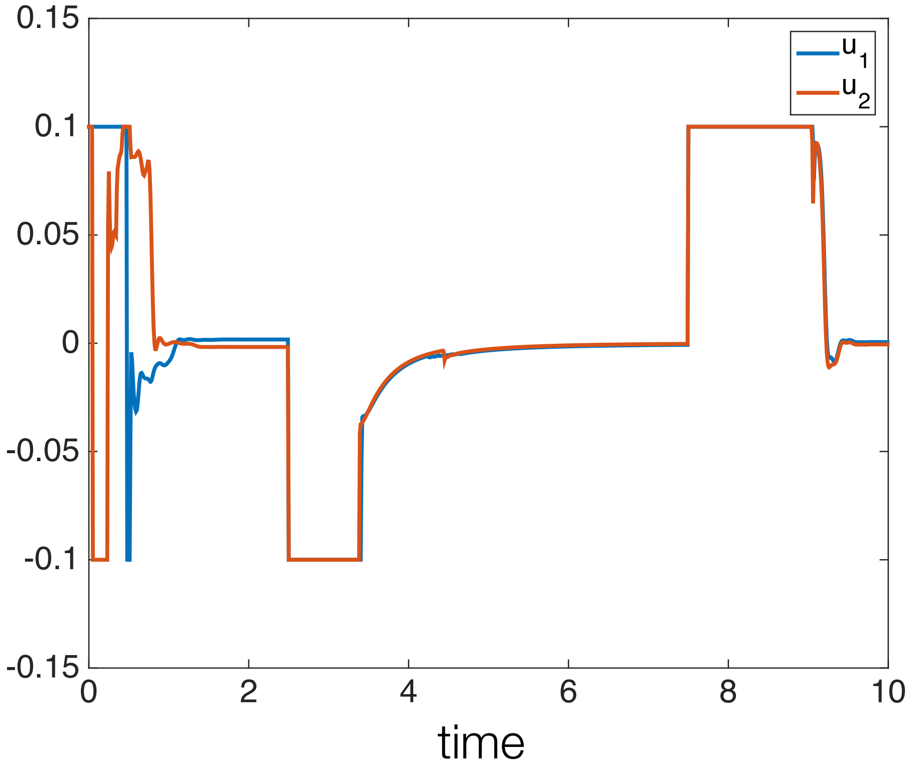
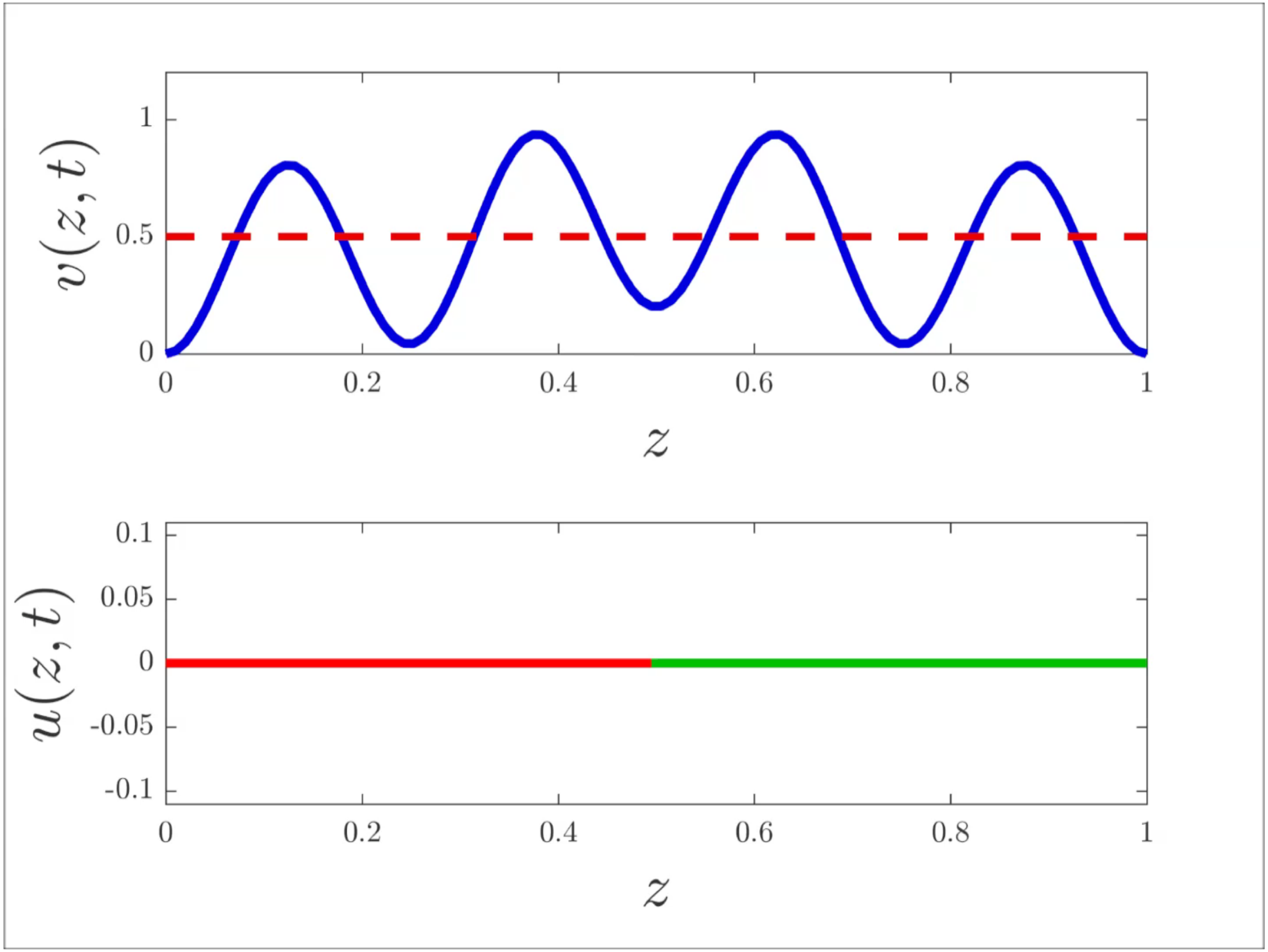
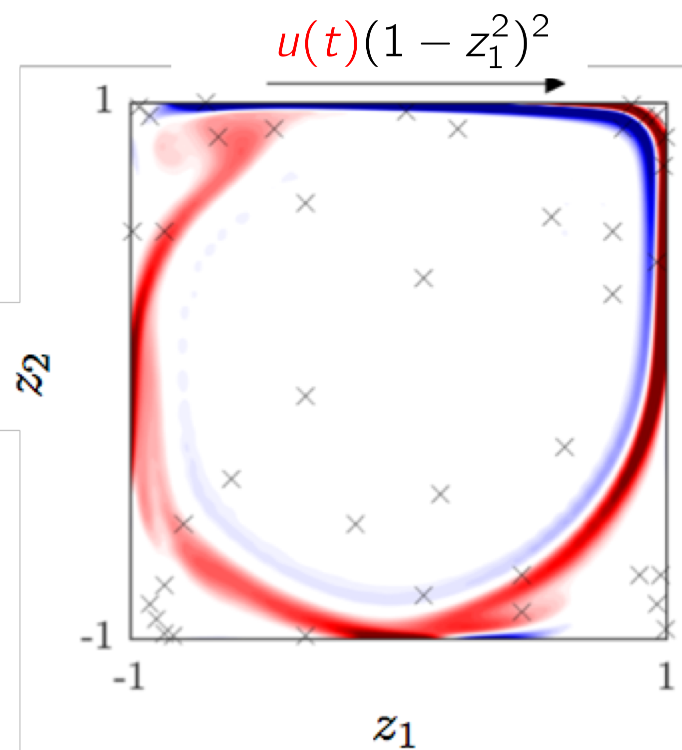




