Design of hybrid control systems for continuous-time plants: from the Clegg integrator to the hybrid H_∞ controller

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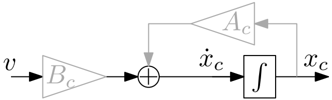
Outline

1. Clegg integrators and First Order Reset Elements (FORE) and an overview of hybrid dynamical systems
2. Exponential stability of FORE control systems
3. Stability/Performance analysis for a larger class of hybrid systems
4. Synthesis of higher order hybrid controllers
5. Conclusions and perspectives

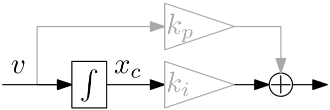
An analog integrator and its Clegg extension (1956)

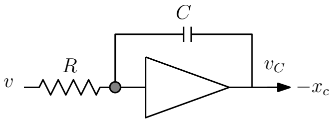
Integrators: core components of dynamical control systems



$$\dot{x}_c = A_c x_c + B_c v$$

Example: PI controller



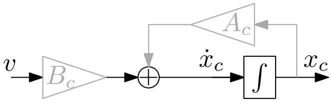


$$\dot{x}_c = \frac{1}{RC} v$$

- In an analog integrator, the state information is stored in a capacitor:

An analog integrator and its Clegg extension (1956)

Integrators: core components of dynamical control systems



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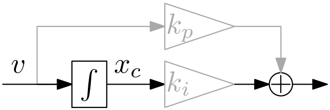
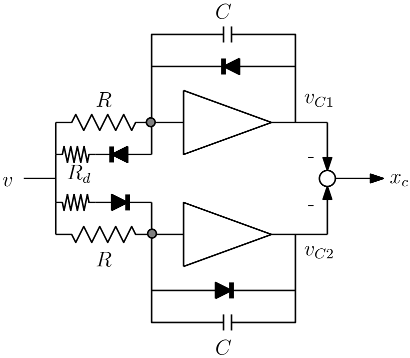




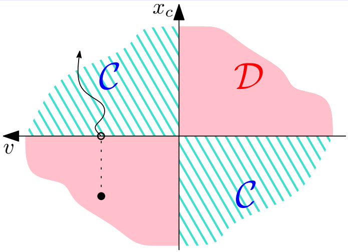
- Clegg's integrator (1956):
  - *feedback diodes*: the **positive** part of x_c is all and only coming from the **upper** capacitor (and viceversa)
  - *input diodes*: when $v \leq 0$ the upper capacitor is reset and the lower one integrates (and viceversa) $[R_d \ll 1]$
- As a consequence $\Rightarrow$ v and x_c **never have opposite signs**

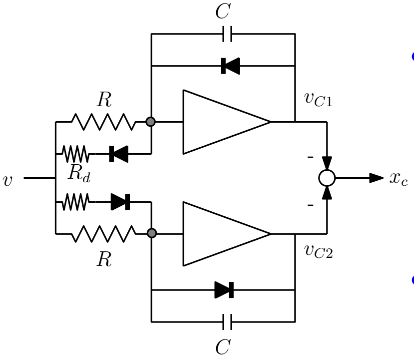
Hybrid dynamics may flow or jump

Hybrid Clegg integrator:

$$\begin{aligned} \dot{x}_c &= \frac{1}{RC} v, & \text{allowed when } x_c v \geq 0, \\ x_c^+ &= 0, & \text{allowed when } x_c v \leq 0, \end{aligned}$$

- Flow set $\mathcal{C}$: where x_c may flow (1st eq'n)
- Jump set $\mathcal{D}$: where x_c may jump (2nd eq'n)





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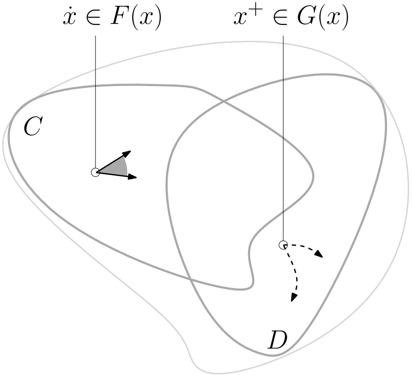
- As a consequence $\Rightarrow$ v and x_c **never have opposite signs**

Hybrid dynamical systems review: **dynamics**

$\mathcal{H} = (\mathcal{C}, \mathcal{D}, F, G)$

- $n \in \mathbb{N}$ (state dimension)
- $\mathcal{C} \subseteq \mathbb{R}^n$ (flow set)
- $\mathcal{D} \subseteq \mathbb{R}^n$ (jump set)
- $F : \mathcal{C} \rightrightarrows \mathbb{R}^n$ (flow map)
- $G : \mathcal{D} \rightrightarrows \mathbb{R}^n$ (jump map)

$$\mathcal{H} : \begin{cases} \dot{x} \in F(x), & x \in \mathcal{C} \\ x^+ \in G(x), & x \in \mathcal{D} \end{cases}$$



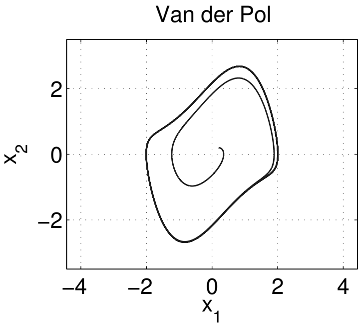
Hybrid dynamical systems review: **continuous dynamics**

$\mathcal{H} = (\mathcal{C}, \mathcal{D}, F, G)$

- $n \in \mathbb{N}$ (state dimension)
- $\mathcal{C} \subseteq \mathbb{R}^n$ (flow set)
- $\mathcal{D} \subseteq \mathbb{R}^n$ (jump set)
- $F : \mathcal{C} \rightrightarrows \mathbb{R}^n$ (flow map)
- $G : \mathcal{D} \rightrightarrows \mathbb{R}^n$ (jump map)

$$\mathcal{H} : \begin{cases} \dot{x} \in F(x), & x \in \mathcal{C} \\ x^+ \in G(x), & x \in \mathcal{D} \end{cases}$$

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + x_2(1 - x_1^2) \end{cases}$$



