

# Strongly Feasible Stochastic Model Predictive Control

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**Abstract**—In this article we develop a systematic approach to enforce strong feasibility of probabilistically constrained stochastic model predictive control problems for linear discrete-time systems under affine disturbance feedback policies. Two approaches are presented, both of which capitalize and extend the machinery of invariant sets to a stochastic environment. The first approach employs an invariant set as a terminal constraint, whereas the second one constrains the first predicted state. Consequently, the second approach turns out to be completely independent of the policy in question and moreover it produces the largest feasible set amongst all admissible policies. As a result, a trade-off between computational complexity and performance can be found without compromising feasibility properties. Our results are demonstrated by means of two numerical examples.

## I. INTRODUCTION

Over the last two decades, the field of constrained model predictive control (MPC) has matured substantially. There is now a solid and very general theoretical foundation for stability and feasibility of nominal as well as robust MPC problems [14, 18]. Nevertheless, the connection to another mature field, stochastic optimal control, is still not fully developed although there has been a considerable research effort in this direction over the last years.

The basic ingredient of any receding horizon policy is finite horizon cost minimization, which is the first direction of recent research. This problem lies at the heart of stochastic optimal control theory and is known to be extremely difficult except for a few special cases (e.g., the linear quadratic problem). Thus, one typically seeks a suboptimal solution in a certain finite dimensional subset of admissible control policies. A popular choice is the affine disturbance feedback [10, 16], which is also the framework of this article. Here, however, we are not primarily concerned with cost minimization itself, but rather closed-loop constraint satisfaction. A more general approach is that of a nonlinear disturbance feedback where decision variables are the coefficients of a linear combination of nonlinear basis functions of the disturbance [11]. In the presence of unbounded disturbances, the nonlinear functions must be bounded whenever bounded control inputs are required. In this article, however, we are dealing with bounded disturbances only.

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Closest to the nature of receding horizon control is the question of enforcing recursive feasibility of probabilistic constraints, which is also the topic of this article. The problem was extensively studied in a series of papers [3, 4, 5, 13, 17], where various types of constraints and disturbance properties were considered, and a number of techniques to tackle these problems were proposed. The common factor of these papers is the use of a perturbed linear state feedback (or pre-stabilization), which necessarily limits the number of degrees of freedom and as a consequence the resulting performance. In this article, in contrast, the use of affine disturbance feedback, where more degrees of freedom are available, brings about performance improvement but also increased computational effort. This can, however, be overcome by imposing additional structural constraints on the feedback matrix, allowing to control the number of degrees of freedom of the optimization variable and as a result find a trade-off between performance and computational burden [16]. Furthermore, the feasibility of the second of the two approaches presented here is independent of the policy employed and in fact provides the largest feasible set amongst all admissible policies. Our approach takes advantage of the notion of controlled invariance, well established in (robust) constrained MPC (see, e.g., [2, 6]), bringing stochastic MPC on a sound footing. In fact, we derive results on strong feasibility and least-restrictiveness (see Definitions 1 and 2) analogous to those of [7, 9, 16] in a stochastic context.

This paper is organized as follows. We set up our notation in Section I-A and state the problem to be solved in Section II. Our main results are in Section III where we present two approaches, one with terminal constraints, one with first step constraints, in Sections III-A and III-B, respectively. Finally, some additional properties of the proposed methods are discussed in Section III-C, and our results are demonstrated via two numerical examples in Section IV.

### A. Notation

Throughout the article  $\mathbb{R}$  denotes the set of reals,  $\mathbb{N}$  the set of positive integers,  $\mathbb{N}_0$  the set of nonnegative integers and  $\mathbb{N}_i^j$  denotes the set of consecutive integers  $\{i, \dots, j\}$ . Random variables are defined on a common probability space with an associated probability measure  $P(\cdot)$ . The symbol  $\|A\|_\infty$  denotes the induced infinity norm of a matrix  $A$ , i.e.,  $\|A\|_\infty = \max_i (\sum_j |A_{ij}|)$ . Note that this notation is also used for row vectors, where it does *not coincide* with the standard infinity norm of a vector, but rather with the 1-norm. Finally, let  $\mathcal{S}^N$  be the Cartesian product of a set  $\mathcal{S}$   $N$ -times with itself.

## II. PROBLEM STATEMENT

We consider the linear time-invariant stochastic dynamic system

$$x_{k+1} = Ax_k + Bu_k + w_k, \quad k \in \mathbb{N}_0 \quad (1)$$

with the state  $x_k \in \mathbb{R}^n$ , the control  $u_k \in \mathbb{R}^m$ , and the i.i.d. disturbance sequence  $w_k \in \mathbb{R}^n$ . It is assumed that the state  $x_k$  is known at time  $k$  for all  $k \in \mathbb{N}_0$ , and that the pair  $(A, B)$  is stabilizable.

The purpose of the paper is to develop a systematic approach to ensure that the closed-loop state trajectory satisfies the probabilistic constraints

$$P(g_j^T x_k \leq h_j) \geq 1 - \alpha_j, \quad j \in \mathbb{N}_1^r, \quad k \in \mathbb{N}, \quad (2)$$

while minimizing a given cost function and satisfying hard input constraints

$$u_k \in \mathcal{U} := \{u \in \mathbb{R}^m \mid \|u\|_\infty \leq U_{\max}\}, \quad k \in \mathbb{N}_0. \quad (3)$$

The allowed probability of violation  $\alpha_j \in [0, 1]$  typically comes directly from application requirements, but it can also be viewed as a tuning parameter tracing a trade-off curve between constraint violation and incurred cost.