1
0.5
0
$v(z,t)$
0
0.2
0.4
0.6
0.8
1
z
0.1
0.05
0
-0.05
-0.1
$u(z,t)$
0
0.2
0.4
0.6
0.8
1
z

Lid-driven cavity flow – problem setup

$$\frac{\partial v}{\partial t} + v \cdot \nabla v = -\nabla p + \frac{1}{Re}\nabla^2 v, \quad \nabla \cdot v = 0$$



Control input: Amplitude of top lid velocity

Measurements: Velocity at randomly selected points

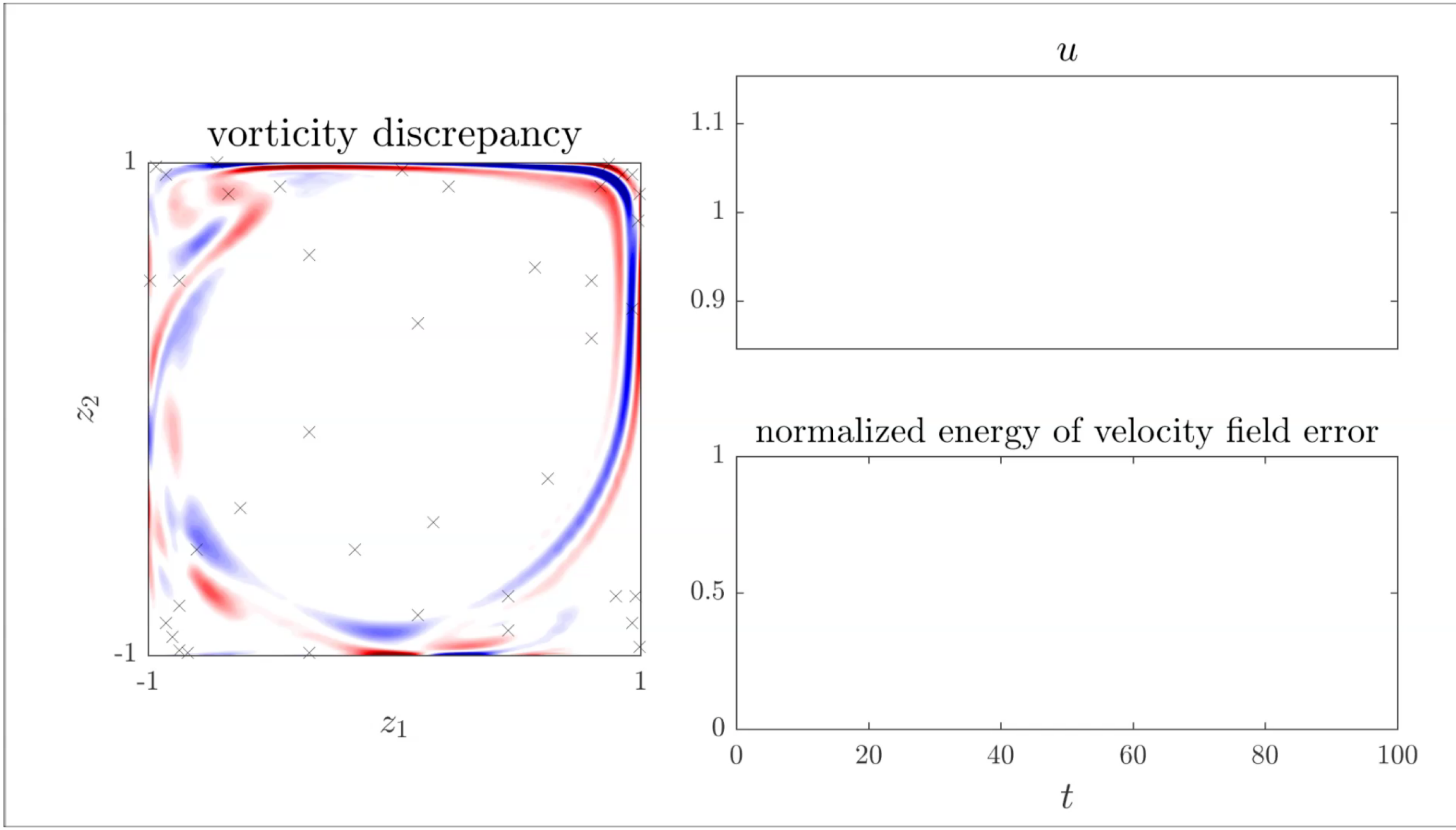
Training data: 300 two-second-long trajectories

Control task: Re 13k (limit cycle) → Re 10k (stable fixed point)

Re 13k (limit cycle) → mean flow (unstable fixed point)