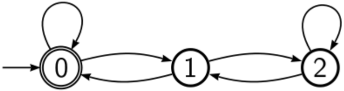
Hybrid dynamical systems review: **discrete dynamics**

$\mathcal{H} = (\mathcal{C}, \mathcal{D}, F, G)$

- $n \in \mathbb{N}$ (state dimension)
- $\mathcal{C} \subseteq \mathbb{R}^n$ (flow set)
- $\mathcal{D} \subseteq \mathbb{R}^n$ (jump set)
- $F : \mathcal{C} \rightrightarrows \mathbb{R}^n$ flow map)
- $G : \mathcal{D} \rightrightarrows \mathbb{R}^n$ (jump map)

$$\mathcal{H} : \begin{cases} \dot{x} \in F(x), & x \in \mathcal{C} \\ x^+ \in G(x), & x \in \mathcal{D} \end{cases}$$

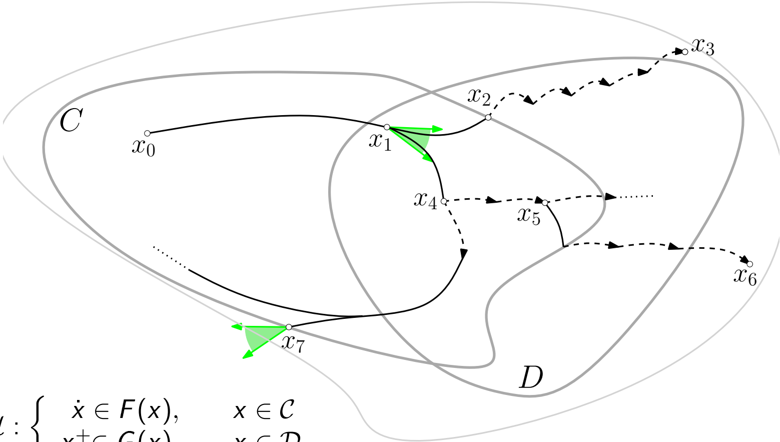
$$x^+ \in \begin{cases} \{0,1\} & \text{if } x = 0 \\ \{0,2\} & \text{if } x = 1 \\ \{1,2\} & \text{if } x = 2 \end{cases}$$



A possible sequence of states from $x_0 = 0$ is:

$$(0 \cdot 1 \cdot 2 \cdot 1)^i \quad i \in N$$

Hybrid dynamical systems review: **trajectories**

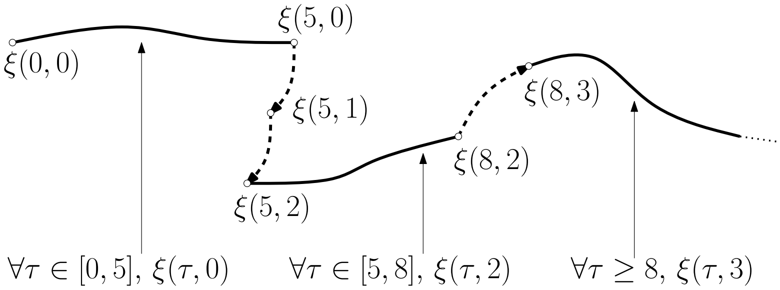


$$\mathcal{H}: \begin{cases} \dot{x} \in F(x), & x \in \mathcal{C} \\ x^+ \in G(x), & x \in \mathcal{D} \end{cases}$$

Hybrid dynamical systems review: **hybrid time**

The motion of the state is parameterized by two parameters:

- $t \in \mathbb{R}_{\geq 0}$, takes into account the elapse of time during the continuous motion of the state;
- $j \in \mathbb{Z}_{\geq 0}$, takes into account the number of jumps during the discrete motion of the state.



Hybrid dynamical systems review: **hybrid time**

$E \subseteq \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is a **compact hybrid time domain** if

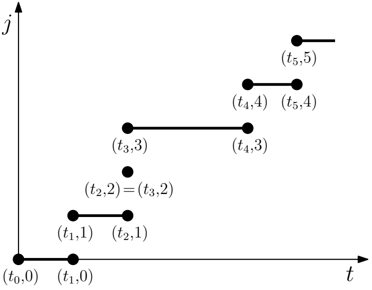
$$E = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}] \times \{j\})$$

where $0 = t_0 \leq t_1 \leq \cdots \leq t_J$.

E is a **hybrid time domain** if for all $(T, J) \in \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$

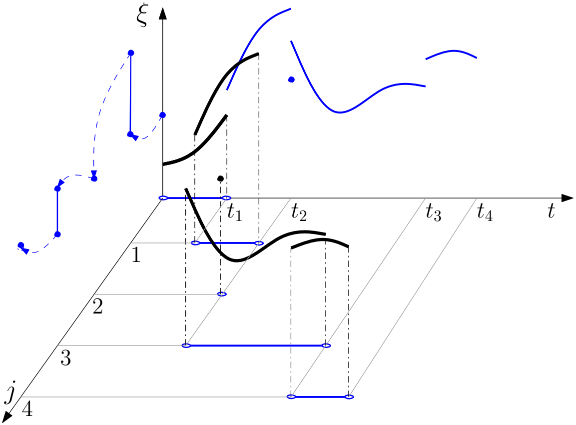
$$E \cap ([0, T] \times \{0, 1, \ldots, J\})$$

is a compact hybrid time domain.



