Lyapunov stability analysis of a string equation coupled with an ordinary differential system

Grenoble / GIPSA-lab

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LAAS - CNRS / Team MAC
Overview about ODE/PDE coupling

- Linear Systems
- Finite Dimension
- Infinite Dimension
- PDEs
- Automatic Control
  - Stability
  - Closed-loop
  - Lyapunov Theory
- Applied Mathematics
  - Existence & Uniqueness
  - Regularity
  - Open-loop
- Boundary Conditions

Lyapunov stability for and ODE/String system
Overview about ODE/PDE coupling

- Linear Systems
- Infinite Dimension
- Finite Dimension
- Delay Systems
- PDEs

• Stability
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  - **Applied Mathematics**
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- **Infinite Dimension**
- **Delay Systems**
- **PDEs**
Overview about ODE/PDE coupling

From Time-delay Systems to the transport equation:

\[
\begin{cases}
\dot{X}(t) = AX(t) + A_d X(t - \tau), & t \geq 0, \\
X(t) = \Phi(t), & t \in (-\tau, 0).
\end{cases}
\]  
(1)

\[
\begin{cases}
\dot{X}(t) = AX(t) + A_d u(1, t), & t \geq 0, \\
u_t(x, t) = \tau^{-1} u_x(x, t), & x \in (0, 1), t \geq 0, \\
u(0, t) = X(t), & t \geq 0, \\
u(x, 0) = \Phi(-\tau x), & x \in (0, 1).
\end{cases}
\]  
(2)

Lyapunov stability for an ODE/String system
Overview about ODE/PDE coupling

From Time-delay Systems to the transport equation:

\[
\begin{align*}
\dot{X}(t) &= AX(t) + A_d X(t - \tau), & t \geq 0, \\
X(t) &= \Phi(t), & t \in (-\tau, 0).
\end{align*}
\]  

\[\Downarrow\]

\[
\begin{align*}
\dot{X}(t) &= AX(t) + A_d u(1, t), & t \geq 0, \\
\frac{\partial u}{\partial t}(x, t) &= \tau^{-1} \frac{\partial u}{\partial x}(x, t), & x \in (0, 1), t \geq 0, \\
u(0, t) &= X(t), & t \geq 0, \\
u(x, 0) &= \Phi(-\tau x), & x \in (0, 1).
\end{align*}
\]

1. Existence and uniqueness for System (2) ?
2. Are we working on the same space ?

Lyapunov stability for and ODE/String system
For the transport equation, this is “easy” because:

1. Systems (1) and (2) are equivalent;
2. There exists a vast literature on Time-delay systems:
   - Exact stability criterion [Olgac and Sipahi, 2004],
   - Complete Lyapunov functional (necessary and sufficient) [Kharitonov and Zhabko, 2003],
   - Accurate inequalities (Jensen, Wirtinger, Bessel [Seuret and Gouaisbaut, 2013, 2015], ...).
More about “Transport equation”

For the transport equation, this is “easy” because:

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   - Exact stability criterion [Olgac and Sipahi, 2004],
   - Complete Lyapunov functional (necessary and sufficient) [Kharitonov and Zhabko, 2003],
   - Accurate inequalities (Jensen, Wirtinger, Bessel [Seuret and Gouaisbaut, 2013, 2015], ...).

Some work done:

1. Using backstepping [Krstic, 2011];
2. Using a hierarchy of LMI conditions [Baudouin & Seuret & Safi, 2016, 2017];
3. ...
What about other PDEs?

1. For the heat equation:
   - Existence, uniqueness, equilibrium points are real problems;
   - Backstepping methodology [Krstic, 2011];
   - Lyapunov methods [Beaudouin, Seuret, Gouaisbaut, 2017].
For the heat equation:
- Existence, uniqueness, equilibrium points are **real** problems;
- Backstepping methodology [Krstic, 2011];
- Lyapunov methods [Beaudouin, Seuret, Gouaisbaut, 2017].