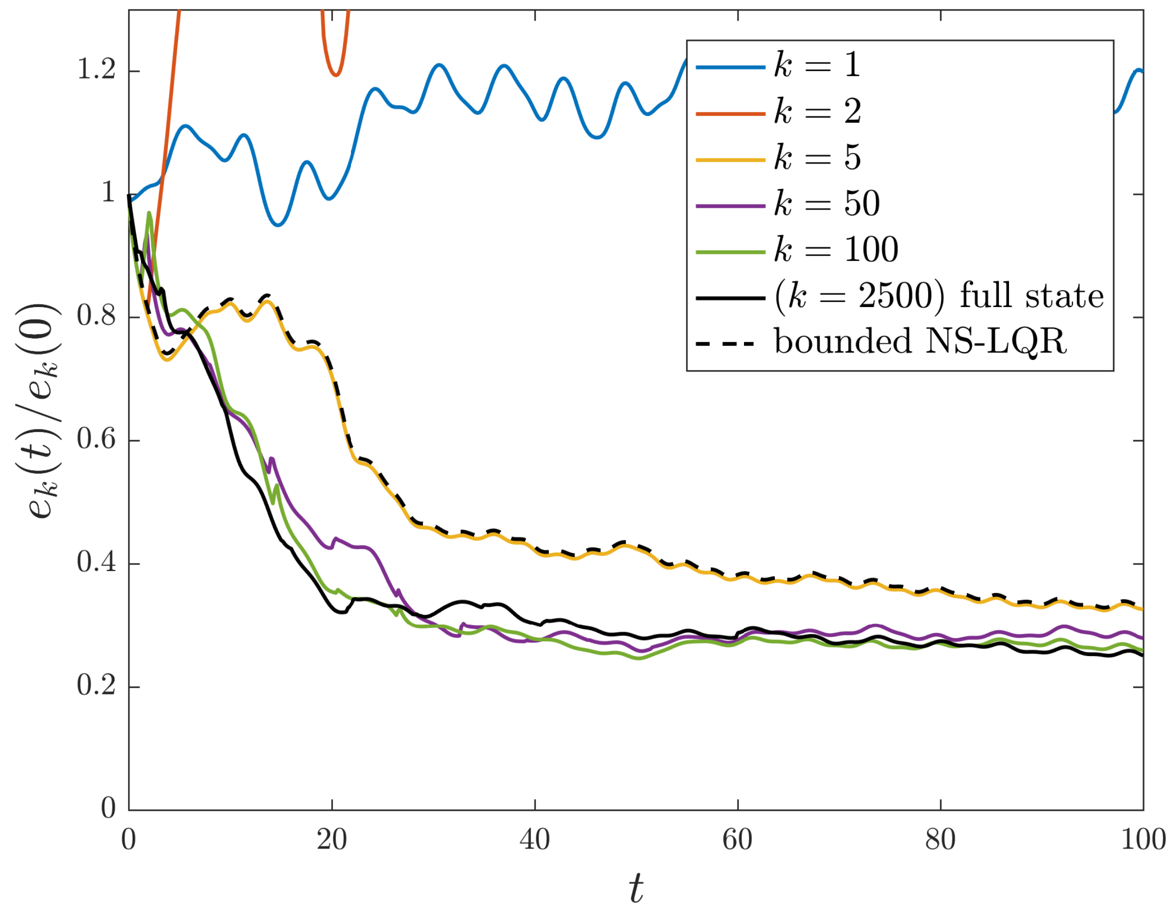
Control performance

Control task: Re 13k (limit cycle) → Re 10k (stable fixed point)



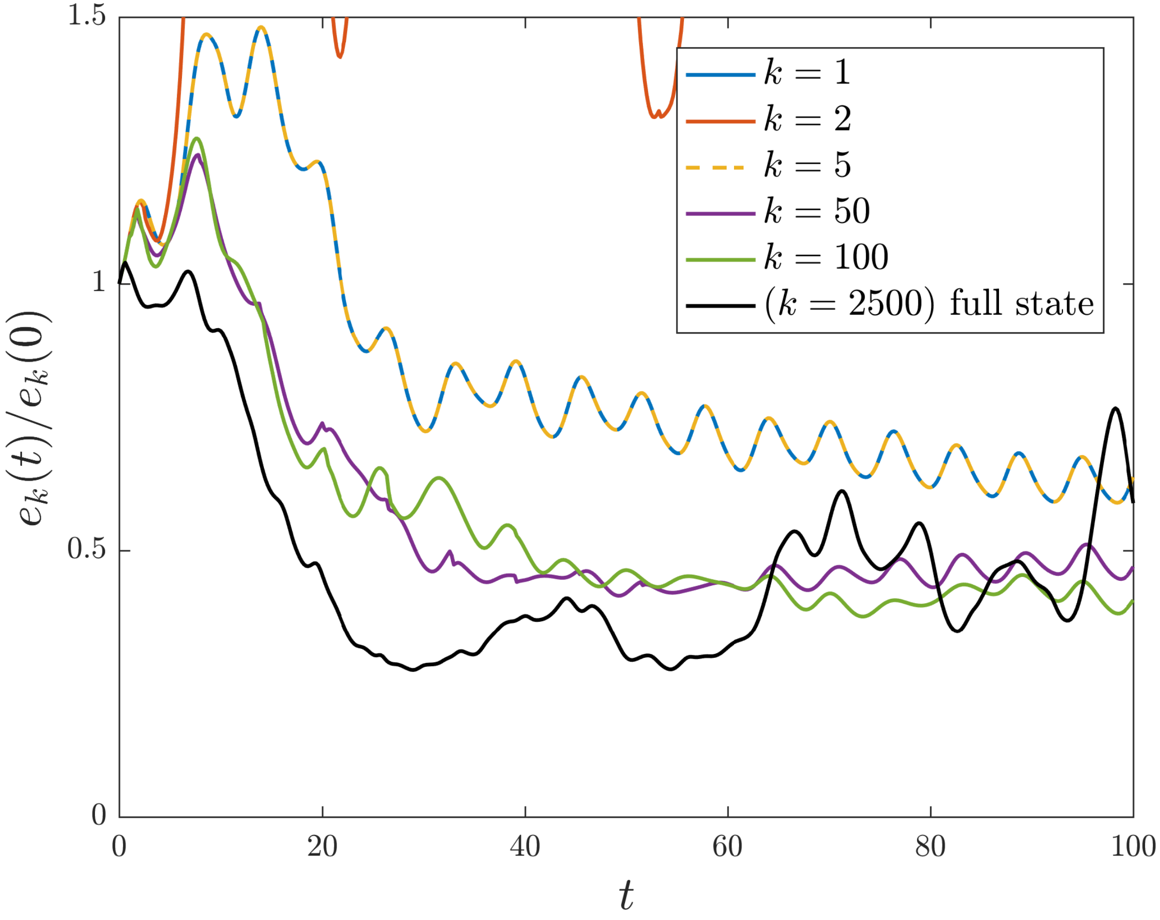
Comparison for different # of measurements

Control task: Re 13k (limit cycle) → Re 10k (locally stable fixed point)



Comparison for different # of measurements

Control task: Re 13k (limit cycle) $\rightarrow$ Mean flow (unstabilizable fixed point)



Data-driven + fast and scalable

Computation time: 10^{-4} second per step

Open problems

- Accuracy of the predictors for finite N

- Choice of the embedding ψ

- Guarantees on the controllers (stability, optimality)

Thank you

Papers & **Code**

(1) M. Korda, I. Mezić. Linear predictors for nonlinear dynamical systems: Koopman operator meets model predictive control. Automatica, 2018

(2) H. Arbabi, M. Korda, I. Mezić. A data-driven Koopman model predictive control framework for nonlinear flows, arxiv 2018.

