Stubborn and Dead-Zone Redesign for State Observers and Dynamic Output Feedback

L. Zaccarian

LAAS-CNRS, Toulouse, France and
University of Trento, Italy

Work in collaboration with A. Alessandri, D. Astolfi, G. Casadei, S. Tarbouriech, M. Cocetti

Nancy, October 3, 2023

Outline

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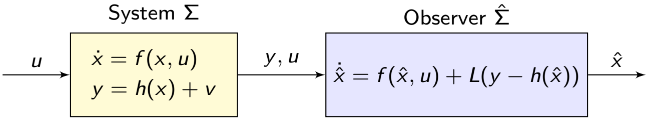
The observation problem



System Σ

- state $x \in \mathbb{R}^n$
- known external input $u \in U$
- measured output $y \in \mathbb{R}^m$

Observer $\widehat{\Sigma}$

- estimate $\hat{x} \in \mathbb{R}^n$
- unknown measurement noise $v \in \mathbb{R}^m$

ISS Observer

The observer $\widehat{\Sigma}$ is ISS (input-to-state stable) if

$$|x(t) - \hat{x}(t)| \leq \beta(|x(0) - \hat{x}(0)|, t) + \gamma\Big(\sup_{s\in[0,t)} |v(s)|\Big)$$

for all $t \geq 0$, for some $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$.

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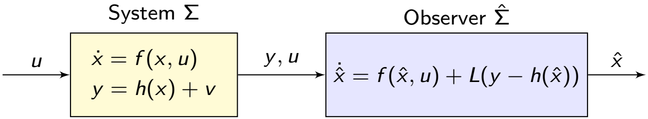

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Properties of an observer

Characteristics of an ISS Observer

- β characterizes the performance in nominal conditions ($v = 0$).
- γ characterizes the robustness with respect to v.

$$|x(t) - \hat{x}(t)| \leq \beta(|x(0) - \hat{x}(0)|, t) + \gamma\Big(\sup_{s\in[0,t)} |v(s)|\Big)$$

- Ideal behavior of an ISS observer:
  - ✓ Fast convergence:

    for example, $\beta(s, t) = ae^{-bt}|s|$, $a, b > 0$, with b "large."
  - ✓ Small Peaking:

    for example, $\beta(s, t) = ae^{-bt}|s|$, $a, b > 0$, with a "small."
  - ✓ Small asymptotic gain:

    for example, $\gamma(s) = \bar{\gamma}|s|$, with $\bar{\gamma} > 0$ "small."

✗ Trade-off between speed of convergence and asymptotic gain

The linear case: an example

Consider a linear system and a linear (Luenberger) observer

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + v \end{cases} \qquad \widehat{\Sigma} : \ \dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

with

- (A, C) detectable pair
- L gain of the observer to be chosen so that $A - LC$ is Hurwitz.

The dynamics of the estimation error $\tilde{x} := x - \hat{x}$ is given by

$$\dot{\tilde{x}} = (A - LC)\tilde{x} - Lv$$

and thus

$$|x(t) - \hat{x}(t)| \leq \left|e^{(A-LC)t}\right| |x(0) - \hat{x}(0)| + \gamma(|v|_\infty), \quad \begin{cases} (s,t) \mapsto \beta(s,t) := \left|e^{(A-LC)t}\right| |s| \\ s \mapsto \gamma(s) := |s||L| \int_0^\infty \left|e^{(A-LC)\tau}\right| d\tau \end{cases}$$

(finite-gain exponentially ISS).

✗ We cannot make both $A - LC$ s.t. $|e^{(A-LC)t}| \leq M \exp(-\alpha t)$ with α large (i.e., fast transient) and $\gamma(\cdot)$ small (i.e. insensitive to noise)

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- $\boldsymbol{L}$ gain of the observer to be chosen so that $A - \boldsymbol{L}C$ is Hurwitz.

The dynamics of the estimation error $\tilde{x} := x - \hat{x}$ is given by

$$\dot{\tilde{x}} = (A - \boldsymbol{L}C)\tilde{x} - \boldsymbol{L}v$$

and thus

$$|x(t) - \hat{x}(t)| \leq \left| e^{(A - \boldsymbol{L}C)t} \right| |x(0) - \hat{x}(0)| + \gamma(|v|_\infty), \quad \begin{cases} (s,t) \mapsto \boldsymbol{\beta}(s,t) := \left| e^{(A - \boldsymbol{L}C)t} \right| |s| \\ s \mapsto \boldsymbol{\gamma}(s) := |s||\boldsymbol{L}| \displaystyle\int_0^\infty \left| e^{(A - \boldsymbol{L}C)\tau} \right| d\tau \end{cases}$$

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Observer gain design for linear systems

- Based on additional hypothesis on v, we can figure out different approaches:
  - H_∞ design allows to minimize γ over all frequencies.
  - If v acts on some known frequencies $\omega \in [\underline{\omega}, \bar{\omega}]$, minimize the gain

$$G(j\omega) = \left[j\omega I - (A - \boldsymbol{L}C)\right]^{-1}\boldsymbol{L}\,.$$

  - Kalman filter for optimal gain design
  - ...

Observer design for nonlinear systems

✓ Exploring some structural properties of the system, many different designs exist

Property	Observer Technique
Detectability	Kazantis-Kravaris Luenberger (KKL) observers
Uniform observability	High-gain (HGO) observers
Lipschitz systems	LMI or circle-criterion approach
Input-affine systems	Riccati-like approach
Local observability	extended Kalman filters
...	...

✗ [Shim, Seo, & Teel, Automatica 2003, p. 890] and reference therein pointed out the fragility (lack of ISS) of certain nonlinear observers

✗ Few tools to analyze the effect of noises in the nonlinear framework:

- ISS gains based on Lyapunov analysis [Alessandri, Mathematics 2020];
- analysis of measurement noise in high-gain observers [Sanfelice & Praly, Automatica 2011], [Astolfi, Marconi, Praly, & Teel, NOLCOS 2016].

✗ Techniques to improve sensitivity to measurement noise are developed *ad hoc* (i.e. for specific classes of systems and/or observers)

Plan of the talk

- We follow a **redesign approach**.
- We focus on two special classes of measurement-noise perturbations v:
  1) outliers (i.e. sporadic impulsive noise);
  2) persistent "small" noise.
- Two techniques will be developed:
  1) dynamic saturation redesign ("stubborn redesign");
  2) dynamic dead-zone redesign.
- We will provide sufficient conditions to apply a general paradigm for the purpose of redesign of
  1) state observers for linear and nonlinear systems;
  2) static output feedback for synchronization of multi-agent systems;
  3) dynamic output feedback of linear plants.

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Bibliography

- The first part of this presentation is based on the following literature:
  - Stubborn redesign for linear observers

    [Alessandri, Zaccarian, "Results on stubborn Luenberger observers for linear time-invariant plants," ECC 2015]

    [Alessandri, Zaccarian, "Stubborn state observers for linear time-invariant systems," Automatica 2018]
  - Stubborn redesign for high-gain observers

    [Astolfi, Alessandri, Zaccarian, "Stubborn ISS redesign for nonlinear high-gain observers," IFAC WC 2017]
  - Dead-zone redesign for linear observers

    [Cocetti, Tarbouriech, Zaccarian, "On dead-zone observers for linear plants," ACC 2018]
  - Dead-zone redesign for high-gain observers

    [Cocetti, Tarbouriech, Zaccarian, "High-gain dead-zone observers for linear and nonlinear plants," IEEE LCSS 2019]
  - Stubborn and dead-zone redesign for nonlinear estimators

    [Astolfi, Alessandri, Zaccarian, "Stubborn and dead-Zone redesign for nonlinear observers and filters," IEEE TAC 2021]

Bibliography

- The second part of this presentation is adapted from

  [Casadei, Astolfi, Alessandri, Zaccarian, "Synchronization of interconnected linear systems via dynamic saturation redesign," IFAC NOLCOS 2019]

  [Casadei, Astolfi, Alessandri, Zaccarian, "Synchronization in networks of identical nonlinear systems via dynamic dead zones," IEEE LCSS 2019]

- The third part of this presentation is based on

  [Tarbouriech, Alessandri, Astolfi, Zaccarian, "LMI-based stubborn and dead-zone redesign in linear dynamic output feedback," 61st CDC and IEEE LCSS 2022]

Outline

Main Assumptions

System Σ

$u \rightarrow$

$$\dot{x} = f(x,u) + \boldsymbol{w}$$
$$y = h(x) + \boldsymbol{v}$$

$\xrightarrow{y,\,u}$

Observer $\widehat{\Sigma}$

$$\dot{z} = \varphi(z,u) + G\kappa(z, y - h(\hat{x})) + G\boldsymbol{d}$$
$$\hat{x} = \psi(z)$$

$\xrightarrow{\hat{x}}$

- state $x \in \mathbb{R}^n$
- known external input $u \in U$
- measured output $y \in \mathbb{R}^m$
- unknown perturbation $\boldsymbol{w} \in W$
- observer state $z \in \mathbb{R}^\varrho$, $\varrho \geq n$
- state estimate $\hat{x} \in \mathbb{R}^n$
- unknown measurement noise $\boldsymbol{v} \in V$
- unknown perturbation $\boldsymbol{d} \in D$

- G is a selection matrix
- κ is the correction term
- ψ maps the state of the observer z in the actual estimate $\hat{x}$
- We will use D^+ to denote the (upper-right) Dini derivative

$$D^+V(t) := \limsup_{h \to 0} \frac{V(t+h) - V(t)}{h}$$

Main Assumptions

System Σ

u

$$\dot{x} = f(x,u) + w$$
$$y = h(x) + v$$

y, u

Observer $\widehat{\Sigma}$

$$\dot{z} = \varphi(z,u) + G\kappa(z, y - h(\hat{x})) + Gd$$
$$\hat{x} = \psi(z)$$

$\hat{x}$

Assumption 1 (ISS Observer with ISS Lyapunov function)

There exists $V : X \times Z \to \mathbb{R}_{\geq 0}$, a function $\psi^{-R} : X \to Z$, $\underline{\alpha}, \bar{\alpha} \in \mathcal{K}_\infty$, and c, $\bar{\kappa}$ $c_v, c_w > 0$ such that

1. $x = \psi(\psi^{-R}(x))$ (pseudo-right-inverse)
2. $\|G\| \leq 1, \quad |\kappa(z, y_1) - \kappa(z, y_2)| \leq \bar{\kappa}|y_1 - y_2|$ (Lipschizianity)
3. $\underline{\alpha}(|x - \psi(z)|) \leq V(x,z) \leq \bar{\alpha}(|\psi^{-R}(x) - z|)$ ("sandwich")
4. $D^+V \leq -cV(x,z) + c_v|v| + c_w|w| + c_d|d|$ ("ISS bound")

for all $x \in X$, $u \in U$, $z \in Z$, $y_1, y_2 \in \mathbb{R}^m$, $(v, w, d) \in V \times W \times D$.