Hybrid dynamical systems review: **solution**

- Formally, a solution satisfies the flow dynamics when flowing and satisfies the jump dynamics when jumping



Hybrid dynamical systems review: **Lyapunov theorem**

Theorem Given the Euclidean norm $|x| = \sqrt{x^T x}$ and a hybrid system

$$\mathcal{H} : \begin{cases} \dot{x} = f(x), & x \in \mathcal{C} \\ x^+ = g(x), & x \in \mathcal{D}, \end{cases}$$

aassume that function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfies for some scalars c_1, c_2 positive and c_3 positive:

$$\begin{aligned} c_1 |x|^2 \leq V(x) \leq c_2 |x|^2, &\qquad \forall x \in \mathcal{C} \cup \mathcal{D} \cup G(\mathcal{D}) \\ \langle \nabla V(x), f(x) \rangle \leq -c_3 |x|^2, &\qquad \forall x \in \mathcal{C}, \\ V(g(x)) - V(x) \leq -c_3 |x|^2, &\qquad \forall x \in \mathcal{D}, \end{aligned}$$

then the origin is uniformly globally exponentially stable (UGES) for $\mathcal{H}$, namely there exist $K, \lambda > 0$ such that all solutions satisfy

$$|\xi(t,j)| \leq K e^{\lambda(t+j)} |\xi(0,0)|, \quad \forall (t,j) \in \operatorname{dom} \xi$$

Note: Lyapunov conditions comprise **flow** and **jump** conditions.
Note: UGAS is characterized in terms of hybrid time (t, j)

Hybrid dynamical systems review: **Lyapunov theorem**

Theorem Given a closed set $\mathcal{A} \subset \mathbb{R}^n$ and a hybrid system

$$\mathcal{H} : \begin{cases} \dot{x} \in F(x), & x \in \mathcal{C} \\ x^+ \in G(x), & x \in \mathcal{D}, \end{cases}$$

aassume that function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfies for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and ρ positive definite:

$$\begin{aligned} &\alpha_1(|x|_\mathcal{A}) \leq V(x) \leq \alpha_2(|x|_\mathcal{A}), && \forall x \in \mathcal{C} \cup \mathcal{D} \cup G(\mathcal{D}) \\ &\langle \nabla V(x), f \rangle \leq -\rho(|x|_\mathcal{A}), && \forall x \in \mathcal{C}, f \in F(x), \\ &V(g) - V(x) \leq -\rho(|x|_\mathcal{A}), && \forall x \in \mathcal{D}, g \in G(x) \end{aligned}$$

then $\mathcal{A}$ is uniformly globally asymptotically stable (UGAS) for $\mathcal{H}$, namely there exists $\beta \in \mathcal{KL}$ such that all solutions satisfy

$$|\xi(t,j)|_\mathcal{A} \leq \beta(|\xi(0,0)|_\mathcal{A}, t+j), \quad \forall (t,j) \in \mathrm{dom}\ \xi$$

Note: Lyapunov conditions comprise **flow** and **jump** conditions.
Note: UGAS is characterized in terms of hybrid time (t,j)

Hybrid dynamics and the Clegg integrator (recall)

Hybrid Clegg integrator:

$$\dot{x}_c = \frac{1}{RC} v, \qquad \text{allowed when } x_c v \geq 0,$$
$$x_c^+ = 0, \qquad \text{allowed when } x_c v \leq 0,$$

- Flow set $\mathcal{C}$: where x_c may flow (1st eq'n)
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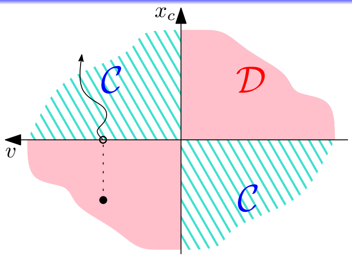
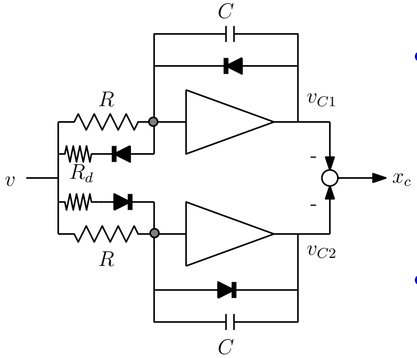




- Clegg's integrator (1956):
  - *feedback diodes*: the **positive** part of x_c is all and only coming from the **upper** capacitor (and viceversa)
  - *input diodes*: when $v \leq 0$ the upper capacitor is reset and the lower one integrates (and viceversa) $[R_d \ll 1]$
- As a consequence $\Rightarrow$ v and x_c **never have opposite signs**

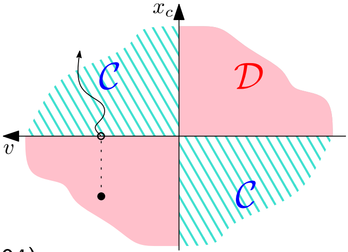
Hybrid dynamics of the Clegg integrator (revisited)

Hybrid Clegg integrator:

$$\dot{x}_c(t,j) = (RC)^{-1} v(t,j), \quad x_c(t,j)v(t,j) \geq 0,$$

$$x_c(t,j+1) = 0, \quad x_c(t,j)v(t,j) \leq 0,$$

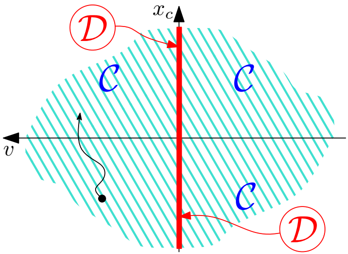
- Flow set $\mathcal{C} := \{(x_c, v) : x_c v \geq 0\}$ is closed
- Jump set $\mathcal{D} := \{(x_c, v) : x_c v \leq 0\}$ is closed
- Stability is robust! (Teel 2006–2012)



Previous models (Clegg '56, Horowitz '73, Hollot '04):

$$\begin{aligned} \dot{x}_c &= (RC)^{-1} v, && \text{if } v \neq 0, \\ x_c^+ &= 0, && \text{if } v = 0, \end{aligned}$$

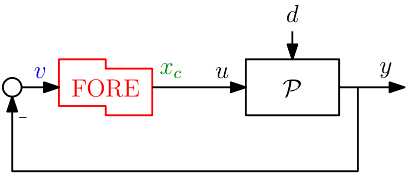
- Imprecise: solutions $\exists$ s.t. $x_c v < 0$, but
  Clegg's x_c and v **always** have same sign!
- Unrobust: $\mathcal{C}$ is almost all $\mathbb{R}^2$
  (arbitrary small noise disastrous)
- Unsuitable: Adds extra solutions
  $\Rightarrow$ Lyapunov results too conservative!