Power grid stabilization

(Join work with Yoshi Susuki)

Problem setup

New England power grid model

$$\dot{\delta}_i = \omega_i$$

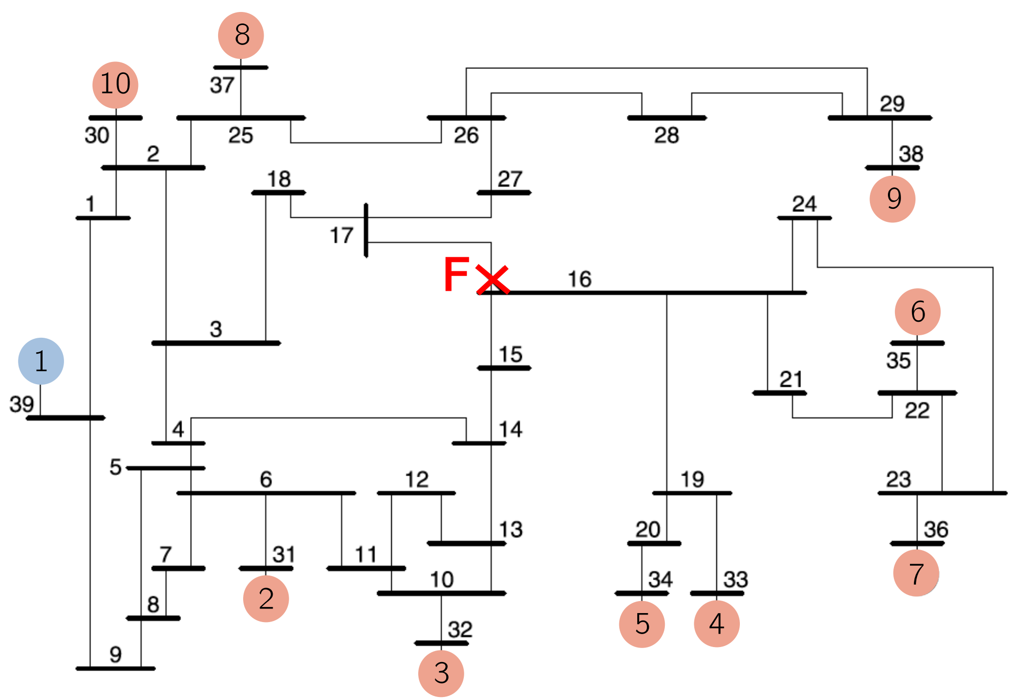
$$\frac{H_i}{\pi f_\mathrm{b}}\dot{\omega}_i = -D_i\omega_i + P_{\mathrm{m}_i}$$

$$-G_{ii}V_i^2 - \sum_{j=1, j\neq i}^{10} V_iV_j\{G_{ij}\cos(\delta_i-\delta_j) + B_{ij}\sin(\delta_i-\delta_j)\}$$

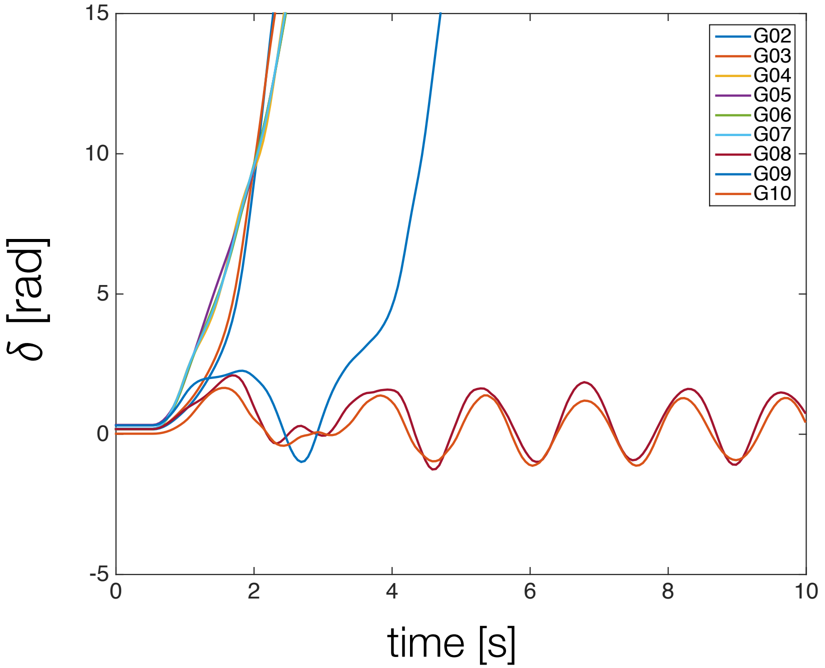
Setup from *[Susuki et al, 2011]*

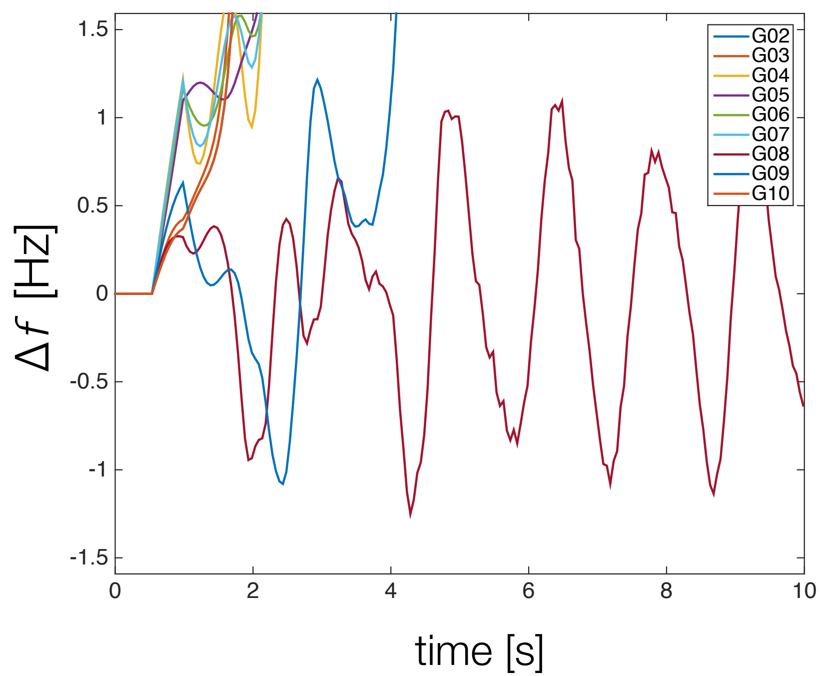
$t = 0.67\,\mathrm{s}$ – fault occurs

$t = 1\,\mathrm{s}$ – faulted line removed



Fault causes instability





Setting up Koopman MPC

New England power grid model

$$\dot{\delta}_i = \omega_i$$

$$\frac{H_i}{\pi f_{\mathrm{b}}}\dot{\omega}_i = -D_i\omega_i + P_{\mathrm{m}_i}$$
$$-G_{ii}V_i^2 - \sum_{j=1, j\neq i}^{10} V_i V_j \{G_{ij}\cos(\delta_i - \delta_j) + B_{ij}\sin(\delta_i - \delta_j)\}$$