For the wave equation:
- Well-posedness is an issue;
- Backstepping methodology [Krstic, 2011];
- Lyapunov methods [Prieur, Tarbouriech, Gomes da Silva, 2016],
  and what I am going to talk now!
1. Wellposedness and Regularity
2. Riemann Invariant and Auxiliary System
3. A First Stability Result
4. Extended Stability Analysis
5. Examples
Lyapunov stability for and ODE/String system
Coupled ODE / String equation

\[
\dot{X}(t) = AX(t) + Bu(1, t), \quad t \geq 0,
\]

\[
u_{tt}(x, t) = c^2 u_{xx}(x, t), \quad x \in (0, 1), t \geq 0,
\]

\[
u(0, t) = KX(t), \quad t \geq 0,
\]

\[
u_x(1, t) = -c_0 u_t(1, t), \quad t \geq 0,
\]

\[
\begin{bmatrix}
X(0) \\
u(x, 0) \\
u_t(x, 0)
\end{bmatrix}
= \begin{bmatrix}
X^0 \\
u^0(x) \\
v^0(x)
\end{bmatrix}, \quad x \in (0, 1).
\]
Wellposedness result

1. When is System (3) well-posed?
2. What is the regularity of the solution?
3. Which is / are the equilibrium point(s)?
Wellposedness result

\[ \mathcal{H} = \mathbb{R}^n \times H^1(0,1) \times L^2(0,1), \]

\[ \| (X, u, v) \|_{\mathcal{H}}^2 = |X|^2_n + \| u \|_{L^2}^2 + c^2 \| u_x \|_{L^2}^2 + \| v \|_{L^2}^2. \]
Wellposedness result

\[ \mathcal{H} = \mathbb{R}^n \times H^1(0, 1) \times L^2(0, 1), \]

\[ \| (X, u, v) \|_\mathcal{H}^2 = |X|^2_n + \| u \|_{L^2}^2 + c^2 \| u_x \|_{L^2}^2 + \| v \|_{L^2}^2. \]

**Well-posedness**

If the two following conditions hold:

1. there exists a norm \( V \) on \( \mathcal{H} \) s.t. \( \frac{d}{dt} V(X(t), u(t), u_t(t)) < 0 \),
2. \( A + BK \) is not singular,

then there exists a unique solution \( (X, u, u_t) \) to System (3) with initial conditions \( (X^0, u^0, v^0) \) \( \in \mathcal{D} \) with

\[ \mathcal{D} = \{ (X, u, v) \in \mathbb{R}^n \times H^2 \times H^1, u(0) = KX, u_x(1) = -c_0 v(1) \} . \]

Moreover, the solution have the following regularity property: \( (X, u, u_t) \in C(0, +\infty, \mathcal{H}) \).
Wellposedness result

\[ \mathcal{H} = \mathbb{R}^n \times H^1(0,1) \times L^2(0,1), \]

\[ \| (X, u, v) \|_{\mathcal{H}}^2 = |X|^2_n + \| u \|_{L^2}^2 + c^2 \| u_x \|_{L^2}^2 + \| v \|_{L^2}^2. \]

Purpose of the presentation

Given \( A, B, K, c \) and \( c_0 \), can we find \( V \) such that:

1. \( V \) is equivalent to the norm \( \| \cdot \|_{\mathcal{H}} \) on \( \mathcal{H} \)
2. and its derivative along the trajectories is negative?
1. Wellposedness and Regularity
2. Riemann Invariant and Auxiliary System
3. A First Stability Result
4. Extended Stability Analysis
5. Examples
\[
\left\{
\begin{array}{l}
  u_{tt}(x, t) = c^2 u_{xx}(x, t), \\
  u(0, t) = 0, \\
  u_x(0, t) = 0.
\end{array}
\right.
\]
Riemann Invariant

\[ \begin{aligned}
    u_{tt}(x, t) &= c^2 u_{xx}(x, t), \\
    u(0, t) &= 0, \\
    u_x(0, t) &= 0.
\end{aligned} \]