Stabilization using hybrid jumps to zero

First Order Reset Element (Horowitz '74):

$$\dot{x}_c = a_c x_c + b_c v, \qquad x_c v \geq 0,$$
$$x_c^+ = 0, \qquad x_c v \leq 0,$$

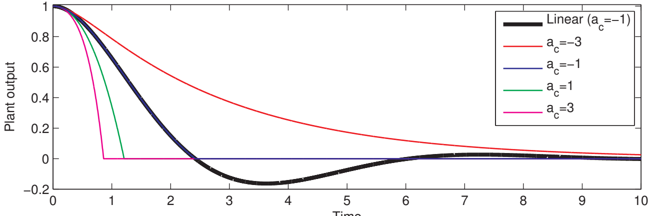


Theorem If $\mathcal{P}$ is linear, minimum phase and relative degree one, **then** a_c, b_c or (a_c, b_c) large enough $\Rightarrow$ uniform global exponential stability

Theorem In the planar case, γ_{dy} shrinks to zero as parameters grow

Simulation uses: $\mathcal{P} = \dfrac{1}{s}$

$b_c = 1$



Interpretation: Resets remove overshoots, instability improves transient

Piecewise quadratic Lyapunov function construction

- Given $N \geq 2$ (number of sectors)
- Patching angles:

$$-\theta_\epsilon = \theta_0 < \theta_1 < \cdots < \theta_N = \frac{\pi}{2} + \theta_\epsilon$$

- Patching hyperplanes ($C_p = [0 \ \cdots \ 0 \ 1]$)

$$\Theta_i = \begin{bmatrix} 0_{1\times(n-2)} & \sin(\theta_i) & \cos(\theta_i) \end{bmatrix}^T$$

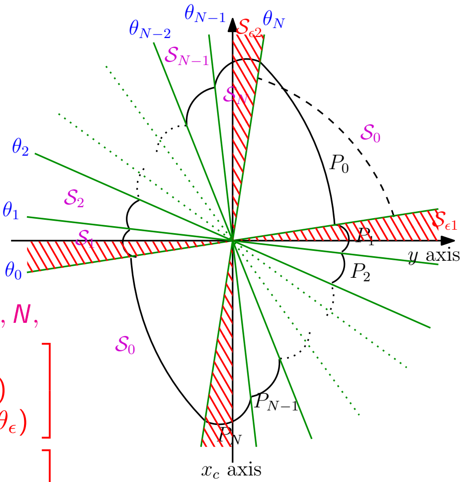
- Sector matrices:

$$S_0 := \Theta_0 \Theta_N^T + \Theta_N \Theta_0^T$$

$$S_i := -(\Theta_i \Theta_{i-1}^T + \Theta_{i-1} \Theta_i^T), \quad i = 1, \ldots, N,$$

$$S_{\epsilon 1} := \begin{bmatrix} 0_{(n-2)\times(n-2)} & 0 & 0 \\ 0 & 0 & \sin(\theta_\epsilon) \\ 0 & \sin(\theta_\epsilon) & -2\cos(\theta_\epsilon) \end{bmatrix}$$

$$S_{\epsilon 2} := \begin{bmatrix} 0_{(n-2)\times(n-2)} & 0 & 0 \\ 0 & -2\cos(\theta_\epsilon) & \sin(\theta_\epsilon) \\ 0 & \sin(\theta_\epsilon) & 0 \end{bmatrix}$$

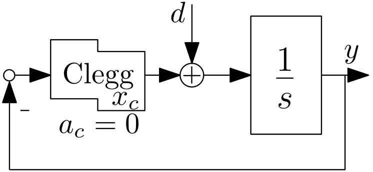


Hybrid closed-loop:

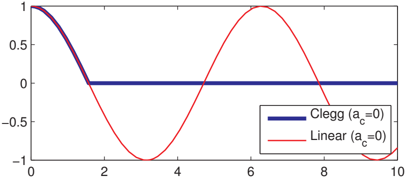
$$\begin{aligned} \dot{x} &= A_F x + B_w w, & x \in \mathcal{C} \\ x^+ &= A_J x, & x \in \mathcal{D} \end{aligned}$$

Example 1: Clegg ($a_c = 0$) connected to an integrator

- Block diagram:



- Output response (overcomes linear systems limitations)

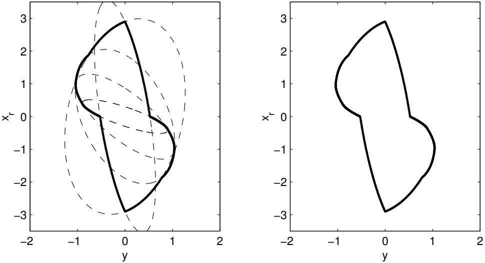


- Quadratic Lyapunov functions are unsuitable

- Gain γ_{dy} estimates ($N = \#$ of sectors)

N	2	4	8	50
gain γ_{dy}	2.834	1.377	0.914	0.87

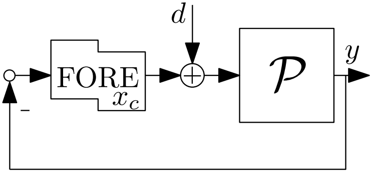
- A lower bound: $\sqrt{\frac{\pi}{8}} \approx 0.626$
- Lyapunov func'n level sets for $N = 4$



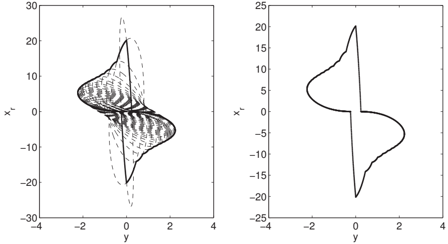
- $P_1, \ldots, P_4$ cover 2nd/4th quadrants
- P_0 covers 1st/3rd quadrants

Example 2: FORE (any a_c) and linear plant (Hollot et al.)