Actuation: P_{m_i} mechanical power

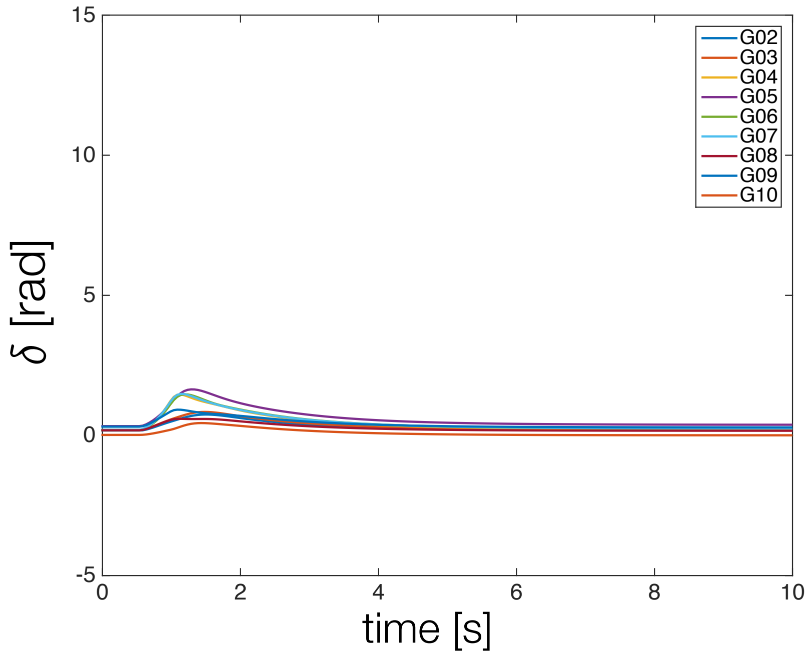
Cost: $\sum_i \omega_i^2$ – frequency deviation

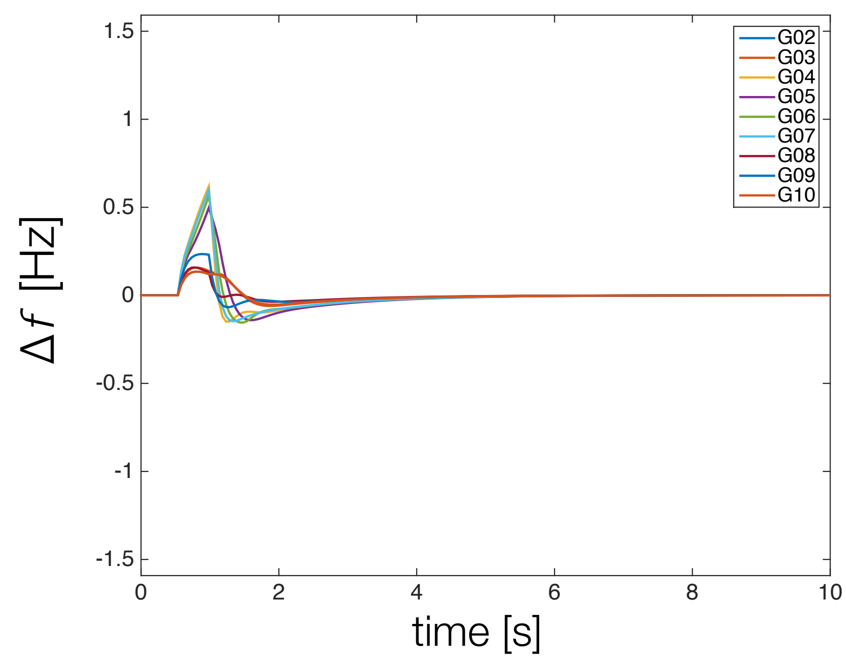
Pred. horizon: 1 second

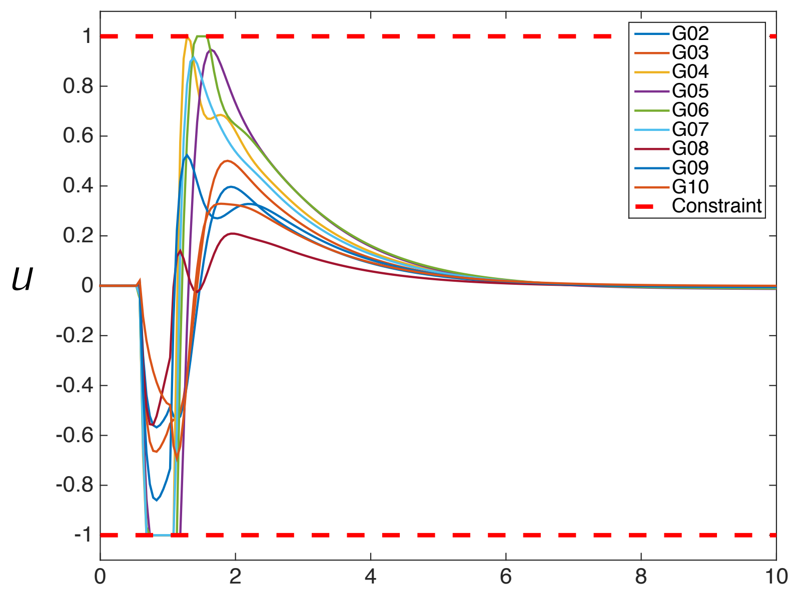
Sampling: 50 ms

Embedding: $\psi = \begin{bmatrix} \cos(\delta) \\ \sin(\delta) \\ \omega \end{bmatrix}$ $\qquad \psi : \mathbb{R}^{18} \to \mathbb{R}^{27}$