The polyhedral intersection of the individual constraints  $g_j^T x \leq h_j$  is referred to as the constraint set and denoted by

$$\mathcal{X} := \{x \in \mathbb{R}^n \mid Gx \leq h\}, \quad (4)$$

where  $g_j^T$  and  $h_j$  form the rows of  $G \in \mathbb{R}^{r \times n}$  and  $h \in \mathbb{R}^r$  respectively.

To ensure satisfaction of (2), it is sufficient to guarantee that

$$P(g_j^T x_{k+1} \leq h_j \mid x_k) \geq 1 - \alpha_j, \quad j \in \mathbb{N}_1^r, \quad (5)$$

for all  $k \in \mathbb{N}_0$ . In the sequel, we will focus on developing techniques to render the constraint (5) recursively feasible, that is, to guarantee its feasibility under a given control policy at each time  $k \in \mathbb{N}_0$ .

**Remark 1.** *The presented approach exhibits a certain degree of conservatism since satisfaction of (5) for all  $k \in \mathbb{N}_0$  is only sufficient for (2). However, (5) offers a tractable and straightforwardly implementable condition, in contrast to (2), which generally cannot be exactly accommodated in a cost-minimization procedure.*

We let

$$\mathbf{u} := [u_0^T, \dots, u_{N-1}^T]^T, \quad \mathbf{w} := [w_0^T, \dots, w_{N-1}^T]^T$$

denote the predicted input sequence and the disturbance sequence along the horizon  $N$ , respectively.

The recursive feasibility is considered with respect to the affine disturbance feedback policy

$$\mathbf{u} = \eta + K\mathbf{w} = \begin{bmatrix} \eta_0 \\ \eta_1 \\ \vdots \\ \eta_{N-1} \end{bmatrix} + \begin{bmatrix} 0 & 0 & \dots & 0 \\ K_{1,1} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \\ K_{N-1,1} & \dots & K_{N-1,N-1} & 0 \end{bmatrix} \mathbf{w}, \quad (6)$$

applied in a receding horizon fashion.

We assume that the common distribution of the disturbance sequence  $w_k$ ,  $k \in \mathbb{N}_0$ , is supported on the compact set

$$\mathcal{W} = \{w \in \mathbb{R}^n \mid \|w\|_\infty \leq \Delta < \infty\}, \quad (7)$$

where the bound  $\Delta$  must be derived from physical understanding of the stochastic factors affecting the system. Note that in general, under some technical assumptions on the constraint set and the system dynamics, it is not possible to enforce recursive satisfaction of the constraint (5) in the face of unbounded additive disturbances.

We let  $w$  denote any random variable having the common distribution of  $w_k$ ,  $k \in \mathbb{N}_0$ .

**Remark 2.** *The presented approach can be immediately generalized to polytopically or quadratically bounded disturbances, still giving rise to tractable convex optimization problems. However, in this paper we do not consider more general disturbance specifications for the sake of brevity.*

Throughout this paper we are concerned with the receding horizon application of the following problem.

**Problem 1.**

$$\underset{\eta, K}{\text{minimize}} \quad J(\eta, K) := \mathbf{E}_{x_0} \left\{ \|Q_N x_N\|_p^p + \sum_{i=0}^{N-1} \|Q x_i\|_p^p + \|R u_i\|_p^p \right\}$$

$$\text{subject to} \quad \mathbf{u} = \eta + K\mathbf{w} \text{ structured as in (6)} \quad (8a)$$

$$x_{i+1} = Ax_i + Bu_i + w_i \quad (8b)$$

$$u_1 \in \mathcal{U} \quad (8c)$$

$$P(g_j^T x_1 \leq h_j \mid x_0) \geq 1 - \alpha_j, \quad j \in \mathbb{N}_1^r, \quad (8d)$$

where  $\mathbf{E}_{x_0}\{\cdot\}$  is the conditional expectation given  $x_0$ . Note that, for notational simplicity, we have here and wherever possible in the sequel shifted time to zero, which means no loss of generality because of the i.i.d. assumption on  $w_k$ .

**Remark 3.** *The particular form of the cost function  $J$  does not affect the theoretical discussion of this paper, because here we are interested only in the feasibility properties of (1) which are independent of  $J$ . We, however, employ the above cost function in the examples of Section IV.*

Leaving aside optimality, a receding horizon application of Problem 1 gives rise to a family of time-invariant state-feedback control policies  $\pi(x) = \eta_0(x)$ , where  $\eta_0(x)$  can come from any feasible point of Problem 1,  $(\eta, K)$ , with the initial state  $x_0 = x$ . The corresponding closed-loop state process  $x_k$  (or rather a family of processes) is then generated by the equation

$$x_{k+1} = Ax_k + B\pi(x_k) + w_k. \quad (9)$$

Provided that Problem 1 is feasible at all times, the closed-loop input process  $u_k = \pi(x_k)$  will satisfy the constraint (3) and the closed-loop state process will satisfy the probabilistic constraint (5) for all  $k \in \mathbb{N}_0$ . Thus, our primary goal is to find a tractable representation for the constraints (8c) and (8d), and to augment Problem 1 by additional constraints that will ensure that the problem remains feasible for all times, when starting from a feasible initial state, due to any admissible disturbance sequence and any sequence of feasible control inputs. This property is known as strong feasibility.

**Definition 1** (Strong feasibility [12]). *A stochastic MPC problem is said to be strongly feasible if for every feasible initial state the closed-loop state process remains feasible due to any admissible disturbance realization and any sequence of feasible control inputs generated in a receding horizon fashion.*

Our secondary goal is to augment Problem 1 in a least-restrictive way.