- The observer is supposed to be ISS also with respect to system disturbances w and observer perturbations d.

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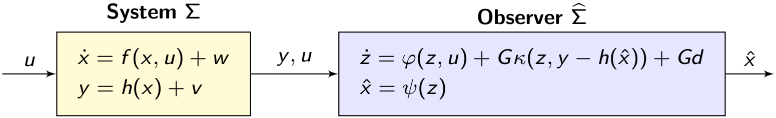

Assumption 2 (output-growth condition)

There exists $\ell_0, \ell_1, \ell_v, \ell_w, \ell_d > 0$ such that

5. $|h(x) - h(\hat{x})| \leq \ell_0 V(x,z)$
6. $|D^+(h(x) - h(\hat{x}))| \leq \ell_1 V(x,z) + \ell_v|v| + \ell_w|w| + \ell_d|d|$

for all $x \in X$, $u \in U$, $z \in Z$, $y_1, y_2 \in \mathbb{R}^m$, $(v, w, d) \in V \times W \times D$.

- Recall that $V(x,z) \geq \alpha(|x - \psi(z)|)$. Hence condition 5 holds if

$$|h(x) - h(\hat{x})| = |h(x) - h(\psi(z))| \leq k_0\, \alpha(|x - \psi(z)|), \qquad k_0 > 0\,.$$

- Condition 6 imposes a growth on the derivative of $y - \hat{y}$, with $\hat{y} := h(\hat{x})$.

Example: Input-affine systems (Besançon et al, 1996)

$$\Sigma : \begin{cases} \dot{x} = A(u)x + Bu \\ y = Cx \end{cases} \qquad \widehat{\Sigma} : \begin{cases} \dot{\hat{x}} = A(u)\hat{x} + Bu + P^{-1}C^\top(y - C\hat{x}) \\ \dot{P} = -2\mu P - A(u)^\top P - PA(u) + 2C^\top C \end{cases}$$

Observer with Lipschitz output injection term

$$\dot{z} = \varphi(z,u) + G\kappa(z, y - h(\hat{x})), \quad \hat{x} = \psi(z)$$

1. $x = \psi(\psi^{-R}(x))$
2. $|G| \leq 1, \qquad |\kappa(z,y_1) - \kappa(z,y_2)| \leq \bar{\kappa}|y_1 - y_2|$

✓ $z = (\hat{x}, \mathrm{vec}(P))$, $\psi(z) = [I\ 0](\hat{x}, \mathrm{vec}(P)) = \hat{x}$, $\psi^{-R}(x) = (x, 0)$

✓ $G = [I; 0]$, $\kappa(z,s) = P^{-1}C^\top s$

✓ Lipschizianity of κ follows from boundedness of P.

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Example: Input-affine systems (Sandwich and Directional Derivative)

$$\Sigma : \begin{cases} \dot{x} = A(u)x + Bu + \mathbf{w} \\ y = Cx + \mathbf{v} \end{cases} \qquad \widehat{\Sigma} : \begin{cases} \dot{\hat{x}} = A(u)\hat{x} + Bu + P^{-1}C^\top(y - C\hat{x}) + \mathbf{d} \\ \dot{P} = -2\mu P - A(u)^\top P - PA(u) + 2C^\top C \end{cases}$$

Observer with ISS Lyapunov function

3. $\underline{\alpha}(|x - \psi(z)|) \le \mathbf{V}(x,z) \le \bar{\alpha}(|\psi^{-R}(x) - z|)$
4. $D^+\mathbf{V} \le -c\mathbf{V}(x,z) + c_v|\mathbf{v}| + c_w|\mathbf{w}| + c_d|\mathbf{d}|$

- Suppose PE is verified $\Longrightarrow \underline{p}I \le P(t) \le \bar{p}I \ \forall\, t \ge 0$.
- Select $\mathbf{V}(x,z) := \sqrt{W(z,x)}$, $W(z,x) := (x - \hat{x})^\top P(x - \hat{x})$.

✓ $\sqrt{\underline{p}}|x - \hat{x}| \le \mathbf{V}(z,x) \le \sqrt{\bar{p}}|x - \hat{x}| \le \sqrt{\bar{p}}|\psi^{-R}(x) - z|$, $\quad \begin{cases} \psi^{-R}(x) = (x, 0) \\ z = (x, \mathrm{vec}(P)) \end{cases}$

✓ Furthermore, $\dot{W} = -2\mu W + 2(x - \hat{x})^\top[P(\mathbf{w} - \mathbf{d}) + C^\top \mathbf{v}]$, which gives

$$D^+\mathbf{V} \le -\mu\mathbf{V} + c_v|\mathbf{v}| + c_w|\mathbf{w}| + c_d|\mathbf{d}|.$$

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Example: Input-affine systems (Output Growth Condition)

$$\Sigma : \begin{cases} \dot{x} = A(u)x + Bu + \mathbf{w} \\ y = Cx + \mathbf{v} \end{cases} \qquad \widehat{\Sigma} : \begin{cases} \dot{\hat{x}} = A(u)\hat{x} + Bu + P^{-1}C^\top(y - C\hat{x}) + \mathbf{d} \\ \dot{P} = -2\mu P - A(u)^\top P - PA(u) + 2C^\top C \end{cases}$$

Observer with ISS Lyapunov function and output growth-condition

5. $|h(x) - h(\hat{x})| \le \ell_0 \mathbf{V}(x, z)$
6. $|D^+(h(x) - h(\hat{x}))| \le \ell_1 \mathbf{V}(x, z) + \ell_v|\mathbf{v}| + \ell_w|\mathbf{w}| + \ell_d|\mathbf{d}|$

✓ $|C(x - \hat{x})| \le \frac{|C|}{\sqrt{\underline{p}}} \mathbf{V}(x, z)$

✓ $$\begin{aligned} |D^+C(x - \hat{x})| &\le |C(A(u) - P^{-1}C^\top C)(x - \hat{x}) - CP^{-1}C^\top \mathbf{v} + C\mathbf{d} + C\mathbf{w}| \\ &\le \underbrace{(|C| \sup |A(u)| + |C|^3 \underline{p}^{-1})(\sqrt{\underline{p}})^{-1}}_{\ell_1} \mathbf{V}(x, z) + \\ &\quad + \underbrace{\underline{p}^{-1}|C|^2}_{\ell_v} |\mathbf{v}| + \underbrace{|C|}_{\ell_w} |\mathbf{w}| + \underbrace{|C|}_{\ell_d} |\mathbf{d}| \end{aligned}$$

Other examples

Assumptions 1-2 are verified by the following observer techniques:

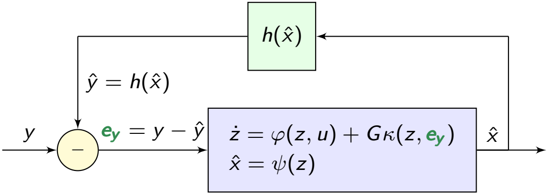
[Astolfi, Alessandri & Zaccarian (2021)]

- Kalman filter and extended Kalman filter [Kalman (1960), others]
- Linear Luenberger observers [Luenberger (1971)]
- Observers for Lipschitz systems based on LMI design or circle criterion [Rajamani (1998), Arcak & Kokotovic (2001)] [Zemouche & Boutayeb (2013)]
- Observers for input-affine systems with Riccati design [Besançon & Bornard & Hammouri (1996)]
- High-gain observers [Tornambé (1991), Khalil (1992)] [Gauthier & Kupka (2001)]
- Kazantzis-Kravaris/Luenberger observer [Andrieu & Praly (2006)]
- ...

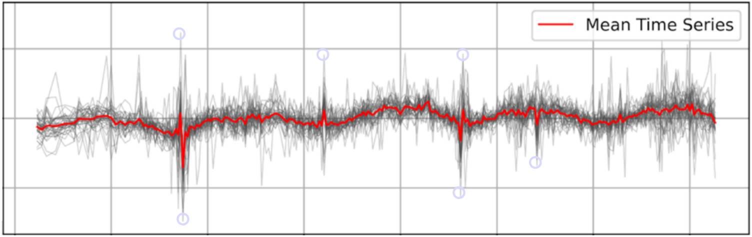
Outline

How to deal with outliers?

Observer:



We have $y = h(x) + v$ where v is supposed to be an outlier (sporadic impulsive perturbations).