Instability suppression





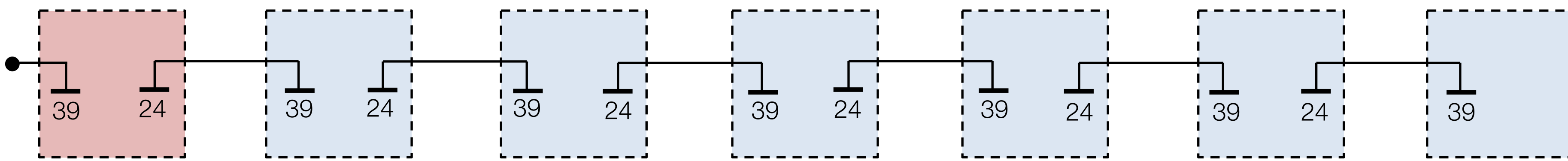


NE grid cascade

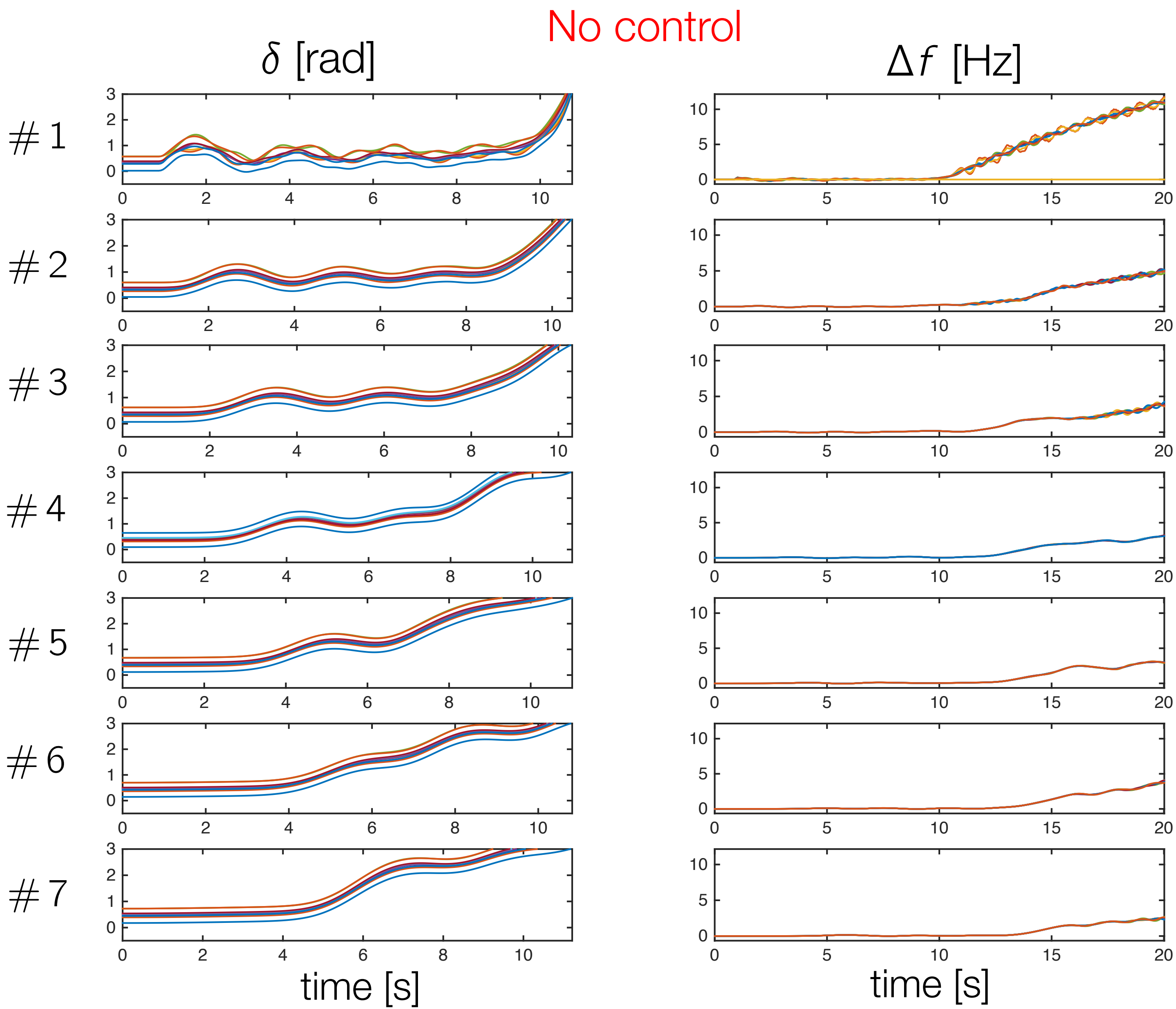
$t = 0.87\,\mathrm{s}$ – fault occurs in grid $\#1$

$t = 1\,\mathrm{s}$ – faulted line removed

Setup from *[Susuki et al, 2012]*

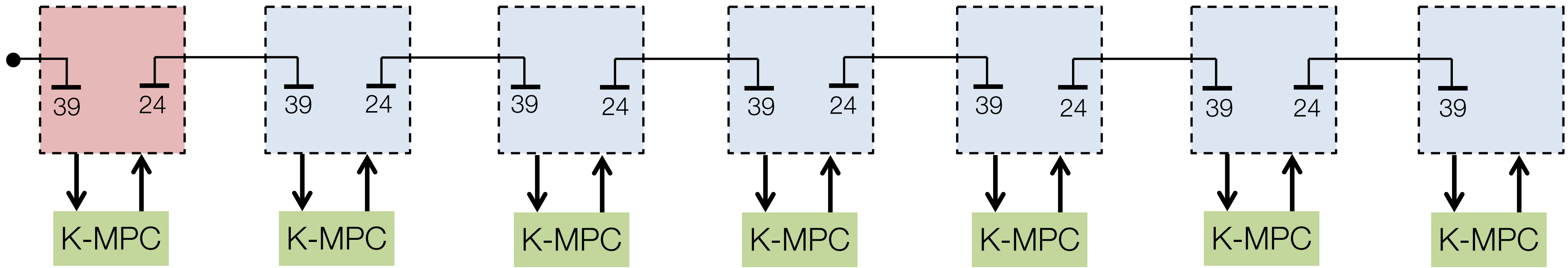


Cascade instability occurs without control

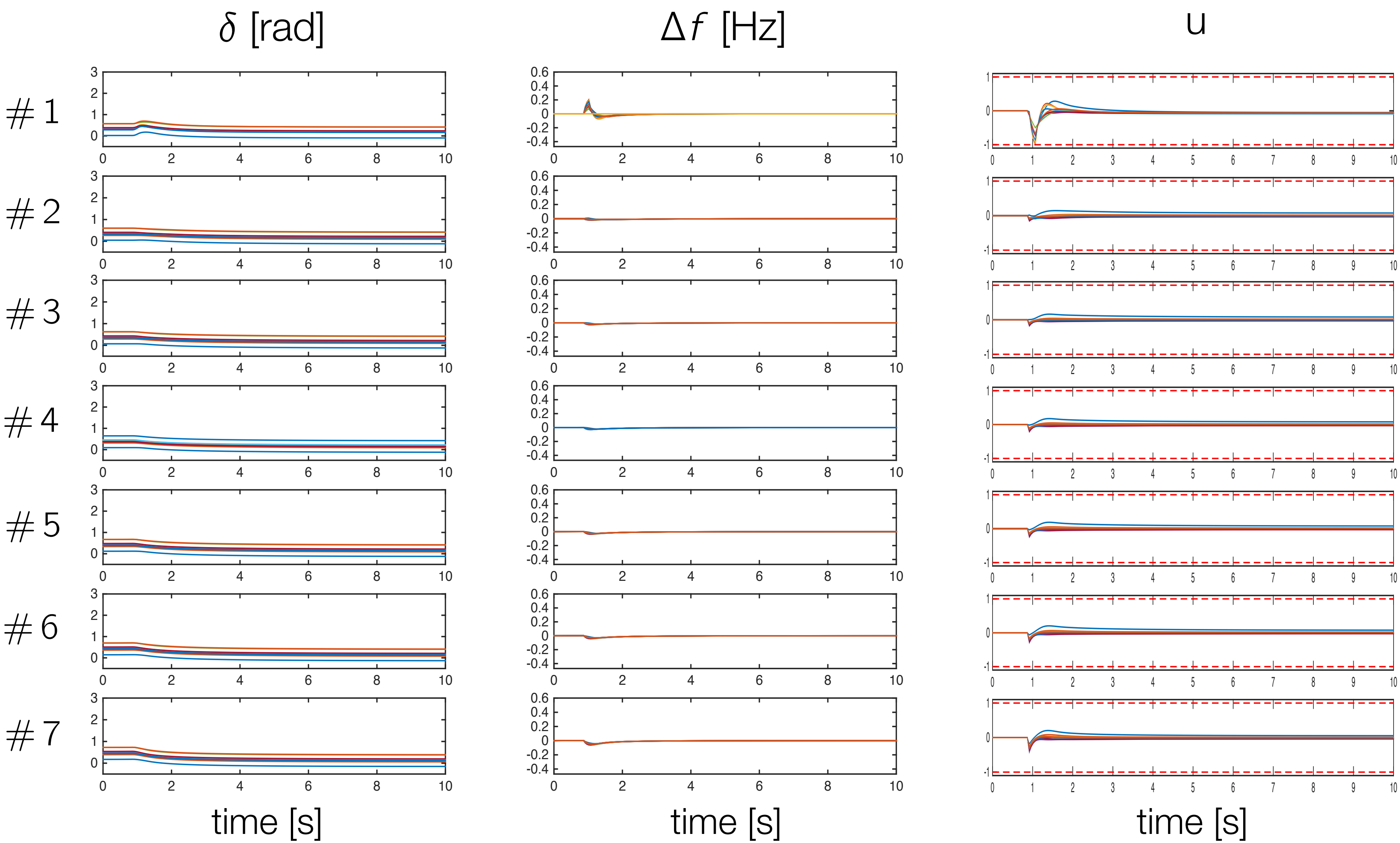


Can we suppress cascade instability?