**Definition 2** (Least restrictiveness). *A stochastic MPC problem is said to be least-restrictive if it is strongly feasible and there is no initial state  $x_0$  outside its feasible set and no policy satisfying the input constraints such that the closed-loop state process, starting from  $x_0$ , generated by that policy satisfies the probabilistic constraint (5) for all  $k \in \mathbb{N}_0$  and for all admissible disturbance realizations  $\{w_k \in \mathcal{W}\}_{k=0}^\infty$ .*

### III. MAIN RESULTS

Two approaches to enforce strong feasibility of Problem 1 are presented, both of which are based on robust invariant sets that have become a standard tool in receding horizon control [2]. We begin with the definition of the feasibility region of the constraint (5), which plays a crucial role in what follows.

**Definition 3** (Stochastic feasibility set). *The stochastic feasibility set of the constraint (5) is*

$$\mathcal{X}_f := \{x \mid \exists u \in \mathcal{U} \text{ s.t. } P(g_j^T(Ax + Bu + w) \leq h_j) \geq 1 - \alpha_j, \forall j \in \mathbb{N}_1^r\}.$$

Being a projection of a polyhedron,  $\mathcal{X}_f$  is also a polyhedron, unless it is empty. Indeed, we have

$$\begin{aligned} \mathcal{X}_f &= \{x \mid \exists u \in \mathcal{U} \text{ s.t.} \\ &F_{g_j^T w}(h_j - g_j^T(Ax + Bu)) \geq 1 - \alpha_j \forall j \in \mathbb{N}_1^r\} \\ &= \{x \mid \exists u \in \mathcal{U} \text{ s.t.} \\ &g_j^T(Ax + Bu) \leq h_j - F_{g_j^T w}^{-1}(1 - \alpha_j) \forall j \in \mathbb{N}_1^r\}, \quad (10) \end{aligned}$$

where  $F_{g_j^T w}(\cdot)$  and  $F_{g_j^T w}^{-1}(\cdot)$  are respectively the cumulative distribution and left quantile function of  $g_j^T w$ . We suppose throughout this paper that  $\mathcal{X}_f$  is nonempty.

**Remark 4.** *The quantiles  $F_{g_j^T w}^{-1}(1 - \alpha_j)$ ,  $j \in \mathbb{N}_1^r$  are the only quantities that need to be computed before standard algorithms for the construction of invariant sets can be employed. The quantiles can be computed offline to virtually arbitrary precision for any reasonable distribution of  $w$ , for instance, by means of Monte Carlo techniques.*

Note also that the stochastic feasibility set  $\mathcal{X}_f$  is, in general, neither a subset nor a superset of the constraint set  $\mathcal{X}$  (see numerical examples).

The main idea is to ensure that the state stays robustly inside  $\mathcal{X}_f$  while using such control inputs that the input and the state-probabilistic constraints (3) and (5), respectively, are satisfied. Both approaches presented achieve this by constraining the state to a robust controlled invariant subset of  $\mathcal{X}_f$ , the first approach implicitly using a dual mode

paradigm with a terminal constraint and the second approach explicitly through a first-step constraint.

#### A. Terminal constraint

First we adopt a dual mode paradigm where the affine disturbance feedback policy (6) is used for predictions in mode 1, that is, at times  $k = 0, \dots, N-1$ , and any stabilizing state feedback in mode 2, that is, at times  $k \geq N$  [14]. For a related approach with pre-stabilization see [13].

In mode 1 we have, given  $x_k$ ,

$$P(g_j^T x_{k+1} \leq h_j \mid x_k) = P(g_j^T(Ax_k + Bu_k + w_k) \leq h_j \mid x_k), \quad j \in \mathbb{N}_1^r.$$

Thus to ensure satisfaction of (5) we require that

$$g_j^T(Ax_k + Bu_k) \leq h_j - F_{g_j^T w}^{-1}(1 - \alpha_j), \quad j \in \mathbb{N}_1^r, \quad k \in \mathbb{N}_0^{N-1} \quad (11)$$

for all possible states  $x_k$  reachable, and all possible control inputs  $u_k$  generated, at prediction step  $k$  by any admissible disturbance sequence up time  $k$ ,  $w_0^{k-1} := (w_0, \dots, w_{k-1})$ , under a given policy in mode 1. Now, from the system dynamics (1) and the definition of the affine disturbance feedback (6), we have

$$x_k = A^k x_0 + \mathcal{B}_k(\eta + K\mathbf{w}) + C_k \mathbf{w}, \quad u_k = \eta_k + K_k \mathbf{w},$$

where

$$\mathcal{B}_k = [A^{k-1}B, \dots, B, 0, \dots, 0], \quad C_k = [A^{k-1}, \dots, I, 0, \dots, 0]$$

and  $\eta_k$  and  $K_k$  denote the  $k$ -th block rows of size  $m$ . Applying this to the left-hand side of (11), we get

$$\begin{aligned} g_j^T(Ax_k + Bu_k) &= \\ &= g_j^T[A(A^k x_0 + \mathcal{B}_k(\eta + K\mathbf{w}) + C_k \mathbf{w}) + B(\eta_k + K_k \mathbf{w})] \\ &= g_j^T(A^{k+1}x_0 + \mathcal{B}_{k+1}\eta) + g_j^T(\mathcal{B}_{k+1}K + AC_k)\mathbf{w}. \end{aligned}$$

Thus, considering the worst-case value of the uncertain term over all disturbances,

$$\max_{\mathbf{w} \in \mathcal{W}^N} g_j^T(\mathcal{B}_{k+1}K + AC_k)\mathbf{w} = \|g_j^T(\mathcal{B}_{k+1}K + AC_k)\|_\infty \Delta, \quad (12)$$

we obtain a sufficient condition for recursive feasibility in mode 1

$$g_j^T(A^{k+1}x_0 + \mathcal{B}_{k+1}\eta) \leq h_j - \|g_j^T(\mathcal{B}_{k+1}K + AC_k)\|_\infty \Delta - F_{g_j^T w}^{-1}(1 - \alpha_j), \quad j \in \mathbb{N}_1^r, \quad k \in \mathbb{N}_0^{N-1}. \quad (13)$$