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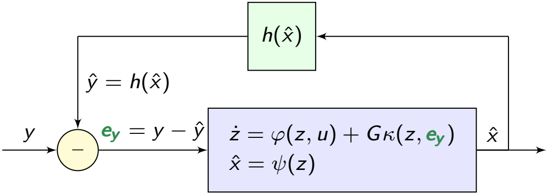

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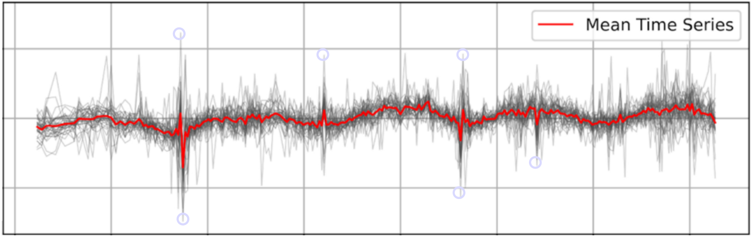

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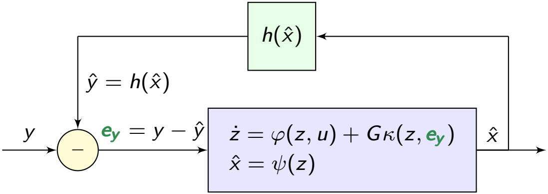

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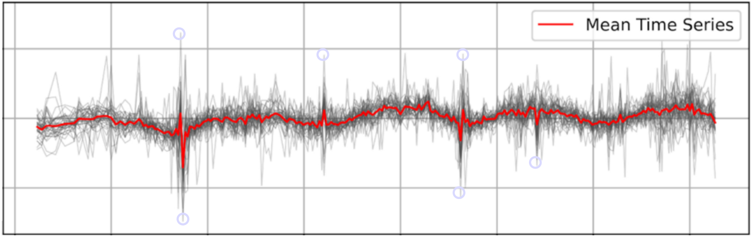

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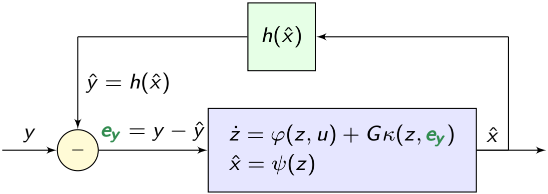

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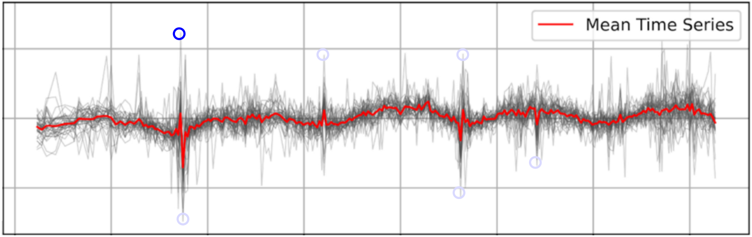

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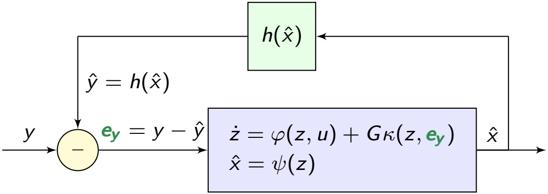

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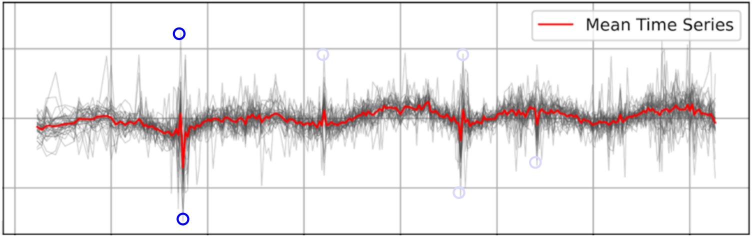

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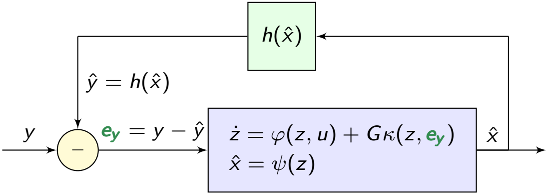

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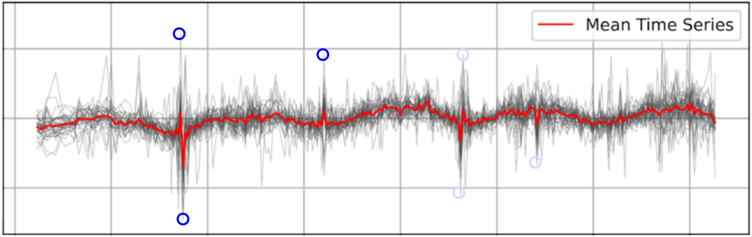

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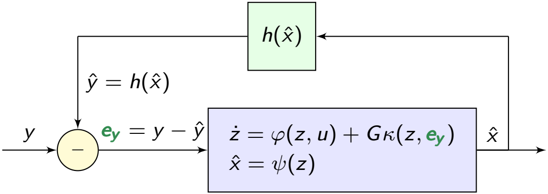

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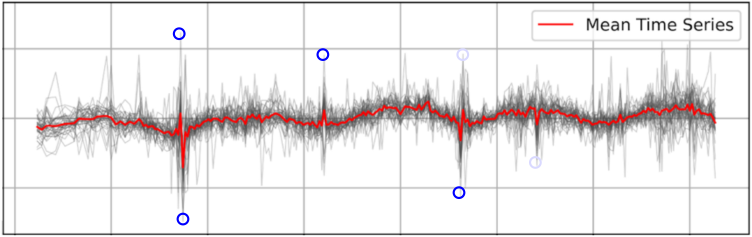

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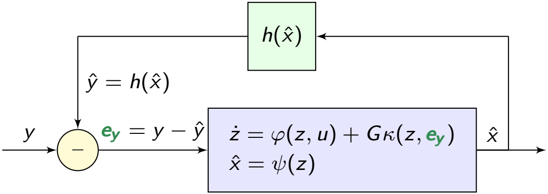

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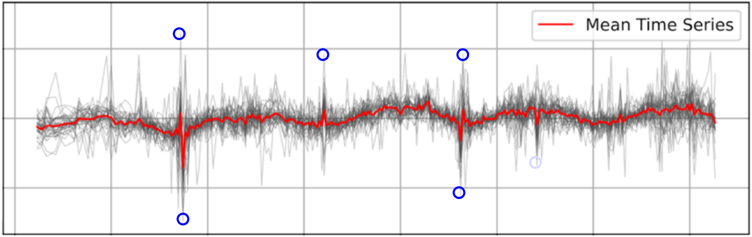

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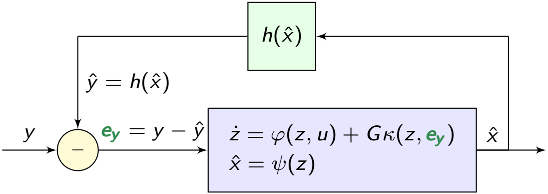

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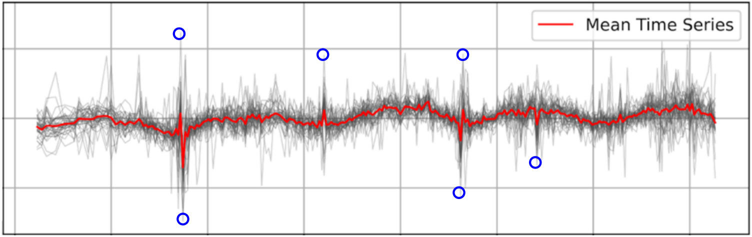

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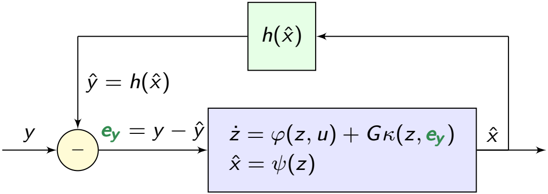

Main idea:

How to deal with outliers?

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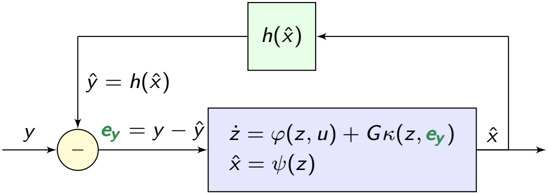

- Suppose e_y is very small for a long amount of time
  $\Longrightarrow$ x and $\hat{x}$ are close to each other
  $\Longrightarrow$ we **don't want to** use the correction term $\kappa(z, e_y)$

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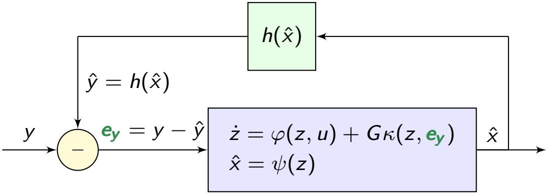

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- Suppose e_y is large but for a very short amount of time
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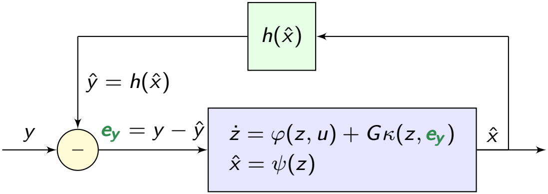

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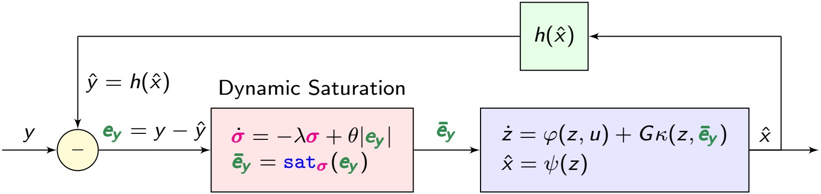
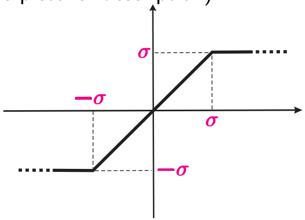
- Suppose e_y is large but for a very short amount of time
  $\Longrightarrow$ it could be an outlier!
  $\Longrightarrow$ we **don't want to** use the correction term $\kappa(z, e_y)$
- If e_y is large for a long amount of time
  $\Longrightarrow$ x and $\hat{x}$ are far from each other
  $\Longrightarrow$ we **need to** use the correction term $\kappa(z, e_y)$

A Dynamic Saturation Scheme

We modify the previous structure by adding a dynamic saturation for $e_y = y - \hat{y}$, i.e.,



where $\mathtt{sat}_\sigma(s) := \max\{-\sigma, \min\{\sigma, s\}\}$ and $\lambda, \theta > 0$ (from now on we consider a scalar output to simplify the pictorial description).