Change of coordinate

\[ \chi(x, t) = \begin{bmatrix}
    u_t(x, t) + cu_x(x, t) \\
    u_t(x, t) - cu_x(x, t)
\end{bmatrix} = \begin{bmatrix}
    \chi^+(x, t) \\
    \chi^-(x, t)
\end{bmatrix} \]
\[
\begin{align*}
\begin{cases}
    u_{tt}(x, t) &= c^2 u_{xx}(x, t), \\
u(0, t) &= 0, \\
u_x(0, t) &= 0.
\end{cases}
\Rightarrow
\begin{cases}
    \chi_t(x, t) &= \begin{bmatrix} c & 0 \\ 0 & -c \end{bmatrix} \chi_x(x, t), \\
\chi(0, t) &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\end{cases}
\end{align*}
\]

Change of coordinate

\[
\chi(x, t) = \begin{bmatrix} u_t(x, t) + cu_x(x, t) \\ u_t(x, t) - cu_x(x, t) \end{bmatrix} = \begin{bmatrix} \chi^+(x, t) \\ \chi^-(x, t) \end{bmatrix}
\]
\[ \begin{cases} u_{tt}(x, t) = c^2 u_{xx}(x, t), \\ u(0, t) = 0, \\ u_x(0, t) = 0. \end{cases} \Rightarrow \begin{cases} \chi_t(x, t) = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \chi_x(x, t), \\ \chi(0, t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \end{cases} \]

**Change of coordinate**

\[ \chi(x, t) = \begin{bmatrix} u_t(x, t) + cu_x(x, t) \\ u_t(1 - x, t) - cu_x(1 - x, t) \end{bmatrix} = \begin{bmatrix} \chi^+(x, t) \\ \chi^-(1 - x, t) \end{bmatrix} \]
Riemann Invariant

\[
\begin{aligned}
&\begin{cases}
  u_{tt}(x, t) = c^2 u_{xx}(x, t), \\
  u(0, t) = 0, \\
  u_x(0, t) = 0.
\end{cases} \\
\Rightarrow \quad &\begin{cases}
  \chi_t(x, t) = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \chi_x(x, t), \\
  \chi(0, t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\end{cases}
\end{aligned}
\]

Change of coordinate

\[
\chi(x, t) = \begin{bmatrix} u_t(x, t) + cu_x(x, t) \\ u_t(1 - x, t) - cu_x(1 - x, t) \end{bmatrix} = \begin{bmatrix} \chi^+(x, t) \\ \chi^-(1 - x, t) \end{bmatrix}
\]

Lyapunov functional [J.M. Coron, 2007]

\[
S, R \in S^2_+, \quad V_{\text{pde}}(\chi) = \int_0^1 \chi^\top(x, t) (S + xR) \chi(x, t) dx
\]

Lyapunov stability for and ODE/String system
But $\chi$ does not appear in the ODE.

$$\dot{X}(t) = AX(t) + Bu(1, t)$$
But $\chi$ does not appear in the ODE.

$$\dot{X}(t) = AX(t) + Bu(1, t) + Bu(0, t) - Bu(0, t)$$
But $\chi$ does not appear in the ODE.

\[
\dot{X}(t) = (A + BK)X(t) + B \int_{0}^{1} u_x(x, t) \, dx
\]
But $\chi$ does not appear in the ODE.

\[
\dot{X}(t) = (A + BK)X(t) + B \int_0^1 u_x(x, t) dx
\]

\[
2c \int_0^1 u_x(x) dx = \int_0^1 \chi^+(x) dx - \int_0^1 \chi^-(x) dx.
\]
But $\chi$ does not appear in the ODE.

\[
\dot{X}(t) = (A + BK)X(t) + B \int_0^1 u_x(x, t) dx
\]

\[
\begin{aligned}
\dot{X}(t) &= (A + BK)X(t) + \tilde{B} \chi_0(t) \\
\tilde{B} &= \frac{1}{2c} B \begin{bmatrix} 1 & -1 \end{bmatrix}, \\
\chi_0(t) &= \int_0^1 \chi(x, t) dx.
\end{aligned}
\]
But $\chi$ does not appear in the ODE.