**Remark 5.** *Although there is the disturbance sequence over the whole horizon  $\mathbf{w}$  in the above computation, only the disturbances  $w_0^{k-1}$  contribute to the worst-case value due to the structure of the matrices  $\mathcal{B}_k$  and  $C_k$ .*

In mode 2 we use a stabilizing state feedback  $u_k = K_s x_k$  with the corresponding strictly stable feedback dynamics matrix  $A + BK_s$ . To ensure strong feasibility we constrain the terminal state  $x_N$  to the maximum robust invariant subset of the stochastic feasibility set  $\mathcal{X}_f$  with respect to the closed-loop dynamics  $x_{k+1} = (A + BK_s)x_k + w_k$ , hard input constraints  $\|K_s x_k\| \in \mathcal{U}$  and the chance constraint

$P(g_j^T(A + BK_s)x_k + w_k \leq h_j) \geq 1 - \alpha_j$ . In other words, we employ a set  $\mathcal{X}_r^{K_s} \subset \mathcal{X}_f$  such that for all  $x \in \mathcal{X}_r^{K_s}$

$$(A + BK_s)x + w \in \mathcal{X}_r^{K_s}, \quad \|K_s x\|_\infty \leq U_{\max}, \quad (14)$$

$$g_j^T(A + BK_s)x \leq h_j - F_{g_j^T w}^{-1}(1 - \alpha_j) \quad \forall w \in \mathcal{W} \quad \forall j \in \mathbb{N}_1^r.$$

It is assumed that the set  $\mathcal{X}_r^{K_s}$  is polyhedral and nonempty in the form  $\mathcal{X}_r^{K_s} = \{x \in \mathbb{R}^n \mid Sx \leq z\}$ . If the set were nonempty but not polyhedral, an inner approximation that is polyhedral can always be constructed. See, [2] and [8] for an algorithm to construct such a set or its polyhedral approximation.

Strong feasibility is now ensured by the requirement that the state  $x_N$  lands robustly inside  $\mathcal{X}_r^{K_s}$ , that is,

$$A^N x_0 + \mathcal{B}_N \eta + (\mathcal{B}_N K + C_N)w \in \mathcal{X}_r^{K_s} \quad \forall w \in \mathcal{W}^N,$$

which is equivalent to

$$s_j^T(A^N x_0 + \mathcal{B}_N \eta) \leq z_j - \|s_j^T(\mathcal{B}_N K + C_N)\|_\infty \Delta \quad \forall j \in \mathbb{N}_1^r, \quad (15)$$

where  $s_j^T$  and  $z_j$  are the rows of the matrices  $S \in \mathbb{R}^{r \times n}$  and  $z \in \mathbb{R}^r$  defining  $\mathcal{X}_r^{K_s}$ .

Hard input constraints are enforced explicitly in mode 1 as

$$|\eta_i| + \Delta \|K_i\|_\infty \leq U_{\max}, \quad i = 1, \dots, mN, \quad (16)$$

and implicitly in mode 2 through the relation (14). Here the subscript  $i$  denotes  $i$ -th row (not block row) of the corresponding matrix.

We can now state and prove the following theorem.

**Theorem 1.** For Problem 1 with the constraints (8c) and (8d) replaced by (13), (15) and (16) the following holds:

- I. The problem is strongly feasible.
- II. The constraints (3) and (5) are satisfied in closed-loop.

*Proof.* I. Given any feasible solution  $(\eta, K)$  (structured as in (6)) at time zero, we are guaranteed to have a feasible point  $(\tilde{\eta}, \tilde{K})$  at time one with  $(\tilde{\eta}, \tilde{K})$  constructed as

$$\tilde{\eta} = \begin{bmatrix} \eta_1 + K_{1,1}w_0 \\ \eta_2 + K_{2,1}w_0 \\ \vdots \\ \eta_{N-1} + K_{N-1,1}w_0 \\ \eta_L \end{bmatrix}, \quad \tilde{K} = \begin{bmatrix} 0 & 0 \\ \hat{K} & 0 \\ K_L & 0 \end{bmatrix},$$

where

$$\hat{K} = \begin{bmatrix} K_{2,2} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \\ K_{N-1,2} & \dots & K_{N-1,N-1} & 0 \end{bmatrix}.$$

The last block rows  $\eta_L$  and  $K_L$  can be determined from the fact that  $K_s x_N$  defines a feasible input at time  $N$  provided that the previous inputs were generated by the policy  $u = \eta + K w$ . Thus for the last block rows we have

$$\eta_L = K_s(A^N x_0 + \mathcal{B}_N \eta) + K_s[(\mathcal{B}_N K + C_N)]_{1:n} \cdot w_0$$

$$K_L = K_s[\mathcal{B}_N K + C_N]_{n+1:nN},$$

where  $[A]_{p,q}$  denotes the sub-matrix of a matrix  $A$  consisting of columns  $p$  through  $q$ . Strong feasibility now follows

by induction.

II. Satisfaction of (8c) and (8d) with any feasible input is ensured by (16) with  $i \in \mathbb{N}_1^m$  and (13) with  $k = 0$ , respectively. Hence the constraints (3) and (5) are satisfied if the problem is strongly feasible, which is guaranteed by I.  $\square$

### B. First-step constraint

An alternative approach to enforce strong feasibility is to constrain at each time step only the predicted state at the very next time instant to a certain invariant set, in our case the maximum stochastic robust controlled invariant set (see Definition 5). This type of technique was recently introduced in the context of nominal as well as robust MPC [7, 16].