What happens in the presence of an outlier?

- If e_y is persistently small then σ becomes small.
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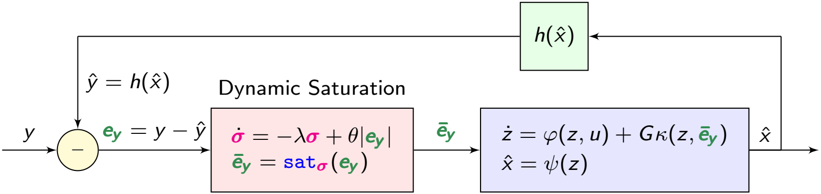

where $\mathtt{sat}_\sigma(s) := \max\{-\sigma, \min\{\sigma, s\}\}$ and $\lambda, \theta > 0$ (from now on we consider a scalar output to simplify the pictorial description).

- The level of the saturation is selected as the current value of σ.
- The level of σ is dynamically adapted based on the norm of $e_y = y - \hat{y}$.

What happens in the presence of an outlier?

- If e_y is persistently small then σ becomes small.
- If an outlier occurs, e_y becomes large but $\bar{e}_y$ is saturated at the current (small) value of σ, thus mitigating the outlier's effect on the estimate.
- If e_y is persistently large then σ becomes large and we desaturate (avoids intrinsic limitations of saturated feedback).

Saturation Redesign: Main Result

Theorem 1 [Astolfi, Alessandri & Zaccarian, IEEE TAC 2021]

Consider a given system Σ and observer $\widehat{\Sigma}$

$$\Sigma : \begin{cases} \dot{x} = f(x,u) + w \\ y = h(x) + v \end{cases} \qquad \widehat{\Sigma} : \begin{cases} \dot{z} = \varphi(z,u) + G\kappa(z, y - h(\hat{x})) + Gd \\ \hat{x} = \psi(z) \end{cases}$$

Suppose that $\widehat{\Sigma}$ is an ISS Observer for Σ satisfying the output-growth condition. Then, for any $\lambda > 0$ there exists a $\theta^\star > 0$ such that, for any $\theta > \theta^\star$, the observer

$$\widehat{\Sigma}_{\mathrm{sat}} : \begin{cases} \dot{z} = \varphi(z,u) + G\kappa(z, \mathtt{sat}_\sigma(y - h(\hat{x}))) + Gd \\ \dot{\sigma} = -\lambda\sigma + \theta|y - h(\hat{x})| \\ \hat{x} = \psi(z) \end{cases}$$

is an ISS Observer for Σ with ISS Lyapunov function satisfying the output-growth condition.

- If $y \in \mathbb{R}^m$, $m \geq 1$, then $\widehat{\Sigma}_{\mathrm{sat}}$ reads

$$\dot{z} = \varphi(z,u) + G\kappa\Big(z, \big[\mathtt{sat}_{\sigma_1}(y_1 - h_1(\hat{x})), \ldots, \mathtt{sat}_{\sigma_m}(y_m - h_m(\hat{x}))\big]\Big)$$
$$\dot{\sigma}_i = -\lambda_i\sigma_i + \theta_i|y_i - h_i(\hat{x})| \qquad i = 1, \ldots, m,$$

- In general, $\lambda \leq \theta$

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Sketch of Proof of Theorem 1

- Rewrite the observer $\widehat{\Sigma}_{\rm sat}$ as

$$\begin{cases} \dot{\boldsymbol{\sigma}} = -\lambda\boldsymbol{\sigma} + \theta|y - h(\hat{x})| \\ \dot{z} = \varphi(z,u) + G\kappa(z, y - h(\hat{x})) + G\boldsymbol{\delta} \\ \boldsymbol{\delta} = \kappa(z, \mathtt{sat}_{\boldsymbol{\sigma}}(y - h(\hat{x}))) - \kappa(z, y - h(\hat{x}))\,. \end{cases}$$

- In light of the Lipschitz properties of κ and saturation function, we have

$$|\boldsymbol{\delta}| \le \bar{\kappa}\left|\mathtt{sat}_{\boldsymbol{\sigma}}(y - h(\hat{x})) - (y - h(\hat{x}))\right| \le \bar{\kappa}|y - h(\hat{x})|\,.$$

- Use the Lyapunov function

$$(x, z, \boldsymbol{\sigma}) \mapsto W(x, z, \sigma) = \boldsymbol{V}(x, z) + \zeta\boldsymbol{\sigma} + (\zeta + \eta)\max\{|y - h(\hat{x})| - \boldsymbol{\sigma}, 0\}$$

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  with $\zeta, \eta > 0$.
- When $|y - h(\hat{x})| \leq \sigma$, then $\boldsymbol{\delta} = 0$, and we have

$$D^+W \leq -cV(x,z) - \zeta\lambda\sigma + \zeta\theta|y - h(\hat{x})|$$

  using the output-growth condition we obtain

$$D^+W \leq -(c - \zeta\theta\ell_0)V(x,z) - \zeta\lambda\sigma \leq 0$$

  for ζ small enough.

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  with $\zeta, \eta > 0$.
- When $|y - h(\hat{x})| > \sigma$, then $\boldsymbol{\delta} \neq 0$, and we have

$$D^+W \leq -cV(x,z) + c_d|\delta| + (\zeta + \eta)|D^+(y - h(\hat{x}))| + \eta\lambda\sigma - \eta\theta|y - h(\hat{x})|)\,.$$

  By using the output-growth condition we obtain

$$\begin{aligned} D^+W &\leq -\left[c - (\zeta + \eta)(\ell_1 + \ell_0\bar{\kappa}\ell_d)\right]V(x,z) - (\theta\eta - \lambda\eta + c_d\bar{\kappa}\ell_0)|y - h(\hat{x})| \\ &\leq 0 \end{aligned}$$

  for ζ, η small enough and θ large enough.

Sketch of Proof of Theorem 1

- Rewrite the observer $\widehat{\Sigma}_{\mathrm{sat}}$ as

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$$|\boldsymbol{\delta}| \leq \bar{\kappa} \left| \mathtt{sat}_{\boldsymbol{\sigma}}(y - h(\hat{x})) - (y - h(\hat{x})) \right| \leq \bar{\kappa} |y - h(\hat{x})| \,.$$

- Use the Lyapunov function

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  with $\zeta, \eta > 0$.

✓ We conclude that $D^+ W \leq -\varepsilon W$ for some $\varepsilon > 0$.

✓ The analysis can be done with $\boldsymbol{w}, \boldsymbol{v}, \boldsymbol{d}$ to show the desired ISS properties.

✓ It is not to hard to verify also the other properties for the redesigned observer.

Noise effect analysis for linear systems

- Consider the linear case

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + \boldsymbol{v} \in \mathbb{R} \end{cases} \qquad \widehat{\Sigma} : \; \dot{\hat{x}} = A\hat{x} + Bu + \boldsymbol{L}(y - C\hat{x})$$

and the (nonlinear) redesigned observer

$$\widehat{\Sigma}_{\rm sat} : \begin{cases} \dot{\hat{x}} = A\hat{x} + Bu + \boldsymbol{L}\ \mathtt{sat}_\sigma(y - C\hat{x}) \\ \dot{\sigma} = -\boldsymbol{\lambda}\sigma + \boldsymbol{\theta}|y - C\hat{x}|\,. \end{cases}$$

- Consider the error variables

$$\begin{aligned} \tilde{x}_0 &:= x - \hat{x}, & \hat{x} &\in \widehat{\Sigma}, \\ \tilde{x}_{\rm sat} &:= x - \hat{x}, & \hat{x} &\in \widehat{\Sigma}_{\rm sat}. \end{aligned}$$

- We are interested in analyzing the effect of two types of measurement noise $\boldsymbol{v}$:
  - impulsive noise (outlier);
  - constant noise.

Effect of outliers in the linear case

Proposition 1 [Astolfi, Alessandri & Zaccarian, IEEE TAC 2021]

- Suppose v is a piecewise constant perturbation of the form

$$v(t) = \delta_\tau(t) = \begin{cases} \frac{1}{\tau} & 0 \le t \le \tau \\ 0 & t > \tau \, . \end{cases}$$

- Suppose A is non-singular.
- Consider the solutions of $\widehat{\Sigma}$ and $\widehat{\Sigma}_{\rm sat}$ with $\tilde{x}_0(0) = \tilde{x}_{\rm sat}(0) = 0$ and $\sigma(0) = 0$.

Then, as τ tends to 0^+, we have

$$|\tilde{x}_{\rm sat}(\tau)| \; \le \; 2\tau\theta|\tilde{x}_0(\tau)| \, .$$

Furthermore, $|\tilde{x}_{\rm sat}(t)|$ converges to zero if $\tau \to 0^+$.

✓ The performance in the presence of outliers is improved.

✓ If the outlier is instantaneous ($\tau = 0$), its effect is completely erased!

- After $t \ge \tau$, there are no more perturbations $v = 0$
  $\Longrightarrow$ the observers evolves with initial conditions such that $|\tilde{x}_{\rm sat}(0)| < |\tilde{x}_0(0)|$.

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Effect of constant noise in the linear case

Proposition 2 [Astolfi, Alessandri & Zaccarian, IEEE TAC 2021]

- Suppose $\boldsymbol{v}$ is constant.
- Suppose $(A - \boldsymbol{L}C)$ is Hurwitz.