Case 1: Each grid controlled separately

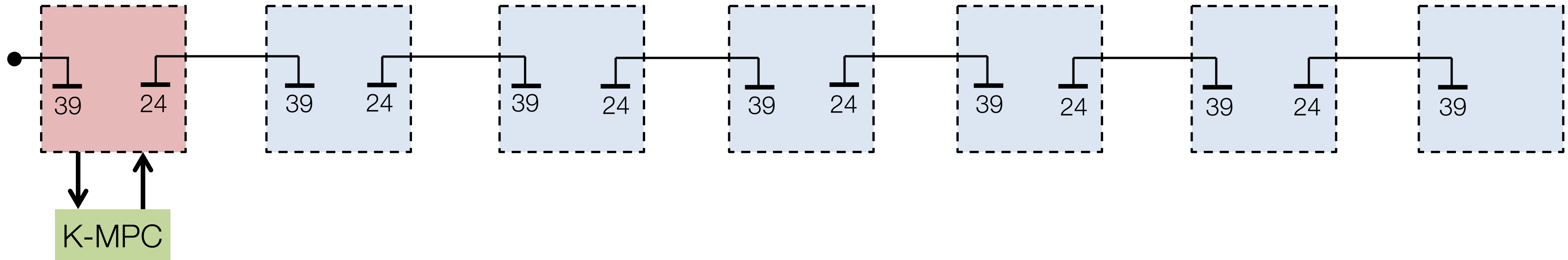


Koopman MPC suppresses the instability

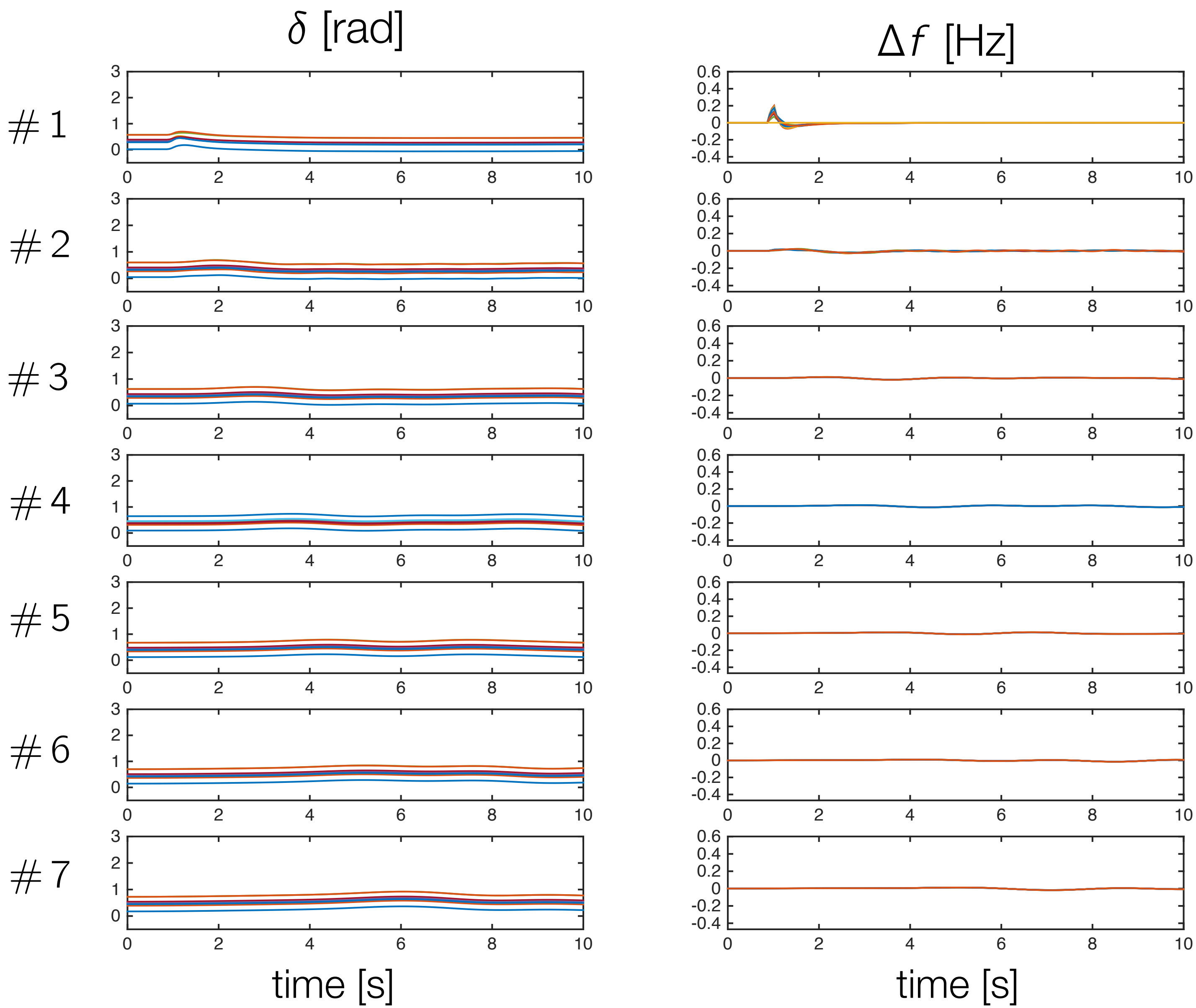


What if only the first grid is controlled?