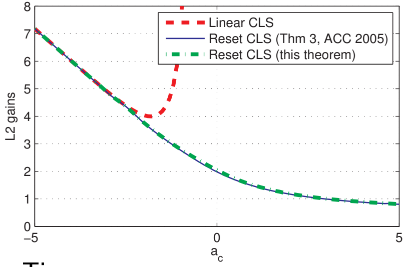
- Block diagram ($\mathcal{P} = \frac{s+1}{s(s+0.2)}$)



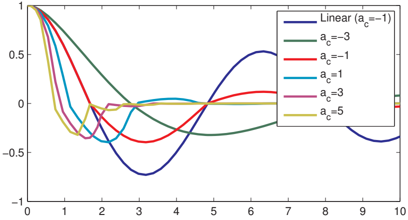
- $a_c = 1$: level set with $N = 50$



- Gain γ_{dy} estimates



- Time responses

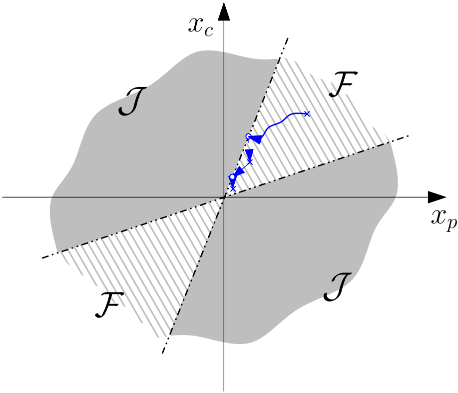


A class of homogeneous hybrid systems

$$\mathcal{H}\begin{cases} \dot{x} &= Ax + Bw \\ \dot{\tau} &= 1 - \mathrm{dz}\left(\frac{\tau}{\rho}\right) \\ x^+ &= Gx \\ \tau^+ &= 0 \\ z &= C_z x + D_{zw} w \end{cases} \quad \begin{matrix} (x,\tau) \in \mathcal{C} \\ \\ (x,\tau) \in \mathcal{D} \end{matrix}$$

$$\mathcal{C} = \{(x,\tau) : x \in \mathcal{F} \text{ or } \tau \in [0,\rho]\}$$
$$\mathcal{D} = \{(x,\tau) : x \in \mathcal{J} \text{ and } \tau \in [\rho, 2\rho]\}$$

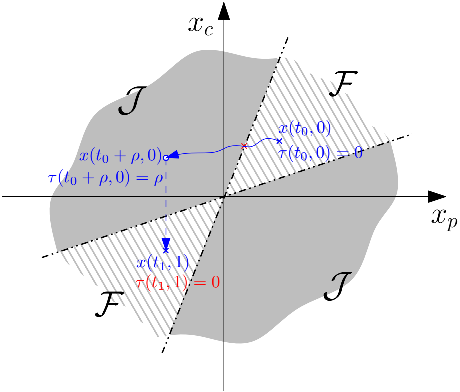
$$\mathcal{F} = \left\{x \in \mathbb{R}^n : x^\top M x \leq 0\right\}$$
$$\mathcal{J} = \left\{x \in \mathbb{R}^n : x^\top M x \geq 0\right\}$$



Ideal behavior

Dwell time can force the solution to flow in the set $\mathcal{J}$

$$\mathcal{H}\begin{cases} \dot{x} = Ax + Bw & (x,\tau) \in \mathcal{C} \\ \dot{\tau} = 1 - \mathrm{dz}\left(\frac{\tau}{\rho}\right) & \\ x^+ = Gx & (x,\tau) \in \mathcal{D} \\ \tau^+ = 0 & \\ z = C_z x + D_{zw} w & \end{cases}$$



$$\mathcal{C} = \{(x,\tau) : x \in \mathcal{F} \text{ or } \tau \in [0,\rho]\}$$
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$$\mathcal{F} = \left\{x \in \mathbb{R}^n : x^\top M x \le 0\right\}$$
$$\mathcal{J} = \left\{x \in \mathbb{R}^n : x^\top M x \ge 0\right\}$$

The dwell time enables flow in the set $\mathcal{J}$ ($t_1 = t_0 + \rho$)

Disadvantage: flow in $\mathcal{J}$

Dwell-time prevents consecutive jumps from being too close

$$\mathcal{H}\begin{cases} \dot{x} = Ax + Bw & \\ \dot{\tau} = 1 - \mathrm{dz}\left(\frac{\tau}{\rho}\right) & (x,\tau) \in \mathcal{C} \\ x^+ = Gx & \\ \tau^+ = 0 & (x,\tau) \in \mathcal{D} \\ z = C_z x + D_{zw} w & \end{cases}$$

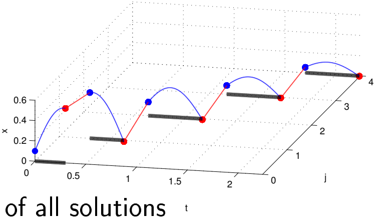
Dwell-time condition: Each solution ξ to $\mathcal{H}$ satisfies

$$t - s \geq \rho$$

for any pair of hybrid times (t,j), $(s,i) \in \mathrm{dom}(\xi)$, $(t,j) \geq (s,i)$

$$\mathcal{C} = \{(x,\tau) : x \in \mathcal{F} \text{ or } \tau \in [0,\rho]\}$$
$$\mathcal{D} = \{(x,\tau) : x \in \mathcal{J} \text{ and } \tau \in [\rho, 2\rho]\}$$

$$\mathcal{F} = \left\{x \in \mathbb{R}^n : x^\top M x \leq 0\right\}$$
$$\mathcal{J} = \left\{x \in \mathbb{R}^n : x^\top M x \geq 0\right\}$$



Advantage: persistent flow of all solutions

From dwell time $\Rightarrow$ performance wrt ordinary time t

Dwell-time allows us to use *classical* performance indexes.

Definition (t-decay rate)

Given a compact set $\mathcal{A} \subset \mathbb{R}^n$ and $w = 0$, $\mathcal{H}$ has t-decay rate $\alpha > 0$ if there exists $K > 0$ such that each solution x satisfies

$$|x(t,j)|_{\mathcal{A}} \leq K \exp(-\alpha t)|x(0,0)|_{\mathcal{A}}, \quad \text{for all } (t,j) \in \mathrm{dom}(x).$$

Definition (t-$\mathcal{L}_2$ gain)

Consider a set $\mathcal{A} \subset \mathbb{R}^n$ uniformly globally asymptotically stable for $\mathcal{H}$. $\mathcal{H}$ is finite t-$\mathcal{L}_2$ gain stable from w to z with gain (upper bounded by) $\gamma > 0$ if any solution x to $\mathcal{H}$ starting from $\mathcal{A}$ satisfies

$$\|x\|_{2t} \leq \gamma \|w\|_{2t}, \quad \text{for all } w \in t\text{-}\mathcal{L}_2.$$