\[
\begin{aligned}
\dot{X}(t) &= (A + BK)X(t) + \tilde{B} \chi_0(t) \\
\tilde{B} &= \frac{1}{2c} B \begin{bmatrix} 1 & -1 \end{bmatrix}, \quad \chi_0(t) = \int_0^1 \chi(x, t) \, dx.
\end{aligned}
\]

\[
\dot{\chi}_0 = c \int_0^1 \chi_x(x) \, dx = c [\chi(1) - \chi(0)].
\]
But $\chi$ does not appear in the ODE.

\[
\begin{cases}
    \begin{bmatrix} \dot{X} \\ X_0 \end{bmatrix}(t) = \begin{bmatrix} A + BK & \tilde{B} \\ 0_{2,n} & 0_2 \end{bmatrix} \begin{bmatrix} X \\ X_0 \end{bmatrix}(t) + \begin{bmatrix} 0_{n,2} \\ cl_2 \end{bmatrix} (\chi(1) - \chi(0)) \\
    \tilde{B} = \frac{1}{2c} B \begin{bmatrix} 1 & -1 \end{bmatrix}, \\
    X_0(t) = \int_0^1 \chi(x, t) dx.
\end{cases}
\]
But $\chi$ does not appear in the ODE.

\[
\begin{cases}
\begin{bmatrix}
\dot{X} \\
\dot{X}_0
\end{bmatrix}(t) = 
\begin{bmatrix}
A + BK & \tilde{B} \\
0_{2,n} & 0_2
\end{bmatrix}
\begin{bmatrix}
X \\
X_0
\end{bmatrix}(t) + 
\begin{bmatrix}
0_{n,2} \\
cl_2
\end{bmatrix}(\chi(1) - \chi(0)) \\
\tilde{B} = \frac{1}{2c}B \begin{bmatrix} 1 & -1 \end{bmatrix}, \\
X_0(t) = \int_0^1 \chi(x, t)dx.
\end{cases}
\]

Lyapunov Functional Candidate for Auxiliary System

\[
P \in \mathbb{S}^{n+2}_+, \quad \mathcal{V}_{ode}(X, u) = 
\begin{bmatrix}
X(t) \\
X_0(t)
\end{bmatrix}^\top P 
\begin{bmatrix}
X(t) \\
X_0(t)
\end{bmatrix}.
\]

Lyapunov stability for and ODE/String system
But $\chi$ does not appear in the ODE.

$$\begin{cases}
\dot{X}(t) = \left[ A + BK \ \tilde{B} \right] X(t) + \begin{bmatrix} 0_{n,2} \end{bmatrix} (\chi(1) - \chi(0)) \\
\tilde{B} = \frac{1}{2c} B \begin{bmatrix} 1 & -1 \end{bmatrix}, \quad X_0(t) = \int_0^1 \chi(x, t) dx.
\end{cases}$$

Lyapunov Functional Candidate for Auxiliary System

$P \in \mathbb{S}_+^{n+2}$, \quad $V_{ode}(X, u) = \begin{bmatrix} X(t) \\ X_0(t) \end{bmatrix}^T P \begin{bmatrix} X(t) \\ X_0(t) \end{bmatrix}.$

Total Lyapunov Functional Candidate

$P \in \mathbb{S}_+^{n+2}$, $R, S \in \mathbb{S}_+^2$, \quad $V_0(X_0, u) = V_{ode}(X_0, u) + V_{pde}(u).$
Lyapunov stability for ODE/String system

System (3) \[
\begin{bmatrix}
X \\
u \\
u_t
\end{bmatrix}
\]

Auxiliary System

Lyapunov Functional Candidate

\[
\begin{bmatrix}
X \\
\chi
\end{bmatrix}
\]
Auxiliary System

Lyapunov Functional Candidate $\begin{bmatrix} X \\ \chi \end{bmatrix}$

Miracle?