**Definition 4.** A set  $\mathcal{X}_{rc} \subset \mathbb{R}^n$  is a stochastic robust controlled invariant set if it satisfies the following condition:

$$\forall x \in \mathcal{X}_{rc} \exists u \in \mathcal{U} \text{ s.t. : } Ax + Bu + w \in \mathcal{X}_{rc} \quad \forall w \in \mathcal{W},$$

$$P(g_j^T(Ax + Bu + w) \leq h_j) \geq 1 - \alpha_j, \quad j \in \mathbb{N}_1^r. \quad (17)$$

**Definition 5** (MSRCI set). The maximum stochastic robust controlled invariant set (MSRCI) is the largest (in the sense of inclusion) set  $\mathcal{X}_{rc}^* \subset \mathbb{R}^n$  that is stochastic robust controlled invariant according to Definition 4. The MSRCI set can be explicitly defined as

$$\mathcal{X}_{rc}^* = \left\{ x_0 \in \mathcal{X}_f \mid \exists \phi : \mathbb{R}^n \rightarrow \mathbb{R}^m \text{ s.t.} \right.$$

$$x_{k+1} = (Ax_k + B\phi(x_k) + w_k) \in \mathcal{X}_f, \quad (18)$$

$$P(g_j^T Ax_k + B\phi(x_k) + w \leq h_j) \geq 1 - \alpha_j,$$

$$\left. \phi(x_k) \in \mathcal{U} \quad \forall j \in \mathbb{N}_1^r \quad \forall k \in \mathbb{N}_0 \quad \forall \{w_k \in \mathcal{W}\}_{k=0}^\infty \right\}.$$

**Remark 6.** It is clear that the MSRCI set  $\mathcal{X}_{rc}^*$  is a superset of the maximum robust controlled invariant subset of  $\mathcal{X}$  for any choice of  $\alpha_j \in [0, 1]$ . However, it is not true in general that the two sets coincide when  $\alpha_j = 0$  for all  $j$ , but rather  $\mathcal{X}_{rc}^*$  is then equal to the set from which the maximum robust controlled invariant subset of  $\mathcal{X}$  can be reached in one step. This implies that those states in  $\mathcal{X}_{rc}^*$  (for  $\alpha_j = 0$ ) that are not in the maximum robust controlled invariant subset of  $\mathcal{X}$  must be outside  $\mathcal{X}$ . See the first numerical example in Section IV.

Using the same argument as with the stochastic feasibility set  $\mathcal{X}_f$  in (10), the stochastic robust controlled invariance condition (17) can be expressed as

$$\forall x \in \mathcal{X}_{rc} \exists u \in \mathcal{U} \text{ s.t. : } Ax + Bu + w \in \mathcal{X}_{rc} \quad \forall w \in \mathcal{W},$$

$$g_j^T(Ax + Bu) \leq h_j - F_{g_j^T w}^{-1}(1 - \alpha_j) \quad \forall j \in \mathbb{N}_1^r,$$

which shows that these sets can be determined by standard algorithms for construction of (maximum) robust controlled invariant sets. Consequently, all of the results for maximum robust controlled invariant sets hold. In particular the set can be expressed as an intersection of possibly infinite number of polyhedra. Hence, the set is convex, and moreover if  $\mathcal{X}_f$  is compact, so is  $\mathcal{X}_{rc}^*$  [2].

Again, it is assumed that the MSRCI set  $\mathcal{X}_{rc}^*$  is polyhedral and nonempty in the form  $\mathcal{X}_{rc}^* = \{x \mid \tilde{S}x \leq \tilde{z}\}$ . If the set were nonempty but not polyhedral, an inner approximation that is stochastic robust controlled invariant and polyhedral can

always be constructed (see, e.g., [8]). This approximation is no longer maximum, rendering the problem more restrictive than the original one, yet still strongly feasible.

Thus, given the initial state  $x_0$ , one can enforce strong feasibility and constraint satisfaction by enforcing the constraints

$$\begin{aligned} Ax_0 + B\eta_0 + w &\in \mathcal{X}_{rc}^* \quad \forall w \in \mathcal{W}, \\ P(g_j^T(Ax_0 + B\eta_0 + w) \leq h_j) &\geq 1 - \alpha_j, \quad j \in \mathbb{N}_1^r, \\ \eta_0 &\in \mathcal{U}, \end{aligned}$$

which translate to

$$\tilde{s}_j^T(Ax_0 + B\eta_0) \leq \tilde{z}_j - \|\tilde{s}_j^T\|_\infty \Delta \quad \forall j \in \mathbb{N}_1^r, \quad (19)$$

$$g_j^T(Ax_0 + B\eta_0) \leq h_j - F_{g_j^T w}^{-1}(1 - \alpha_j) \quad \forall j \in \mathbb{N}_1^r, \quad (20)$$

$$\|\eta_0\|_\infty \leq U_{\max}, \quad (21)$$

where  $\tilde{s}_j^T$  and  $\tilde{z}_j$  are the rows of the matrices  $\tilde{S} \in \mathbb{R}^{\tilde{r} \times n}$  and  $\tilde{z} \in \mathbb{R}^{\tilde{r}}$  defining  $\mathcal{X}_{rc}^*$ .

The following theorem now summarizes these observations.