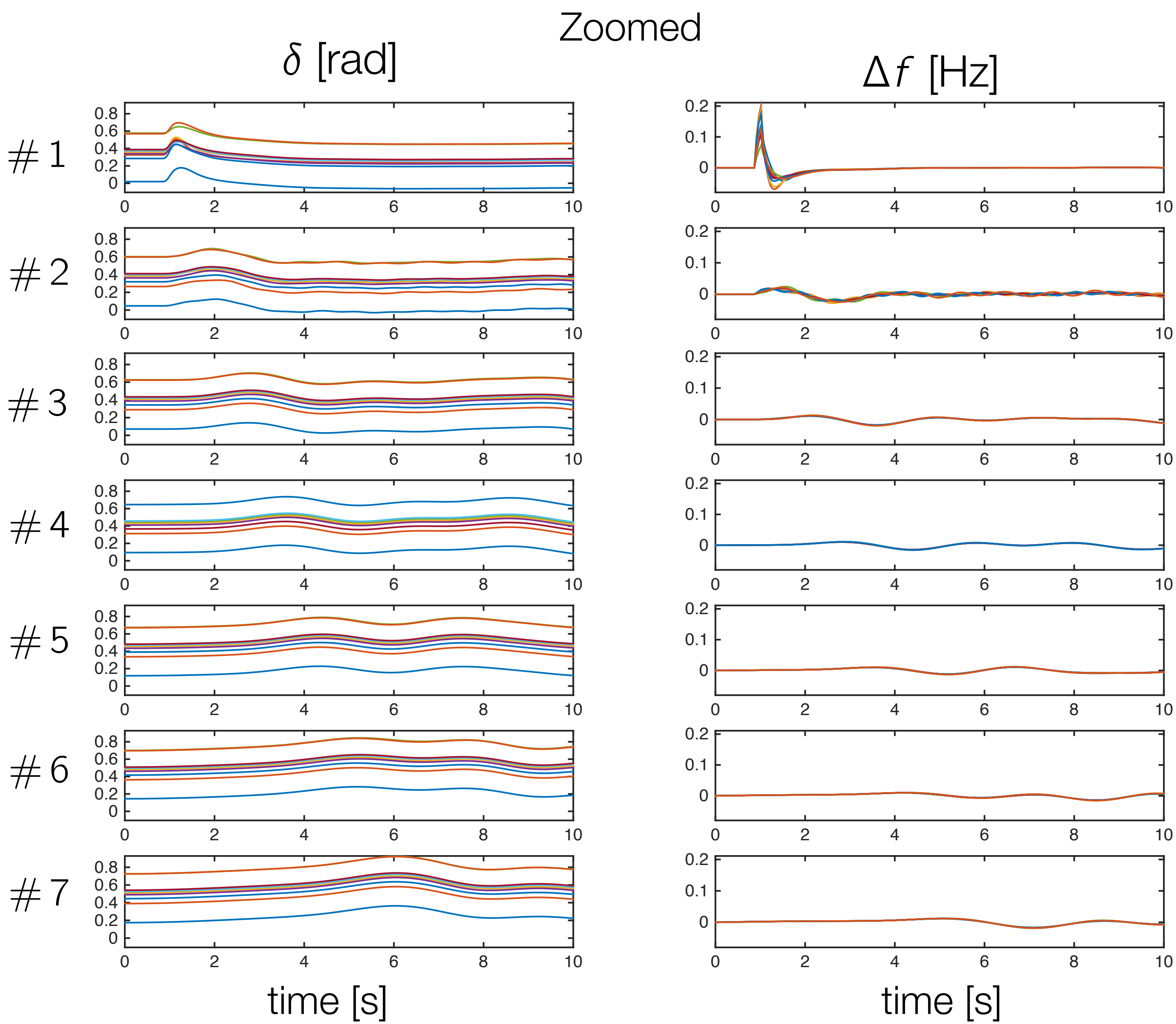
Case 2: Only the grid where the fault occurred controlled



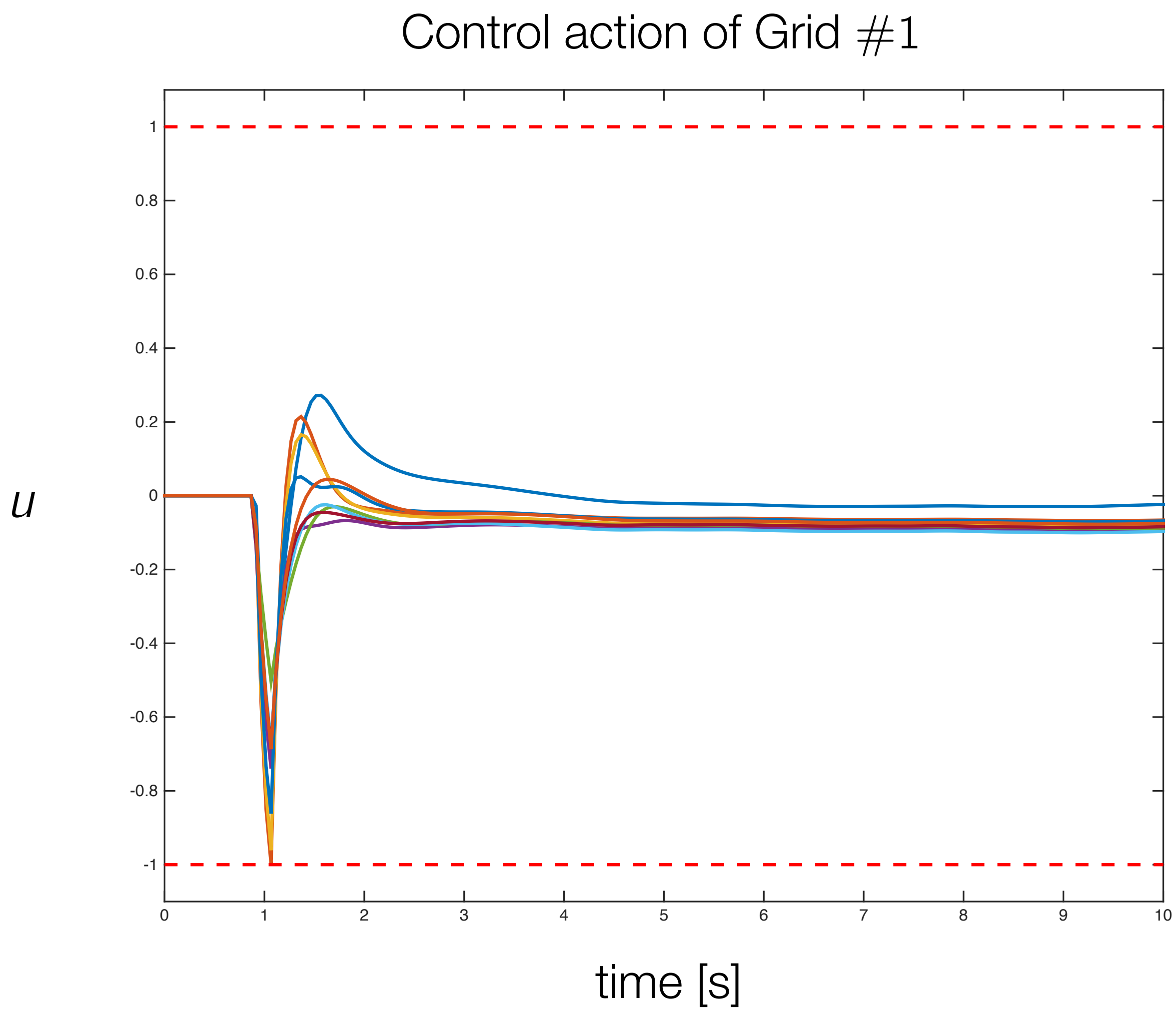
Even one grid control suppresses the instability



Even one grid control suppresses the instability



Control input



Conclusion

- Koopman MPC applied to power grid
  - Data-driven
  - Simple

Future work

- Use generator voltage instead of mechanical power for actuation

- Statistical / Robustness analysis

- Better embedding

Thank you