System (3) $\begin{bmatrix} X \\ u \\ u_t \end{bmatrix}$

Auxiliary System $\begin{bmatrix} X \\ \chi \end{bmatrix}$

Lyapunov stability for and ODE/String system
$V_0(X_0, u) = X_0^T P X_0 + \int_0^1 \chi^T (x) (S + xR) \chi(x) dx$

This Lyapunov functional falls actually into three terms:

1. The quadratic term in $X$ introduced by the ODE;
2. The functional $\nu_{pde}$ for the stability of the string equation;
3. A cross-term between $X_0$ and $X$ described by the extended state $X_0$. 
1. Wellposedness and Regularity
2. Riemann Invariant and Auxiliary System
3. A First Stability Result
4. Extended Stability Analysis
5. Examples
A First Stability Result

**Theorem (1)**

Assume there exist \( P_0 \in S^{n+2}_+ \) and \( R, S \in S^2_+ \) such that the following linear matrix inequality\(^a\) holds

\[
\Psi_0 \prec 0,
\]

then there exists a unique solution to System (3) and it is exponentially stable in the sense of norm \( \| \cdot \|_\mathcal{H} \) i.e. there exist \( \gamma \geq 1, \delta > 0 \) such that the following estimate holds:

\[
\forall t > 0, \|(X(t), u(t), u_t(t))\|^2_\mathcal{H} \leq \gamma e^{-\delta t} \|(X^0, u^0, v^0)\|^2_\mathcal{H}.
\]

\(^a\) The expression of \( \Psi_0 \) is given in the article.
Sketch of proof

The proof follows this line:
If \( \psi_0 \prec 0 \) then there exists \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \mathbb{R}^+ \) s.t.

\[
\varepsilon_1 \|(X, u, u_t)\|_{\mathcal{H}}^2 \leq V_0(X_0, u) \leq \varepsilon_2 \|(X, u, u_t)\|_{\mathcal{H}}^2, \\
\dot{V}_0(X_0, u) \leq -\varepsilon_3 \|(X, u, u_t)\|_{\mathcal{H}}^2.
\]

To find \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \), we need Jensen’s inequality.
Sketch of proof

The proof follows this line:
If $\Psi_0 \prec 0$ then there exists $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \mathbb{R}^+$ s.t.

\[
\begin{align*}
\varepsilon_1 \| (X, u, u_t) \|^2_H & \leq V_0(X_0, u) \leq \varepsilon_2 \| (X, u, u_t) \|^2_H, \\
\dot{V}_0(X_0, u) & \leq -\varepsilon_3 \| (X, u, u_t) \|^2_H.
\end{align*}
\]

To find $\varepsilon_1, \varepsilon_2, \varepsilon_3$, we need Jensen’s inequality.

1. Then, $V$ is equivalent to $\| \cdot \|_H$ with $\dot{V} < 0$

2. $\Psi_0 \prec 0 \Rightarrow \text{He} \left( (A + BK)^\top Q \right) \prec 0$ with $Q \in \mathbb{R}^{n \times n}$

$\Rightarrow$ there exists a unique solution to System (3) converging exponentially to $0_H$. 

Lyapunov stability for and ODE/String system
**Remark**

The condition $He((A + BK)^T Q) \prec 0$ with $Q \in \mathbb{R}^{n \times n}$ does not imply $A + BK$ Hurwitz because $Q \notin S^n_+$. 
Some remarks about the proof

Remark

The condition $He \left( (A + BK)^{\top} Q \right) \prec 0$ with $Q \in \mathbb{R}^{n \times n}$ does not imply $A + BK$ Hurwitz because $Q \notin S^n_+$. 

Remark

Another necessary condition coming from $\Psi_0 \prec 0$ is $c_0 > 0$. The string must be damped to ensure the exponential stability of System (3). [Helmicki, 1991] shows that exponential stability cannot be reached if $c_0 \leq 0$. 