For any $\boldsymbol{\theta} \geq \boldsymbol{\lambda}$ the disturbance-to-error DC-gains between $\boldsymbol{v}$ and $|\tilde{x}_0|$, and between $\boldsymbol{v}$ and $|\tilde{x}_{\mathrm{sat}}|$, coincide.

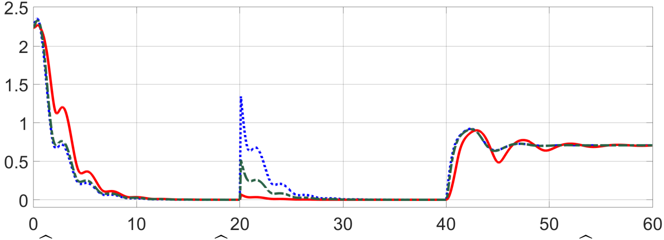
- For constant $\boldsymbol{v}$, the redesigned observer cannot do worse than the nominal one.

A Numerical Example

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + v \end{cases} \qquad \widehat{\Sigma}_{\rm sat} : \begin{cases} \dot{\hat{x}} = A\hat{x} + Bu + L\, \mathtt{sat}_\sigma(y - C\hat{x}) \\ \dot{\sigma} = -\lambda\sigma + \theta|y - C\hat{x}| \end{cases}$$

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad x(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \hat{x}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

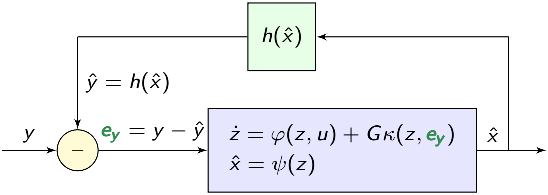
$$v(t) = \begin{cases} 0 & 0 \le t \le 20, \\ \delta_\tau(t) & 20 \le t \le 40, \\ 1 & 40 \le t \le 60, \end{cases} \qquad \tau = 0.01$$



• $|\tilde{x}_0|(t)$ of $\widehat{\Sigma}$ • $|\tilde{x}_{\rm sat}(t)|$ of $\widehat{\Sigma}_{\rm sat}$ with $\lambda = \theta = 10$ • $|\tilde{x}_{\rm sat}(t)|$ of $\widehat{\Sigma}_{\rm sat}$ with $\lambda = \theta = 1$

Outline

Another scenario: persistent small measurement noise



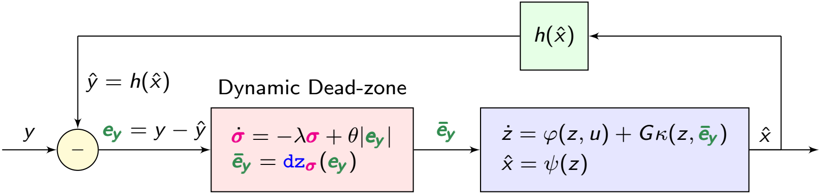
- Suppose $y = h(x) + v$, with v a small persistent measurement noise.
- When e_y is persistently small, we have $e_y \simeq v$:

  $\Longrightarrow$ the information given by e_y is not reliable

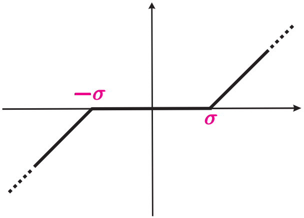
  $\Longrightarrow$ we want to trim out e_y.

A Dynamic Dead-Zone Scheme

We modify the previous structure by adding a dynamic dead-zone for $e_y = y - \hat{y}$, i.e.,



where $\mathtt{dz}_\sigma(s) := s - \mathtt{sat}_\sigma s$ and $\lambda, \theta > 0$.



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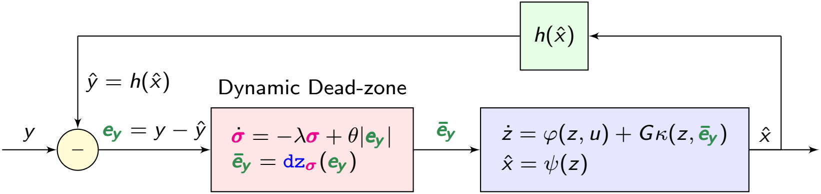
We modify the previous structure by adding a dynamic dead-zone for $e_y = y - \hat{y}$, i.e.,



where $\mathtt{dz}_\sigma(s) := s - \mathtt{sat}_\sigma s$ and $\lambda, \theta > 0$.

- The level of the dead-zone is selected as the current value of σ.
- If $|e_y| > \sigma$ then $|\mathtt{dz}_\sigma(e_y)| = |e_y|\left(1 - \frac{\sigma}{|e_y|}\right) \leq |e_y|$.
- If $|e_y| < \sigma$ then $\mathtt{dz}_\sigma(e_y) = 0$.
- We obtain a trimming effect of e_y.

Dead-Zone Redesign: Main Result

Theorem 2 [Astolfi, Alessandri & Zaccarian, IEEE TAC 2021]

Consider a given system Σ and observer $\widehat{\Sigma}$

$$\Sigma : \begin{cases} \dot{x} = f(x,u) + w \\ y = h(x) + v \end{cases} \qquad \widehat{\Sigma} : \begin{cases} \dot{z} = \varphi(z,u) + G\kappa(z, y - h(\hat{x})) + Gd \\ \hat{x} = \psi(z)\,. \end{cases}$$

Suppose that $\widehat{\Sigma}$ is an ISS observer for Σ satisfying the output-growth condition. Then, for any $\theta > 0$ there exists a $\lambda^\star > 0$ such that, for any $\lambda > \lambda^\star$, the observer

$$\widehat{\Sigma}_{\rm dz} : \begin{cases} \dot{z} = \varphi(z,u) + G\kappa(z, \mathtt{dz}_\sigma(y - h(\hat{x}))) + Gd \\ \dot{\sigma} = -\lambda\sigma + \theta|y - h(\hat{x})| \\ \hat{x} = \psi(z) \end{cases}$$

is an ISS observer for Σ with ISS Lyapunov function satisfying the output-growth condition.

- If $y \in \mathbb{R}^m$, $m \geq 1$, then $\widehat{\Sigma}_{\rm dz}$ reads

$$\begin{aligned} \dot{z} &= \varphi(z,u) + G\kappa\Big(z, \big[\mathtt{dz}_{\sigma_1}(y_1 - h_1(\hat{x})), \ldots, \mathtt{dz}_{\sigma_m}(y_m - h_m(\hat{x}))\big]\Big) \\ \dot{\sigma}_i &= -\lambda_i\sigma_i + \theta_i|y_i - h_i(\hat{x})| \qquad i = 1, \ldots, m. \end{aligned}$$

- In general, $\lambda \geq \theta$.

Sketch of Proof of Theorem 2

- Rewrite the observer $\widehat{\Sigma}_{\rm dz}$ as

$$\begin{cases} \dot{\sigma} = -\lambda\sigma + \theta|y - h(\hat{x})| \\ \dot{z} = \varphi(z,u) + G\kappa(z, y - h(\hat{x})) + G\delta \\ \delta = \kappa(z, \mathtt{dz}_{\sigma}(y - h(\hat{x}))) - \kappa(z, y - h(\hat{x}))\,. \end{cases}$$

- In light of the Lipschitz properties of κ and the dead-zone function, we have

$$|\delta| \leq \bar{\kappa}\left|\mathtt{dz}_{\sigma}(y - h(\hat{x})) - (y - h(\hat{x}))\right| \leq \bar{\kappa}\,\sigma\,.$$

- Use the Lyapunov function

$$(x, z, \sigma) \mapsto W(x, z, \sigma) = V(x, z) + \zeta\,\sigma$$

with $\zeta > 0$.

Sketch of Proof of Theorem 2

- Rewrite the observer $\widehat{\Sigma}_{\mathrm{dz}}$ as

$$\begin{cases} \dot{\boldsymbol{\sigma}} = -\lambda\boldsymbol{\sigma} + \theta|y - h(\hat{x})| \\ \dot{z} = \varphi(z,u) + G\kappa(z, y - h(\hat{x})) + G\boldsymbol{\delta} \\ \boldsymbol{\delta} = \kappa(z, \mathtt{dz}_{\boldsymbol{\sigma}}(y - h(\hat{x}))) - \kappa(z, y - h(\hat{x}))\,. \end{cases}$$

- In light of the Lipschitz properties of κ and the dead-zone function, we have

$$|\boldsymbol{\delta}| \le \bar{\kappa}\left|\mathtt{dz}_{\boldsymbol{\sigma}}(y - h(\hat{x})) - (y - h(\hat{x}))\right| \le \bar{\kappa}\,\boldsymbol{\sigma}\,.$$

- Use the Lyapunov function

$$(x, z, \boldsymbol{\sigma}) \mapsto W(x, z, \boldsymbol{\sigma}) = V(x,z) + \zeta\,\boldsymbol{\sigma}$$

with $\zeta > 0$.
- When $|y - h(\hat{x})| > \sigma$ then $\boldsymbol{\delta} = 0$ and we have

$$D^+W \le -cV(x,z) - \zeta\lambda\sigma + \zeta\theta|y - h(\hat{x})|$$

using the output-growht condition

$$D^+W \le -(c - \zeta\theta\ell_0)V(x,z) - \zeta\lambda\sigma \le 0$$

for ζ small enough (θ is fixed).

Sketch of Proof of Theorem 2

- Rewrite the observer $\widehat{\Sigma}_{\mathrm{dz}}$ as

$$\begin{cases} \dot{\boldsymbol{\sigma}} = -\lambda\boldsymbol{\sigma} + \theta|y - h(\hat{x})| \\ \dot{z} = \varphi(z,u) + G\kappa(z, y - h(\hat{x})) + G\boldsymbol{\delta} \\ \boldsymbol{\delta} = \kappa(z, \mathtt{dz}_{\boldsymbol{\sigma}}(y - h(\hat{x}))) - \kappa(z, y - h(\hat{x}))\,. \end{cases}$$

- In light of the Lipschitz properties of κ and the dead-zone function, we have

$$|\boldsymbol{\delta}| \le \bar{\kappa}\left|\mathtt{dz}_{\boldsymbol{\sigma}}(y - h(\hat{x})) - (y - h(\hat{x}))\right| \le \bar{\kappa}\,\boldsymbol{\sigma}\,.$$

- Use the Lyapunov function

$$(x, z, \boldsymbol{\sigma}) \mapsto W(x, z, \boldsymbol{\sigma}) = V(x,z) + \zeta\,\boldsymbol{\sigma}$$

with $\zeta > 0$.