**Theorem 2.** *For Problem 1 with the constraint (8c) and (8d) replaced by (19), (20) and (21) the following holds:*

- I. *The problem is strongly feasible.*
- II. *The problem is least-restrictive with the feasibility set equal to the associated MSRCI set  $\mathcal{X}_{rc}^*$ .*
- III. *The constraints (3) and (5) are satisfied in closed-loop.*

*Proof.* I. Given constraints (19), (20), (21), strong feasibility follows immediately by construction of the MSRCI set as follows. Given initial state  $x_0 \in \mathcal{X}_{rc}^*$  and any feasible point  $(\eta, K)$  at time zero, the constraint (19) guarantees that the state at the next time instant stays robustly in  $\mathcal{X}_{rc}^*$  after application of the first control move  $\eta_0$ . The result now follows by induction.

II. The least-restrictiveness follows from the equivalent characterization of  $\mathcal{X}_{rc}^*$  (18). The fact that  $\mathcal{X}_{rc}^*$  is the feasible set of the problem is clear from the maximality of the MSRCI and the problem constraints.

III. Satisfaction of (8c) and (8d) with any feasible input follows from (21) and (20), respectively. Strong feasibility now ensures closed-loop satisfaction of (3) and (5).  $\square$

*1) Mode 1 constraints:* Theorem 2 tells us that if the stochastic *maximum* robust controlled invariant set is employed, the problem is feasible at time zero (and then by induction at all times) if and only if  $x_0 \in \mathcal{X}_{rc}^*$ . Even though constraints (19), (20) and (21) are sufficient, it may be beneficial for the sake of cost minimization to also include the mode 1 constraints (13) and (16). Adding the state mode 1 constraints (13) can, however, unnecessarily reduce the size of the set of feasible initial states. Indeed, the additional constraints employ explicitly the affine disturbance feedback policy, whereas  $\mathcal{X}_{rc}^*$  is maximum with respect to all policies. A remedy proposed in [16, 7] is to relax the additional constraints in a minimal way such that the feasible

set remains unchanged. This amounts to replacing (13) and (16) with

$$\begin{aligned} g_j^T(A^{k+1}x_0 + \mathcal{B}_{k+1}\eta) &\leq h_j - \|g_j^T A(\mathcal{B}_k K + C_k)\|_\infty \Delta \\ &\quad - F_{g_j^T w_k}^{-1}(1 - \alpha_j) + \zeta_j, \end{aligned} \quad (22)$$

and

$$|\eta_i| + \Delta \|K_i\|_\infty \leq U_{\max} + \xi_i, \quad (23)$$

where  $\zeta = [\zeta_1, \dots, \zeta_r]^T$  and  $\xi = [\xi_1, \dots, \xi_{mN}]^T$  are minimal in some sense (e.g., in the 2-norm) such that the set of feasible initial states does not shrink. It is shown in [7] that computation of such a minimal relaxation gives rise to a convex problem where enumeration of all vertices of  $\mathcal{X}_{rc}^*$  is necessary, which can quickly become prohibitive in larger dimensions. If this is the case, one can, however, always resort to a soft relaxation, that is, to keep  $\zeta$  and  $\xi$  as optimization variables, and add regularization terms to the cost. If the 2-norms of  $\zeta$  and  $\xi$  are of interest, this approach leads to the cost of the form

$$\tilde{J}(\eta, K) = J(\eta, K) + \gamma_1 \|\zeta\|_2^2 + \gamma_2 \|\xi\|_2^2 \quad (24)$$

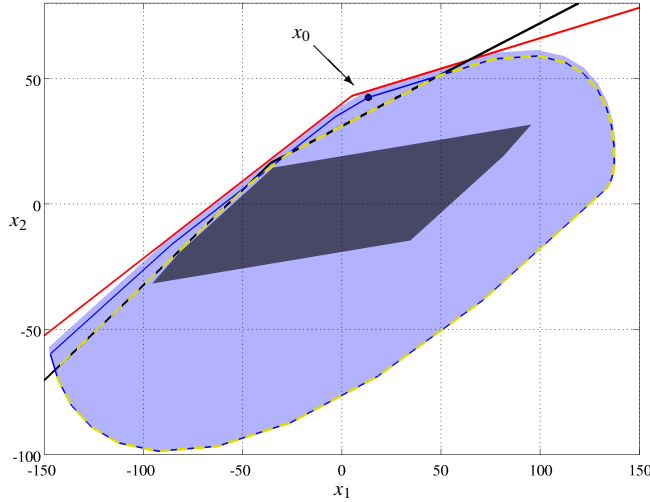
with some positive  $\gamma_1$  and  $\gamma_2$ .

*2) Structural constraints:* The second approach is particularly useful when additional structure is imposed on the matrix  $K$  and/or  $\eta$  in order to reduce the number of decision variables, and consequently the computational burden. A typical structure of the matrix  $K$  might be block-banded, i.e., allowing only a limited recourse via the disturbance sequence in the sense that  $K_{i,j} = 0$  for  $j < i - j_0$  for some fixed  $j_0 \geq 0$ . Another viable structure is a diagonal one for which  $K_{i,j} = K_{i+1,j+1}$ . See [16] for a comparison of various blocking strategies in the context of robust MPC.

It can be seen from the proof of Theorem 1 that this additional structural constraint cannot be accommodated within the first approach. On the other hand, the MSRCI set in the second approach, and hence its feasibility properties, remain completely unaffected as long as the first control move is free. This is a major advantage of the second approach since it allows for a trade-off between performance and complexity of the resulting problem while retaining (least-restrictive) strong feasibility. This is in fact one of the motivations behind the results of [7, 8] in the context of the standard move-blocking strategies widely employed in receding horizon control.