Performance analysis result: $V(x) = x^\top P x$

Proposition: Consider system $\mathcal{H}$. If there exist matrices $P = P^\top > 0$, $\widetilde{M} = \widetilde{M}^\top$, non-negative scalars τ_F, τ_C, $\tau_R \in \mathbb{R}_{\geq 0}$ and positive scalars ϵ, $\bar{\gamma}$, such that

$$\begin{pmatrix} A^\top P + PA - (\widetilde{M} - \epsilon I) & PB & C_z^\top \\ B^\top P & -\bar{\gamma} I & D_{zw}^\top \\ C_z & D_{zw} & -\bar{\gamma} I \end{pmatrix} < 0, \tag{1a}$$

$$G^\top P G - P + \tau_R M \leq 0, \tag{1b}$$

$$\widetilde{M} - \tau_F M \leq \epsilon I, \tag{1c}$$

$$G^\top \widetilde{M} G + \tau_C M \leq 0. \tag{1d}$$

Then for any γ satisfying

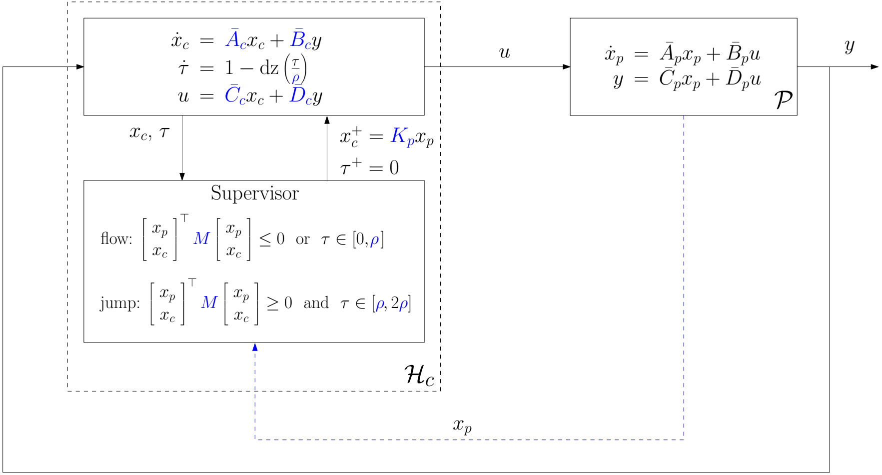
$$\gamma \geq \bar{\gamma}, \quad \gamma > \sqrt{2}|D_{zw}|, \tag{2}$$

there exists $\overline{\rho} > 0$ such that for any $\rho \in (0, \overline{\rho})$:

1) the set $\mathcal{A} = \{0\} \times [0, 2\rho]$ is uniformly globally exponentially stable for the hybrid system $\mathcal{H}$ with $w = 0$;
2) the t-$\mathcal{L}_2$ gain from w to z is less than or equal to γ, for all $w \in t$-$\mathcal{L}_2$.

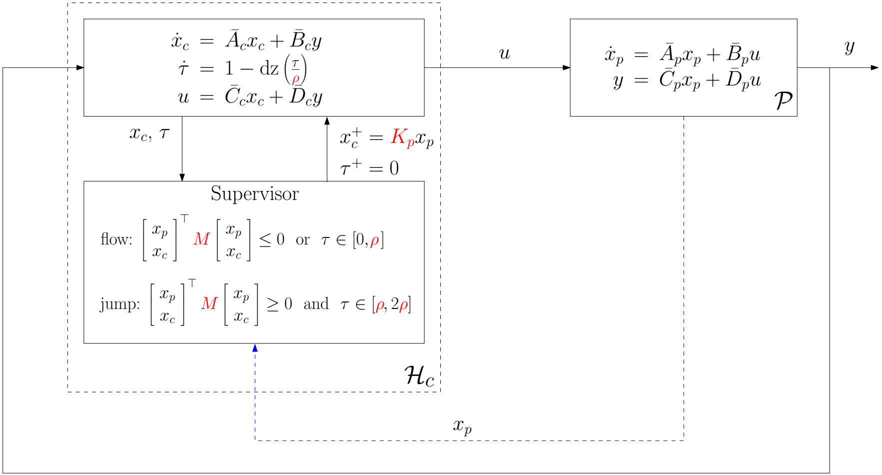
An architecture for homogeneous reset controller deisgn

May design the reset rules K_p, M, ρ only (case 1) or the whole dynamics (case 2)



Case 1: Flow dynamics is given, design Jump sets and rules

Matrices $\bar{A}_c, \bar{B}_c, \bar{C}_c, \bar{D}_c$ are given. Deisgn K_p, M and ρ



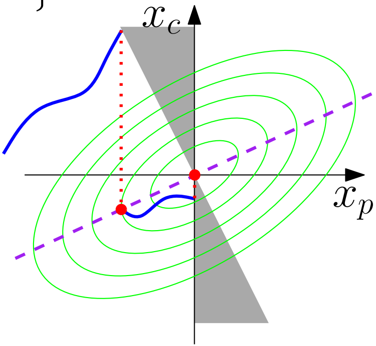
Design paradigm is inherited from Clegg integrator

- Reset to the minimizer of the hybrid Lyapunov function:

$$x_c^+ = \phi(x_p) := \operatorname*{argmin}_{x_c} V(x_p, x_c)$$

- Reset whenever the function $V_p(x_p) := V(x_p, \phi(x_p))$ is nondecreasing:

$$\begin{aligned} \mathcal{J} &= \left\{ \left[\begin{smallmatrix} x_p \\ x_c \end{smallmatrix}\right] \in \mathbb{R}^n : \left[\begin{smallmatrix} x_p \\ x_c \end{smallmatrix}\right]^\top M \left[\begin{smallmatrix} x_p \\ x_c \end{smallmatrix}\right] \geq 0 \right\}, \\ \left[\begin{smallmatrix} x_p \\ x_c \end{smallmatrix}\right]^\top M \left[\begin{smallmatrix} x_p \\ x_c \end{smallmatrix}\right] &= \langle \nabla V_p(x_p), A_p x_p + B_p x_c \rangle \\ &= \dot{V}_p(x_p, x_c) \end{aligned}$$