- When $|y - h(\hat{x})| \le \sigma$ then $\boldsymbol{\delta} \ne 0$ and we have

$$D^+W \le -cV(x,z) + c_d|\delta| - \zeta\lambda\sigma + \zeta\theta|y - h(\hat{x})|$$

using the output-growth condition

$$D^+W \le -cV(x,z) - (\zeta\lambda - \zeta\theta - \bar{\kappa}c_d)\sigma \le 0$$

for λ large enough (ζ, θ are fixed).

Sketch of Proof of Theorem 2

- Rewrite the observer $\widehat{\Sigma}_{\mathrm{dz}}$ as

$$\begin{cases} \dot{\boldsymbol{\sigma}} = -\lambda \boldsymbol{\sigma} + \theta |y - h(\hat{x})| \\ \dot{z} = \varphi(z, u) + G\kappa(z, y - h(\hat{x})) + G\boldsymbol{\delta} \\ \boldsymbol{\delta} = \kappa(z, \mathtt{dz}_{\boldsymbol{\sigma}}(y - h(\hat{x}))) - \kappa(z, y - h(\hat{x})) \,. \end{cases}$$

- In light of the Lipschitz properties of κ and the dead-zone function, we have

$$|\boldsymbol{\delta}| \le \bar{\kappa} \left| \mathtt{dz}_{\boldsymbol{\sigma}}(y - h(\hat{x})) - (y - h(\hat{x})) \right| \le \bar{\kappa}\, \boldsymbol{\sigma} \,.$$

- Use the Lyapunov function

$$(x, z, \boldsymbol{\sigma}) \mapsto W(x, z, \boldsymbol{\sigma}) = V(x, z) + \zeta \boldsymbol{\sigma}$$

  with $\zeta > 0$.

✓ We conclude that $D^+W \le -\varepsilon W$ for some $\varepsilon > 0$.

✓ The analysis can be done with $\boldsymbol{w}, \boldsymbol{v}, \boldsymbol{d}$ to show the desired ISS properties.

✓ It is not to hard to verify also the other properties for the redesigned observer.

Noise effect analysis for linear systems

- Consider the linear case

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + v \in \mathbb{R} \end{cases} \qquad \widehat{\Sigma} : \; \dot{\hat{x}} = A\hat{x} + Bu + \boldsymbol{L}(y - C\hat{x})$$

  and the (nonlinear) redesigned observer

$$\widehat{\Sigma}_{\mathrm{dz}} : \begin{cases} \dot{\hat{x}} = A\hat{x} + Bu + \boldsymbol{L}\, \mathtt{dz}_{\sigma}(y - C\hat{x}) \\ \dot{\sigma} = -\boldsymbol{\lambda}\sigma + \boldsymbol{\theta}|y - C\hat{x}|\,. \end{cases}$$

- Consider the error variables

$$\begin{aligned} \tilde{x}_0 &:= x - \hat{x}, \qquad && \hat{x} \in \widehat{\Sigma}, \\ \tilde{x}_{\mathrm{dz}} &:= x - \hat{x}, \qquad && \hat{x} \in \widehat{\Sigma}_{\mathrm{dz}}. \end{aligned}$$

- We are interested in analyzing the effect of constant perturbations (approximation of persistently small noise).

Effect of constant noise in the linear case

Proposition 3 [Astolfi, Alessandri & Zaccarian, IEEE TAC 2021]

- Suppose $\boldsymbol{v}$ is constant.
- Suppose $(A - \boldsymbol{L}C)$ is Hurwitz and $CA^{-1}\boldsymbol{L} < 1$.

The disturbance-to-error DC-gains between $\boldsymbol{v}$ and $|\tilde{x}_0|$, denoted as k_0, and between $\boldsymbol{v}$ and $|\tilde{x}_{\mathrm{dz}}|$, denoted as k_{dz}, satisfy

$$k_{\mathrm{dz}} \leq \left[1 - \tilde{k}\left(\frac{\boldsymbol{\theta}}{\boldsymbol{\lambda}}\right)\right] \; k_0$$

for any $\boldsymbol{\lambda} > \boldsymbol{\theta} \geq 0$ and for some $\tilde{k} \in \mathcal{K}$.

- The DC-gain is always reduced thus improving the rejection to measurement noise.
- The condition $CA^{-1}\boldsymbol{L} < 1$ is always verified if both A and $(A - \boldsymbol{L}C)$ are Hurwitz.

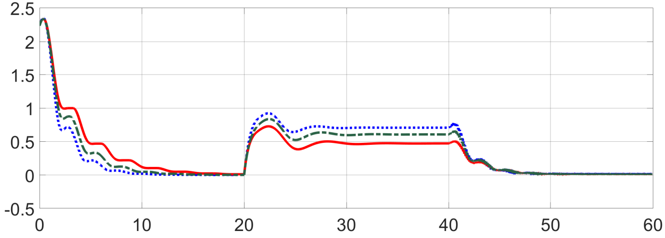
A Numerical Example

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + v \end{cases} \qquad \widehat{\Sigma}_{\text{sat}} : \begin{cases} \dot{\hat{x}} = A\hat{x} + Bu + L\,\mathtt{dz}_\sigma(y - C\hat{x}) \\ \dot{\sigma} = -\lambda\sigma + \theta|y - C\hat{x}| \end{cases}$$

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad x(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \hat{x}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$v(t) = \begin{cases} 0 & 0 \le t \le 20, \\ 1 & 20 \le t \le 40 \\ \sin(50t) & 40 \le t \le 60, \end{cases}$$

- $\sup_\infty |\tilde{x}_0(t)| \le 0.028$
- $\sup_\infty |\tilde{x}_{\mathrm{dz}}(t)| \le 0.022$
- $\sup_\infty |\tilde{x}_{\mathrm{dz}}(t)| \le 0.016$



- $|\tilde{x}_0(t)|$ of $\widehat{\Sigma}$
- $|\tilde{x}_{\mathrm{dz}}(t)|$ of $\widehat{\Sigma}_{\text{sat}}$ with $\lambda = 4$, $\theta = 1$
- $|\tilde{x}_{\mathrm{dz}}(t)|$ of $\widehat{\Sigma}_{\text{sat}}$ with $\lambda = 2$, $\theta = 1$

Application to vehicle lateral speed estimation

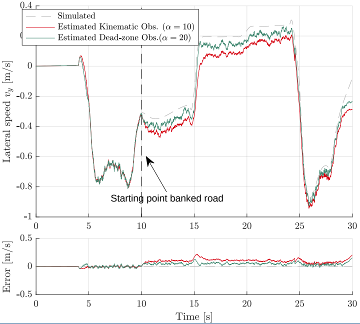
- linear parameter-varying model (parameter r = yaw rate)

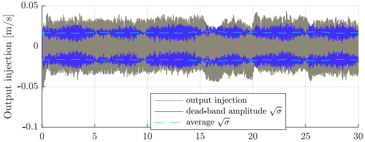
$$\begin{cases} \dot{x} = A(r)x + u \\ y = Cx, \end{cases} \quad A(r) := \begin{bmatrix} 0 & r \\ -r & 0 \end{bmatrix}, \quad C := \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

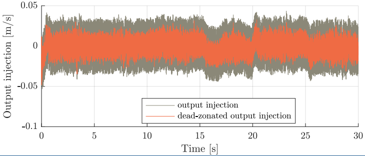
- yaw-rate dependent kinematic observer

$$\begin{cases} \dot{\hat{x}} = A(r)\hat{x} + u + L(r)(\hat{y} - y) \\ \hat{y} = C\hat{x}, \end{cases} \quad L(r) := \begin{bmatrix} -2\alpha|r| \\ (1-\alpha^2)r \end{bmatrix},$$

- deadzonated observer reduces the sensitivity to noise



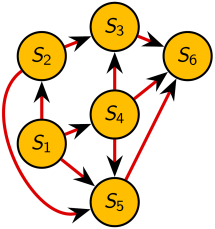




Outline

A synchronization problem for multi-agent systems

- Consider a synchronization problem among N identical agents



$$\dot{x}_i = Ax_i + \phi(x_i) + u_i \qquad i = 1, \dots, N,$$

$$y_i = Cx_i \qquad i = 1, \dots, N,$$

$$x_i \in \mathbb{R}^n, \quad u_i \in \mathbb{R}, \quad y_i \in \mathbb{R}.$$

- We want to achieve consensus among all the states

$$\lim_{t\to\infty} |x_i(t) - x_j(t)| = 0 \qquad \forall\ i, j \in \{1, \dots, N\}.$$

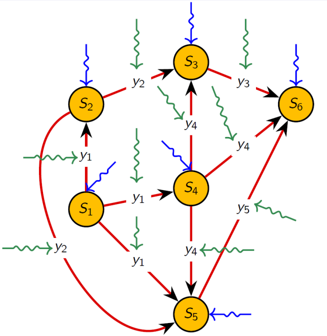
- We want to use a distributed control law.
- A typical solution is to use a diffusive coupling

$$u_i = -K \sum_{j=1}^{N} \ell_{ij}\, y_j \qquad \ell_{ij} \in L$$

where L is the Laplacian matrix of the graph.