Lyapunov stability for and ODE/String system
Remark

The condition $He \left( (A + BK)^\top Q \right) \prec 0$ with $Q \in \mathbb{R}^{n \times n}$ does not imply $A + BK$ Hurwitz because $Q \not\in \mathbb{S}^n_+$. 

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But one question still remains...
Some remarks about the proof

Remark

The condition $He \left( (A + BK)^\top Q \right) \prec 0$ with $Q \in \mathbb{R}^{n \times n}$ does not imply $A + BK$ Hurwitz because $Q \notin S^n_+$. 

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Why the subscript $0$ everywhere?
Wellposedness and Regularity

Riemann Invariant and Auxiliary System

A First Stability Result

Extended Stability Analysis

Examples
Projection of the state

<table>
<thead>
<tr>
<th>Auxiliary System</th>
<th>State in Theorem 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \begin{bmatrix} X \ \chi \end{bmatrix} )</td>
<td>( \begin{bmatrix} X \ \int_0^1 \chi(x, \cdot)dx \end{bmatrix} )</td>
</tr>
<tr>
<td>( \mathbb{R}^n \times L^2 \times L^2 )</td>
<td>( \mathbb{R}^n \times \mathbb{R}^2 )</td>
</tr>
</tbody>
</table>

Lyapunov stability for an ODE/String system
Projection of the state

**Auxiliary System**

\[
\begin{bmatrix}
X \\
\chi
\end{bmatrix}
\]

\(\mathbb{R}^n \times L^2 \times L^2\)

**State in Theorem 1**

\[
\begin{bmatrix}
X \\
\int_0^1 \chi(x, \cdot) \, dx
\end{bmatrix}
\]

\(\mathbb{R}^n \times \mathbb{R}^2\)

Projection from \(L^2\) on \(\mathbb{R}\) using its average.

Idea: Enlarge the projection space! [Seuret and Gouaisbaut, 2015]
Bessel Inequality

Let \( \{ e_k \}_{k \in [0, N]} \) be an orthonormal basis of \( L^2 \) with respect to the dot product \( \langle \cdot, \cdot \rangle \) in \( L^2 \). For any function \( f \in L^2 \), the following integral inequality holds for all \( N \in \mathbb{N} \):

\[
\langle f, f \rangle \geq \sum_{k=0}^{N} \langle f, e_k \rangle^2.
\]

Furthermore, there is equality if \( N \to +\infty \) (Parseval).
Projection of the state

Bessel Inequality

Let \( \{ e_k \}_{k \in [0,N]} \) be an orthonormal basis of \( L^2 \) with respect to the dot product \( \langle \cdot, \cdot \rangle \) in \( L^2 \). For any function \( f \in L^2 \), the following integral inequality holds for all \( N \in \mathbb{N} \):

\[
\langle f, f \rangle \geq \sum_{k=0}^{N} \langle f, e_k \rangle^2.
\]

Furthermore, there is equality if \( N \to +\infty \) (Perseval).

Let’s choose \( \{ L_k \}_{k \in \mathbb{N}} \) the orthonormal family of Legendre polynomials on \( L^2(0, 1) \).
Projection of the state

Auxiliary System

State in Theorem 2

\[ X(t) \]

\[ \chi(x, t) \]

\[ X_N(t) = \begin{bmatrix}
     \langle \chi(\cdot, t), L_0 \rangle \\
     \vdots \\
     \langle \chi(\cdot, t), L_N \rangle
\end{bmatrix} \]

\[ \mathbb{R}^n \times L^2 \times L^2 \]

\[ \mathbb{R}^n \times \mathbb{R}^{2(N+1)} \]}
Projection of the state

<table>
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<th>State in Theorem 2</th>
</tr>
</thead>
</table>
| \([X(t)  
\chi(x, t)]\) | \([X_N(t) = \begin{bmatrix} X(t) 
\langle \chi(\cdot, t), L_0 \rangle 
\vdots 
\langle \chi(\cdot, t), L_N \rangle \end{bmatrix}\) |

\[V_N(X, u) = X_N^T P_N X_N + \int_0^1 \chi^T(x) (S + xR) \chi(x) dx.\]
Extended Stability Result