Design paradigm for the quadratic case $V(x) = x^T P x$

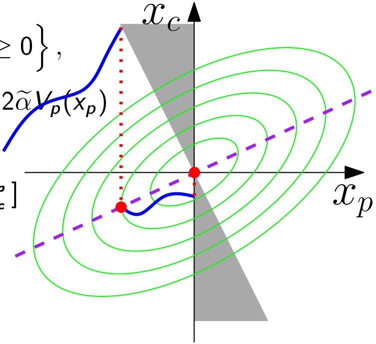
- Reset to the minimizer of the hybrid Lyapunov function:

$$x_c^+ = \phi(x_p) := \underset{x_c}{\operatorname{argmin}} \begin{bmatrix} x_p \\ x_c \end{bmatrix}^\top \begin{bmatrix} P_p & P_{pc} \\ P_{pc}^T & P_c \end{bmatrix} \begin{bmatrix} x_p \\ x_c \end{bmatrix} = -P_c^{-1} P_{pc}^T x_p = K_p x_p$$

- Reset to ensure nonincrease of

$$V_p(x_p) := V(x_p, \phi(x_p)) = x_p^T \underbrace{\left(P_p - P_{pc} P_c^{-1} P_{pc}^T\right)}_{\bar{P}_p} x_p$$

$$\begin{aligned}
\mathcal{J} &= \left\{ \begin{bmatrix} x_p \\ x_c \end{bmatrix} \in \mathbb{R}^n : \begin{bmatrix} x_p \\ x_c \end{bmatrix}^\top M \begin{bmatrix} x_p \\ x_c \end{bmatrix} \geq 0 \right\}, \\
\begin{bmatrix} x_p \\ x_c \end{bmatrix}^\top M \begin{bmatrix} x_p \\ x_c \end{bmatrix} &= \langle \nabla V_p(x_p), A_p x_p + B_p x_c \rangle + 2\tilde{\alpha} V_p(x_p) \\
&= \dot{V}_p(x_p, x_c) + 2\tilde{\alpha} V_p(x_p) \\
&= \begin{bmatrix} x_p \\ x_c \end{bmatrix}^\top \underbrace{2 \begin{bmatrix} \bar{P}_p(A_p + \tilde{\alpha} I) & \bar{P}_p B_p \\ 0 & 0 \end{bmatrix}}_{M} \begin{bmatrix} x_p \\ x_c \end{bmatrix}
\end{aligned}$$



Construction leaves many degrees of freedom

Theorem: Consider system $\mathcal{H}$ and assume that

$$\mathrm{He}\left(\bar{P}_p(A_p + B_p K_p) + \frac{\alpha}{2}\bar{P}_p\right) < 0, \quad \bar{P}_p = \bar{P}_p^\top > 0, \quad \alpha > 0. \qquad (3)$$

Then for each $\widetilde{\alpha} \in (0, \alpha]$, there exists a small enough $\rho > 0$ such that controller $\mathcal{H}_c$ with

$$M = 2\begin{bmatrix} \bar{P}_p(A_p + \widetilde{\alpha} I) & \bar{P}_p B_p \\ 0 & 0 \end{bmatrix}$$

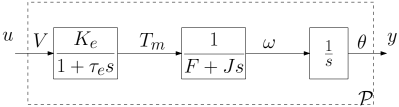
guarantees that:

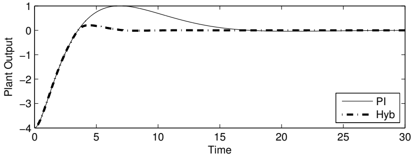
- the set $\mathcal{A} = \{0\} \times [0,\ 2\rho]$ is globally exponentially stable for $\mathcal{H}$;
- any solution with $x_c(0,0) = 0$ satisfies

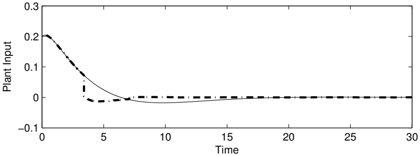
$$|x_p(t,j)| \leq K \exp\left(-\frac{\widetilde{\alpha}}{2}t\right)|x_p(0,0)|, \quad \forall (t,j) \in \mathrm{dom}(\xi). \qquad (4)$$

Positioning system: overshoot reduction

A DC motor controlled by a first oder filter



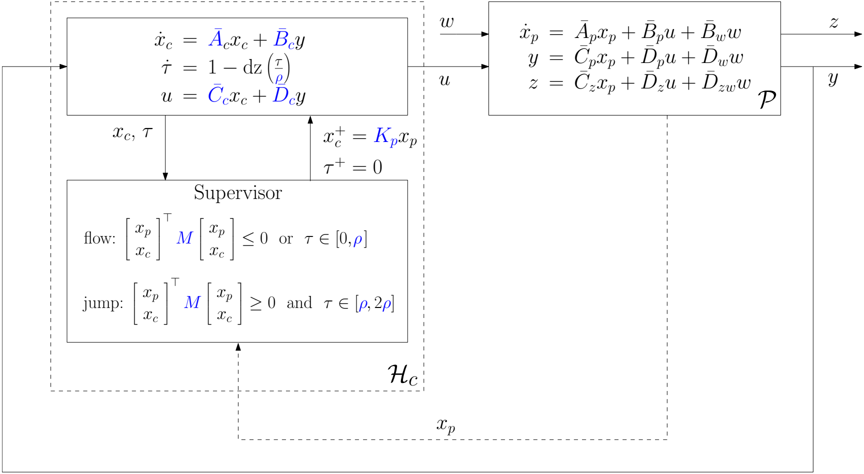




- optimal synthesis for **overshoot reduction** $x_p^T \overline{P}_p x_p \approx |y|^2$ or improvement of the **convergence rate** (using $\widetilde{\alpha}$)

Multi-objective hybrid $\mathcal{H}_\infty$ controller synthesis

Design all blue parameters $\bar{A}_c$, $\bar{B}_c$, $\bar{C}_c$, $\bar{D}_c$, K_p, M and ρ



Multi-objective hybrid $\mathcal{H}_\infty$: t-$\mathcal{L}_2$ gain and t-decay rate