Perturbations in networks

- What happens in the presence of perturbations?



$$\dot{x}_i = Ax_i + \phi(x_i) + u_i$$

$$y_i = Cx_i + \boldsymbol{v}_i$$

- If we have some ISS property then

$$\limsup_{t\to\infty} |x_i(t) - x_j(t)| \leq \boldsymbol{\gamma} \left(\sum_{i=1}^{N} |\boldsymbol{v}_i| \right) \qquad \forall\; i,j \in \{1,\dots,N\} \qquad \boldsymbol{\gamma} \in \mathcal{K}\,.$$

Perturbations in networks

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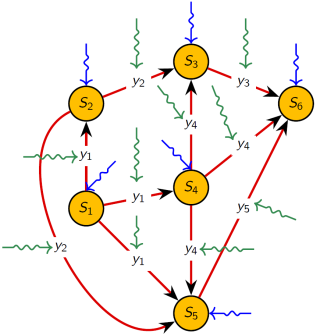


$$\dot{x}_i = Ax_i + \phi(x_i) + u_i$$

$$y_i = Cx_i + \boldsymbol{v}_i$$

✗ Design of diffusing coupling for nonlinear systems in the presence of output perturbations $\boldsymbol{v}_i$?

Saturation and Dead-Zone Redesign for Diffusing Coupling

- In the presence of impulsive disturbances (e.g. outliers, switching topologies, ...) we use a **dynamic saturation redesign**

$$\dot{\boldsymbol{\sigma}}_i = -\lambda \boldsymbol{\sigma}_i + \theta_i \left| \sum_{i=1}^{N} \ell_{ij}\, y_j \right| \qquad i = 1, \ldots, N$$
$$u_i = -\boldsymbol{K}\, \mathtt{sat}_{\boldsymbol{\sigma}_i} \left(\sum_{i=1}^{N} \ell_{ij}\, y_j \right) \qquad i = 1, \ldots, N\,.$$

- In the presence of persistent disturbances (communication networks noise) we use a **dynamic dead-zone redesign**

$$\dot{\boldsymbol{\sigma}}_i = -\lambda \boldsymbol{\sigma}_i + \theta_i \left| \sum_{i=1}^{N} \ell_{ij}\, y_j \right| \qquad i = 1, \ldots, N$$
$$u_i = -\boldsymbol{K}\, \mathtt{dz}_{\boldsymbol{\sigma}_i} \left(\sum_{i=1}^{N} \ell_{ij}\, y_j \right) \qquad i = 1, \ldots, N\,.$$

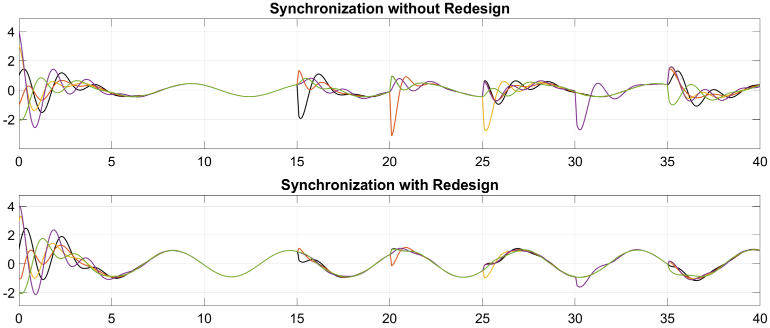
- Synchronization in nominal conditions $\boldsymbol{v}_i = 0$ is preserved.
- Performance in the presence of noise is improved.
- Proofs and philosophy design are very similar to the observer design.

Simulation Example for Saturation Redesign

Consider a network of $N = 6$ linear oscillators

$$\dot{x}_i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x_i + u_i, \qquad y_i = \begin{pmatrix} 1 & 0 \end{pmatrix} x_i + \boldsymbol{v}_i$$

in the presence of outliers $\boldsymbol{v}_i$.



Simulation Example for Saturation Redesign

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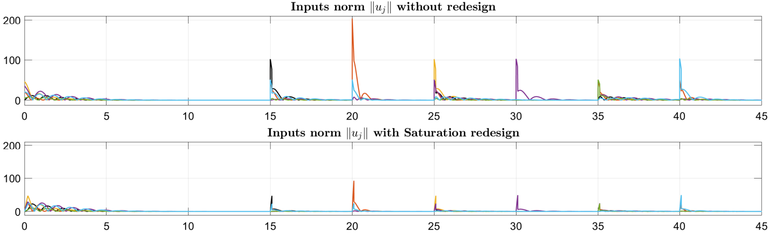
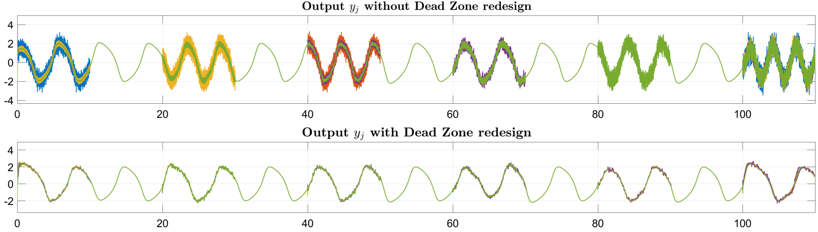


Simulation Example for Dead-Zone Redesign

Consider a network of $N = 5$ Van der Pol oscillators

$$\dot{x}_i = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix} x_i + \begin{pmatrix} 0 \\ -\mu \, x_{i_1}^2 x_{i_2} \end{pmatrix} + u_i, \qquad y_i = \begin{pmatrix} 1 & 0 \end{pmatrix} x_i + \boldsymbol{v}_i$$

in the presence of white noise $\boldsymbol{v}_i$.



Simulation Example for Dead-Zone Redesign

Consider a network of $N = 5$ Van der Pol oscillators

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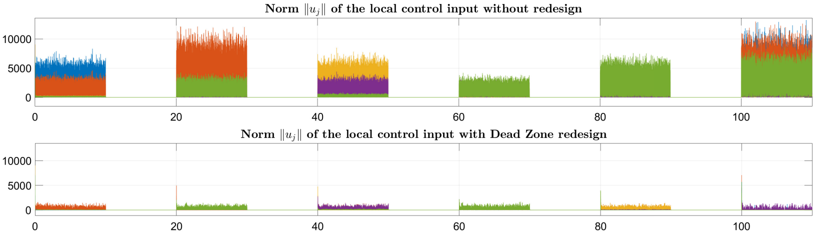
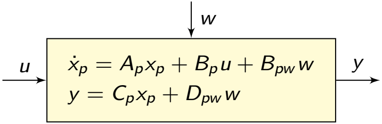
in the presence of white noise $\boldsymbol{v}_i$.



Outline

Dynamic Output Feedback

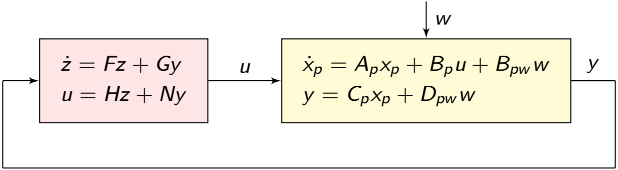


Plant

- state $x_p \in \mathbb{R}^{n_p}$
- known external input $u \in \mathbb{R}^m$
- measured output $y \in \mathbb{R}^p$
- unknown system and measurement disturbances $w \in \mathbb{R}^{n_d}$

Controller

- state $z \in \mathbb{R}^{n_c}$



Assumption 1

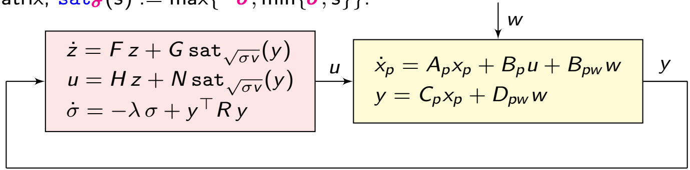
The linear closed-loop system with $w \equiv 0$ is globally exponentially stable to the origin.

Stubborn Redesgin to handle Measurment Outliers

- The controller dynamics is augmented with a new non-negative state $\sigma \in \mathbb{R}_{\geq 0}$:

$$\dot{z} = F\,z + G\,\texttt{sat}_{\sqrt{\sigma v}}(y)$$
$$u = H\,z + N\,\texttt{sat}_{\sqrt{\sigma v}}(y)$$
$$\dot{\sigma} = -\lambda\,\sigma + y^\top R\,y$$

where $\sqrt{\sigma v}$ are the componentwise square-roots of the elements of the non-negative vector v scaled by σ; $\lambda > 0$; R is a symmetric, positive definite matrix; $\texttt{sat}_{\sigma}(s) := \max\{-\sigma, \min\{\sigma, s\}\}$.



- Notice:
  - ✓ The level of the saturation is a scaled square root of the current value of σ.
  - ✓ The value of σ is dynamically adapted according to $y^\top R\,y$.
- What happens in presence of an outlier?
  - ✓ If y is persistently constant, then σ tends to a constant value.
  - ✓ If an outlier occurs, y becomes large but the effect of y on the loop is saturated on the current (small) value of $\sqrt{\sigma v}$, mitigating its effect on the closed-loop system.
  - ✓ If y grows or is persistently affected by outliers, then σ increases, thus generating desaturation.