**Theorem (2)**

Let $N \geq 0$. Assume there exist $P_N \in \mathbb{S}_+^{n+2(N+1)}$ and $R, S \in \mathbb{S}_+^2$ such that the following LMI holds

$$
\Psi_N \prec 0,
$$

then there exists a unique solution to System (3) and it is exponentially stable in the sense of norm $\| \cdot \|_H$ i.e. there exist $\gamma \geq 1, \delta > 0$ such that the following estimate holds:

$$
\forall t > 0, \| (X(t), u(t), u_t(t)) \|_H^2 \leq \gamma e^{-\delta t} \| (X^0, u^0, v^0) \|_H^2.
$$

\[a\] The expression of $\Psi_N$ is given in the article.
Remark

Two necessary conditions for $\Psi_N$ to be in $\mathcal{S}_-$ are:

1. $A + BK$ not singular;
2. $c_0 > 0$. 
Some remarks on Theorem 2

Remark

Two necessary conditions for $\Psi_N$ to be in $\mathbb{S}_-$ are:

1. $A + BK$ not singular;
2. $c_0 > 0$.

Remark

The proof of this theorem relies on Bessel Inequality and follows the same lines than for Theorem 1. For $N = 0$, we recover exactly Theorem 1. If we denote by

$$C_N = \left\{ c > 0 \text{ s.t. } \exists P_N \in \mathbb{S}_+^{n+2(N+1)}, S, R \in \mathbb{S}_+^2, \Psi_N \prec 0 \right\},$$

it can be proven that $C_N \subseteq C_{N+1}$. There is a hierarchy of LMI conditions.
1. Wellposedness and Regularity
2. Riemann Invariant and Auxiliary System
3. A First Stability Result
4. Extended Stability Analysis
5. Examples
First Example: $A$ and $A + BK$ Hurwitz

\[ A = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad K = \begin{bmatrix} 0 & -2 \end{bmatrix}. \]

**Figure** – Stability chart for $A$ and $A + BK$ Hurwitz.
First Example: $A$ and $A + BK$ Hurwitz

\[
A = \begin{bmatrix}
-2 & 1 \\ 0 & -1 
\end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad K = \begin{bmatrix} 0 & -2 \end{bmatrix}.
\]

Stable open-loop and closed-loop systems are still stable for some tuples $(c, c_0)$.

⇒ The PDE is seen as a perturbation which needs to be fast enough to keep the system stable.
Second Example: only $A + BK$ Hurwitz

\[
A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad K = \begin{bmatrix} -10 & 2 \end{bmatrix}.
\]

**Figure** – Stability chart for $A$ unstable and $A + BK$ Hurwitz.
Last Example: neither $A$ nor $A + BK$ Hurwitz.

\[
A = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad K = \begin{bmatrix} 1 & 0 \end{bmatrix}.
\]

**Figure** – Stability chart with neither $A$ nor $A + BK$ Hurwitz.
Last Example: neither $A$ nor $A + BK$ Hurwitz.

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad K = [1 \ 0].$$

Unstable open-loop and closed-loop systems can be stabilized using some tuples $(c, c_0)$.

$\Rightarrow$ The PDE is not a perturbation anymore, it helps stabilizing.
Conclusion & Perspectives

Summary:

1. We derived 2 stability Theorems using an extended system and a state augmentation inside a **Lyapunov framework**;
2. Examples show **good estimation** of stability area for small $N$;
3. But an important and growing number of variables;
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1. We derived 2 stability Theorems using an extended system and a state augmentation inside a **Lyapunov framework**;
2. Examples show **good estimation** of stability area for small $N$;
3. But an important and growing number of variables;

Perspectives:

1. Finish the paper on frequency analysis;
2. Change the boundary conditions;
3. Extension to the synthesis of $K$ for a given $A, B, c, c_0$;
4. Include saturations, sampling, event-triggered...
Thanks for your attention!

Questions?