### C. Discussion

First note that the first-step approach is completely independent of the policy in question as long as the additional mode 1 constraints (13) and (16) are not used. In fact, the MSRCI set depends only on the probabilistic constraints (2), the disturbance set  $\mathcal{W}$  and the set of admissible controls  $\mathcal{U}$ . Hence, the first-step approach extends readily to arbitrary control policies, for instance to the nonlinear disturbance feedback considered for example in [19]. The terminal-constraint approach, by contrast, does not extend straightforwardly to general nonlinear disturbance feedback policies,



**Fig. 1:** Constraint set  $\mathcal{X}$  (below solid black line), feasible set  $\mathcal{X}_f$  (below solid red line), MSRCI set  $\mathcal{X}_{rc}^*$  (light blue interior), linear state-feedback positively invariant set  $\mathcal{X}_r^{K_s}$  (dark blue interior), maximum robust controlled invariant set  $\mathcal{X}_{rob}$  (dashed yellow boundary), zero violation ( $\alpha_j = 0$ ) MSRCI set  $\mathcal{X}'_{rob}$  (blue solid boundary). Note that  $\mathcal{X}_{rc}^* \not\subset \mathcal{X}$  and  $\mathcal{X}_{rob} \neq \mathcal{X}'_{rob}$ , but  $\mathcal{X}_{rob} \subset \mathcal{X}'_{rob} \subset \mathcal{X}_{rc}^* \subset \mathcal{X}_f$  and  $\mathcal{X}_{rob} \subset \mathcal{X}$ .

as the terminal constraint set is associated with a *linear* controller.

Furthermore, Theorem 2 states that the feasible set of the first-step approach is maximal amongst all admissible policies. Thus, the feasible set of the terminal-constraint formulation is necessarily a subset of, or equal to, the feasible set of the first-step formulation. On the other hand, construction of the robust invariant set with respect to the linear state feedback in the terminal-constraint approach is substantially less computationally demanding than the construction of the MSRCI set in the first-step approach. At this point it should, however, be emphasized that both sets are computed offline.

Lastly, we note that the recursively feasible chance constraint (2) translates to *affine* constraints on  $\eta$  and  $K$  regardless of the disturbance distribution, which is in sharp contrast to the traditional “open-loop” chance constraints that lead to second-order-cone constraints for Gaussian disturbances (with the affine disturbance feedback) and have typically no exact representation otherwise [1, 15]. This is not completely unexpected since in (5) the stochastic nature of the problem comes into play at the last step only (from  $k$  to  $k+1$ ), whereas all of the previous disturbances have to be treated robustly.

#### IV. NUMERICAL EXAMPLES

In our first example we compare both the terminal (AD-T) and the first-step (without mode 1 constraints) (AD-F) affine disturbance feedback policies against the perturbed linear state feedback stochastic MPC (P-SMPC) of [13] and the robust affine disturbance feedback (AD-R). The additional parameters for the P-SMPC policy are  $\hat{N} = 40$  and  $n^* = 1$  (see [13] for the meaning of the parameters). As the first step constraint for AD-R we used the MSRCI set with a zero probability of violation (i.e.,  $\alpha_j = 0$  for all  $j$ ),  $\mathcal{X}'_{rob}$ , which is in general *not* the maximum robust controlled invariant subset of  $\mathcal{X}$ ,  $\mathcal{X}_{rob}$  (see Remark 6 and Figure 1).

We consider system (1) with  $A = \begin{bmatrix} 1.25 & -0.15 \\ 0.25 & 1.02 \end{bmatrix}$ ,  $B = \begin{bmatrix} 0.14 \\ 0.12 \end{bmatrix}$ , and  $w_k$  an i.i.d. sequence obtained by truncating the standard normal distribution at  $\Delta = \|w_k\|_\infty \leq 3$ . We chose a quadratic ( $p = 2$ ) cost function  $J$ , which can be evaluated exactly for all of the policies considered. The weighting matrices were set to  $Q = I$ ,  $R = 1$  and  $Q_N$  to the solution to the corresponding algebraic Riccati equation. The bound on control authority was  $U_{max} = 250$ . The constraint set  $\mathcal{X}$  is given by two constraints  $g_1^T x \leq h_1$  and  $g_2^T x \leq h_2$  with  $g_1 = [-0.41, 1]^T$ ,  $h_1 = 31$  and  $g_2 = [-0.7593, 1]^T$ ,  $h_2 = 43.494$ , and the allowed probability of violation  $\alpha_1 = \alpha_2 = 0.1$ . All of the policies were applied in a receding horizon fashion with the prediction horizon  $N = 8$ . We chose  $K_s = [1.73, -13.10]$  as the mode 2 controller for AD-T as well as the base policy for P-SMPC. Note that the LQ optimal state feedback cannot be used in this case since then  $\mathcal{X}_r^{K_s}$  turns out to be empty and as a consequence both policies are globally infeasible. For the sake of comparison we also included the LQ-optimal policy itself. The initial state  $x_0 = [13.34, 42.46]^T$  was chosen to lie on the boundary of  $\mathcal{X}'_{rob}$ . The various sets considered and the initial state are depicted in Figure 1.

Performance and constraint violation was evaluated over 500 Monte Carlo runs; the results over the simulation horizon  $T = 20$  are summarized in Table I. The two proposed strongly feasible MPC formulations outperform the P-SMPC and AD-R policies and, naturally, perform worse than the LQ-optimal policy. We also observe tight satisfaction of the chance constraint at the time  $k = 1$  with our policies: the constraint violation is 9.5 % for both, which is close to, but within, the prescribed 10 % limit. The P-SMPC and AD-R policies are more conservative here, exhibiting zero violation. The LQ-optimal control, in contrast, violated the constraint in 85.0 % of the 500 runs performed. Violations at other time steps were zero or negligible for all investigated policies.