Closed-Loop Input-to-State Stability After Redesign

Theorem 3 [Tarbouriech, Alessandri, Astolfi, Zaccarian, IEEE LCSS 2022]

Assume that there exist a scalar $\lambda > 0$, a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$, a symmetric positive semi-definite matrix $R \in \mathbb{R}^{p\times p}$, two diagonal positive definite matrices $U_g \in \mathbb{R}^{p\times p}$, $U_\ell \in \mathbb{R}^{p\times p}$, and a matrix $Y \in \mathbb{R}^{p\times n}$ such that inequalities

$$M_g := \mathrm{He}\begin{bmatrix} PA - \frac{1}{2}\lambda C^\top RC + \frac{1}{2}\lambda P & -PB \\ U_g C & -U_g \end{bmatrix} < 0$$

$$M_\ell := \mathrm{He}\begin{bmatrix} PA & -PB \\ U_\ell C + Y & -U_\ell \end{bmatrix} < 0$$

$$\begin{bmatrix} P & Y_{(i)}^\top \\ Y_{(i)} & \lambda^{-1} u_{\ell,i} \end{bmatrix} \geq 0, \quad \forall i = 1, \dots, p,$$

are satisfied, where $\mathrm{He}(\star) := \star + \star^\top$. Then, the closed loop system with dynamically **saturated output regulator** having entries with a vector v as the inverse of the diagonal elements of U_ℓ (namely $\mathrm{diag}(v) = U_\ell^{-1}$), is finite-gain exponentially ISS from w to x, namely there exist positive scalars M, $\alpha >$ and γ such that all solutions satisfy

$$\left|(x(t), \sqrt{\sigma(t)})\right| \leq M e^{-\alpha t} \left|(x(0), \sqrt{\sigma(0)})\right| + \gamma \sup_{\tau\in[0,t]} |w(\tau)|,$$

for all $t \geq 0$.

Sketch of Proof and Feasibility

- The proof of Theorem 3 follows from using the Lyapunov function

$$\mathcal{V}(x,\sigma) = x^\top Px + \zeta\sigma + \mu \max\{x^\top Px - \lambda\sigma, 0\}$$

  with P symmetric positive semi-definite matrix and constants $\zeta, \mu > 0$ to be suitably chosen.

- It is an ISS Lyapunov function and it is **not continuosly differentiable**.

- For an overview on nonsmooth Lyapunov functions, see [Della Rossa, Goebel, Tanwani, Zaccarian, "Piecewise structure Lyapunov functions and densely checked decrease conditions for hybrid systems," MCSS 2021].

- The LMIs involved by Theorem 3 are homogeneus, thus they can be solved with the additional condition $P > I$ for increased numerical robustness

Proposition 4 [Tarbouriech, Alessandri, Astolfi, Zaccarian, IEEE LCSS 2022]

Under Assumption 1 there exist parameters P, R, U_ℓ, U_g , Y and λ satisfying the conditions of Theorem 3.

Simulation Case Study

- Linearization about an equilibrium point of the **longitudinal dynamics of fixed-wing aircrafts** flying at high speed [Astolfi, Praly, "Integral action in output feedback for multi-input multi-output nonlinear systems," IEEE TAC, 2017]:

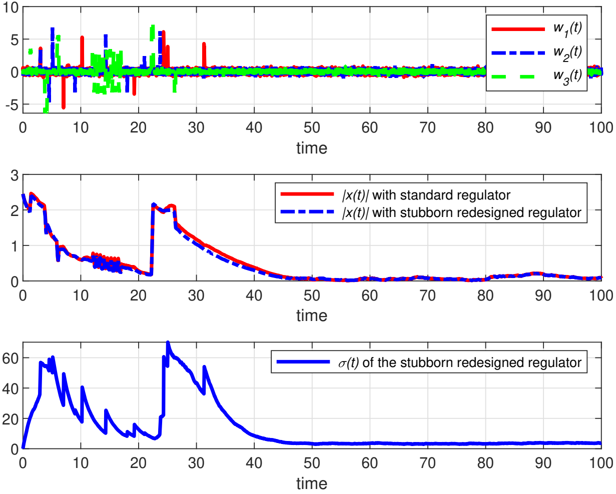
$$\dot{v} = e - g\sin(\gamma) + w_1$$
$$\dot{\gamma} = \ell v \sin(\theta - \gamma) - \frac{g\cos(\gamma)}{v}$$
$$\dot{\theta} = q$$

  where v is the modulus of the speed, γ is the flight path angle, θ is the pitch angle, q is the pitch rate, e is the propulsive balance, g is the standard gravitational acceleration, ℓ is an aerodynamic lift coefficient, w_1 is a wind perturbation.
- The signals e, q are regarded as control inputs and γ, θ as measured outputs.
- The measurement noises w_2, w_3 affect the outputs.
- The linearization around an equilibrium $(v_0, 0, 0)$ of this model provides matrices A_p, B_p, C_p as follows:

$$\left[\begin{array}{c|c|c} A_p & B_p & B_{pw} \\ \hline C_p & & D_{pw} \end{array}\right] = \left[\begin{array}{ccc|cc|ccc} 0 & -g & 0 & 1 & 0 & 1 & 0 & 0 \\ gv_0^{-2} & -\ell v_0 & \ell v_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & & & 0 & 1 & 0 \\ 0 & 0 & 1 & & & 0 & 0 & 1 \end{array}\right].$$

- We choose $g = 1$, $v_0 = 2$, $\ell = 0.1$ and used pole placement to select closed-loop poles having real part in $[-3, -0.1]$.

Simulation Results



Integrals of $\lvert x(t)\rvert$, $t \in [0, 60]$	
Standard reg.	Stubborn reg.
42.4572	39.4415

✓ The adpaptive saturation attenuates the effect of the outliers upon their sudden rise.

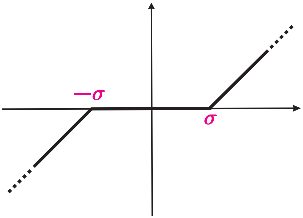
✗ Small noises are not reduced at steady state.

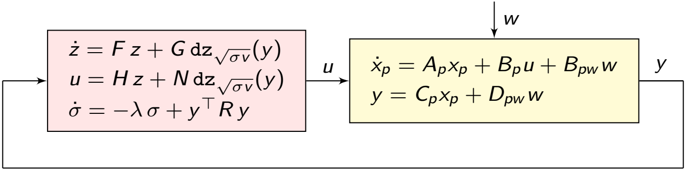
Deadzone Redesign to handle persistent measurement noise

- The controller dynamics, augmented with a new non-negative state $\sigma \in \mathbb{R}_{\geq 0}$, is given by

$$\dot{z} = F\,z + G\,\mathtt{dz}_{\sqrt{\sigma v}}(y)$$
$$u = H\,z + N\,\mathtt{dz}_{\sqrt{\sigma v}}(y)$$
$$\dot{\sigma} = -\lambda\sigma + y^\top R\,y\,,$$

where we modify the original structure by adding a dynamic dead-zone.





✓ The dead-zone provides a trimming effect on y, which denoises the feedback loop when the output is close enough to zero.

Closed-Loop Input-to-State Stability After Redesign

Theorem 5 [Tarbouriech, Alessandri, Astolfi, Zaccarian, IEEE LCSS 2022]

Assume that there exist a scalar $\lambda > 0$, a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$, a symmetric positive semi-definite matrix $R \in \mathbb{R}^{p\times p}$, a diagonal positive definite matrices $U_g \in \mathbb{R}^{p\times p}$ such that

$$M_g := \mathrm{He}\begin{bmatrix} PA + \frac{1}{2}C^\top RC & -PB \\ U_g C & -U_g(1+\lambda) \end{bmatrix} < 0,$$

is satisfied, where $\mathrm{He}(\star) := \star + \star^\top$. Then, the closed loop system with dynamically **dead-zonated output regulator** having entries with a vector v as the inverse of the diagonal elements of U_g (namely $\mathrm{diag}(v) = U_g^{-1}$), is finite-gain exponentially ISS from w to x, namely there exist positive scalars M, $\alpha >$ and γ such that all solutions satisfy

$$\left|(x(t), \sqrt{\sigma(t)})\right| \leq M\mathrm{e}^{-\alpha t}\left|(x(0), \sqrt{\sigma(0)})\right| + \gamma \sup_{\tau\in[0,t]} |w(\tau)|,$$

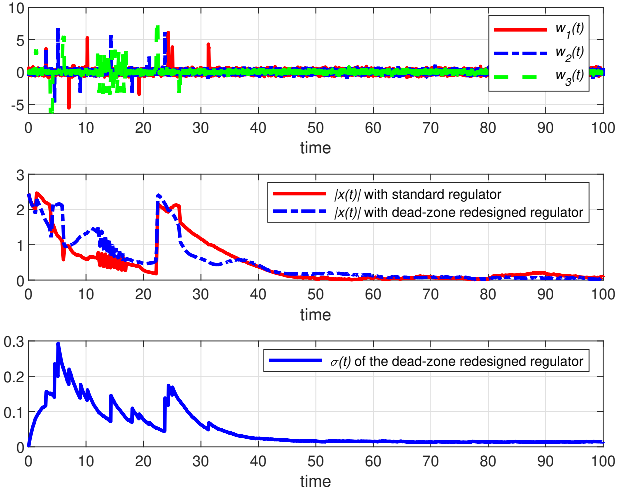
for all $t \geq 0$.

- The proof of Theorem 5 follows from using the Lyapunov function $\mathcal{V}(x,\sigma) = x^\top Px + 2\sigma$ with $P = P^\top > 0$ to be suitably chosen.

Proposition 6 [Tarbouriech, Alessandri, Astolfi, Zaccarian, IEEE LCSS 2022]

Under Assumption 1 there exist parameters P, R, U_g, and λ satisfying the conditions of Theorem 5.

Simulation Results



Integrals of $\|x(t)\|$, $t \in [60, 100]$	
Standard reg.	Dead-zone reg.
3.6459	2.5558

✓ The dead-zone adaptation attenuates the effect of small noises at steady state.

✗ Outliers deteriorate performance.

Outline

Conclusions

Summary

- Adaptive nonlinearities such as **saturation and dead-zone can improve the performance** of state observers and controllers in the presence of measurement noise.
- General and flexible approach to **redesign** ISS observers and output feedback regulators while **preserving ISS**.
- The effectiveness of the redesign approach follows from devising the **appropriate ISS Lyapunov functions**.
- For linear systems, **design conditions based on LMIs** are established that can be treated by means of convex optimization tools.

Next Goals

- redesign in output feedback control for **nonlinear continuous-time systems**;
- extension of the redesign approach for estimation and control of **discrete-time systems**;
- experimental validation of the approach on suitable **case studies**.