Analysis conditions:
t-$\mathcal{L}_2$ gain

$$\begin{pmatrix} A^\top P + PA - (\widetilde{M} - \epsilon I) & PB & C_z^\top \\ B^\top P & -\bar{\gamma} I & D_{zw}^\top \\ C_z & D_{zw} & -\bar{\gamma} I \end{pmatrix} < 0$$

$$G^\top P G - P + \tau_R M \leq 0$$

$$\widetilde{M} - \tau_F M \leq \epsilon I$$

$$G^\top \widetilde{M} G + \tau_C M \leq 0$$

Reset controller:
t-decay rate

$$\mathrm{He}\left(\bar{P}_p(A_p + B_p K_p) + \frac{\alpha}{2}\bar{P}_p\right) < 0$$

Nonlinear coupling constraints:

$$P = \begin{bmatrix} P_p & P_{pc} \\ P_{pc}^T & P_c \end{bmatrix}$$

$$\bar{P}_p = P_p - P_{pc} P_c^{-1} P_{pc}^T$$

$\Longrightarrow$ Change of coordinates from Scherer, Gahinet, Chilali 1997 leads to

$$P := \begin{bmatrix} W & -W \\ -W & W + Z^{-1} \end{bmatrix}, \quad P^{-1} := \begin{bmatrix} Y & Z \\ Z & Z \end{bmatrix}, \quad \bar{P}_p = Y^{-1}$$

$$\Pi = \begin{bmatrix} Y & Z \\ I & 0 \end{bmatrix}, \quad \Pi P = \begin{bmatrix} I & 0 \\ W & -W \end{bmatrix}$$

Multi-objective synthesis then becomes an LMI problem

Theorem: Consider plant $\mathcal{P}$ and any solution to LMIs:

$$\begin{bmatrix} Y & I \\ I & W \end{bmatrix} > 0$$

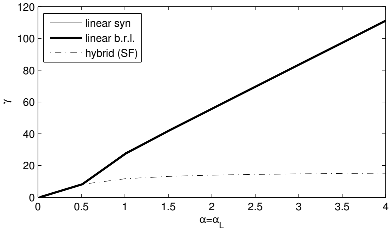
$$\mathrm{He}\left(\left[\begin{array}{cc|cc} \bar{A}_p Y + \bar{B}_p \hat{C} & \bar{A}_p + \bar{B}_p \hat{D} \bar{C}_p & \bar{B}_w + \bar{B}_p \hat{D} \bar{D}_w & Y \bar{C}_z^\top + \hat{C}^\top \bar{D}_z^\top \\ \hat{A} & W \bar{A}_p + \hat{B} \bar{C}_p & W \bar{B}_w + \hat{B} \bar{D}_w & \bar{C}_z^\top + \bar{C}_p^\top \hat{D}^\top \bar{D}_z^\top \\ \hline 0 & 0 & -\frac{\gamma}{2} I & \bar{D}_{zw}^\top + \bar{D}_w^\top \hat{D}^\top \bar{D}_z^\top \\ 0 & 0 & 0 & -\frac{\gamma}{2} I \end{array}\right]\right) < 0$$

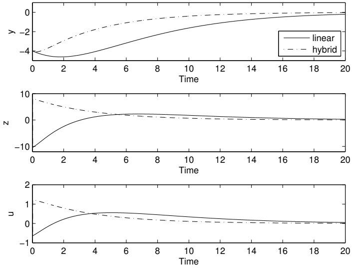
$$\mathrm{He}\left(\bar{A}_p Y + \bar{B}_p \hat{C} + \tfrac{\alpha}{2} Y\right) < 0$$

Then there exists a hybrid controller $\mathcal{H}_c$ such that:

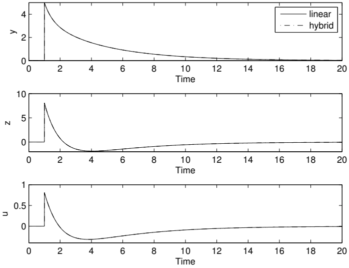
- the t-decay rate is equal to $\widetilde{\alpha}/2$, with $\widetilde{\alpha} \in (0, \alpha]$;
- the t-$\mathcal{L}_2$ gain from w to z less than or equal to γ, for all $w \in t$-$\mathcal{L}_2$.

DC motor: optimal t-$\mathcal{L}_2$ gain for a given $\widetilde{\alpha} = \alpha = 0.5$



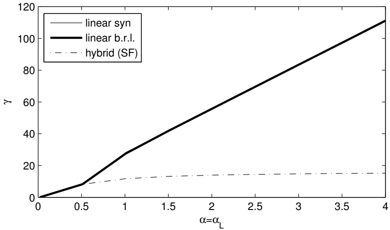


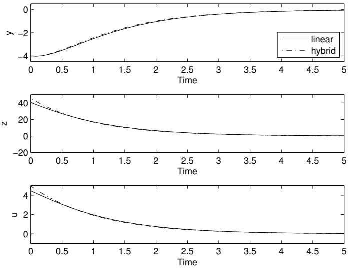
Free response



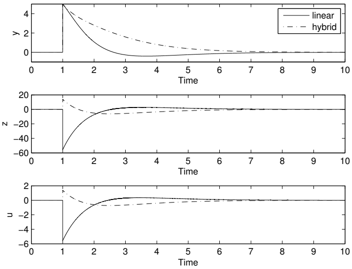
Forced response

DC motor: optimal t-$\mathcal{L}_2$ gain for a given $\widetilde{\alpha} = \alpha = 2$





Free response



Forced response

Conclusions and perspectives

1. Objective and motivation
   - exploit hybrid tools to push further the initial idea of Clegg in 1956
2. Revisiting Clegg and FORE
   - new modeling paradigm: flow only in half of the state space
   - can now give Lyapunov guarantees of exponential stability
   - exp instability before reset promises high performance
   - experimental tests on EGR valve control (Diesel engines)
3. Generalized reset controllers
   - A Lyapunov framework for stability and performance analysis
   - A hybrid $\mathcal{H}_\infty$ controller design
4. Perspectives
   - feedback from observed state (not covered here, partially done)
   - overcome performance limitations
   - improve synthesis scheme to allow for unstable continuous dynamics