In the above example there are no constraint violations after stationarity is reached. The next example shows that it is possible to achieve repeated constraint violations in stationarity, and thus to obtain significant performance improvement compared to the robust MPC by fully exploiting the probabilistic nature of constraints over a long period of time. We consider system (1) with  $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ ,  $B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ , and with the i.i.d. disturbance sequence  $w_k$  having the standard normal distribution truncated at  $\Delta = 3$ . The cost  $J$  is again quadratic ( $p = 2$ ) with weighting matrices  $Q = \text{diag}(0, 1)$ ,  $R = 0$  and  $Q_N$  equal to the solution to the corresponding algebraic Riccati equation. The initial state was set to  $x_0 = [5, 5]^T$ . The only constraint on the state is  $P(x_2 \geq 0) \geq 1 - \alpha$ , while the control authority is bounded by  $U_{max} = 12$ . Simulations

**TABLE I:** Comparison of control policies over the optimization horizon  $T = 20$ . First row: percentage increase over the LQ-optimal policy. Second row: the probability of violating the state constraints at time one. The final cost of the LQ-optimal policy,  $J_{LQ}$ , is  $1.5504 \cdot 10^5$ .

Policy	LQ	AD-F	AD-T	AD-R	P-SMPC
$100 \frac{J - J_{LQ}}{J_{LQ}}$	0	2.01	2.20	8.31	9.19
$P(g_1^T x_1 > h_1)$	0.85	0.095	0.095	0	0

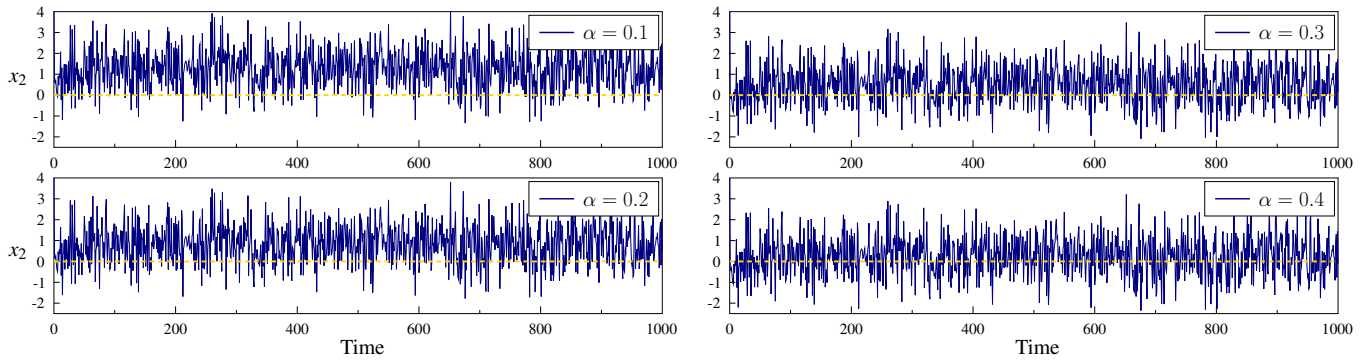


Fig. 2: Sample paths of  $x_2$  over the first 1000 time steps for four different values of  $\alpha$ .

were carried out for four values of the allowed probability of constraint violation:  $\alpha = 0.1$ ,  $\alpha = 0.2$ ,  $\alpha = 0.3$  and  $\alpha = 0.4$ . We compared the first-step affine disturbance feedback (AD-F) with the robust affine disturbance feedback policy (AD-R) and the LQ optimal policy. The prediction horizon was  $N = 8$  for both disturbance feedback policies.

Instead of Monte Carlo analysis, we examined constraint violations over a single, but very long (10000 time steps), trajectory. Simulation results are depicted in Figures 2 and 3. Table II then summarizes the results. For all four values of  $\alpha$ , the closed-loop trajectory under the first-step affine disturbance feedback tightly satisfies the probabilistic constraint, and as a result achieves a significant performance improvement over the robust affine disturbance feedback policy. The LQ optimal policy, which is oblivious to all constraints, naturally outperforms both policies, but violates the probabilistic constraint substantially.

TABLE II: Comparison of control policies over the simulation time  $T = 10000$ .

policy	LQ	$\alpha = 0.4$	$\alpha = 0.3$	$\alpha = 0.2$	$\alpha = 0.1$	Robust
$J/J_{LQ}$	1	1.07	1.29	1.73	2.68	10.23
#violations	4920	3916	2942	1983	992	0

## V. CONCLUSION

In this article we developed a systematic approach to enforce strong feasibility of MPC problems with probabilistic constraints and affine disturbance feedback policies. The first approach employs the well established notion of positively invariant terminal constraint sets, whereas the second one takes advantage of the more recently developed first-step constraint. Both approaches turn out to have direct analogies in a stochastic environment carrying over their advantages and disadvantages. In particular the first approach is policy-dependent and hence not amenable to imposing additional structural constraints on the affine disturbance feedback matrix. In contrast, the

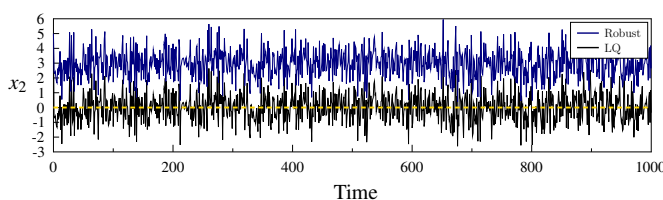


Fig. 3: Sample paths of  $x_2$  for LQ and robust control policies.

second approach is policy-independent and results in the largest feasible set amongst all admissible policies.

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