Optimizing shifted equilibria for enlarged basins of attraction in quadratically stabilizing saturated feedback

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42nd Benelux Meeting on Systems and Control, March 21, 2023

Motivation: Stabilization of Saturated linear systems

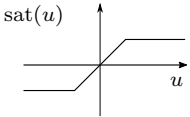
- Saturated linear systems:

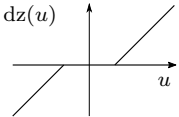
$$\dot{x} = Ax + B\,\mathrm{sat}_{[u^-,u^+]}(u), \quad (x \in \mathbb{R}^n,\ u \in \mathbb{R}^m)$$

- (Asymmetric) actuator saturation: $(u^-, u^+ \in \mathbb{R}^m_{>0})$

$$\mathrm{sat}_{[u^-,u^+]}(u) = \max\{-u^-, \min\{u^+, u\}\}$$

- Deadzone: $\mathrm{dz}(u) = u - \mathrm{sat}(u)$





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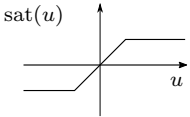
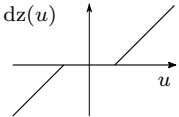




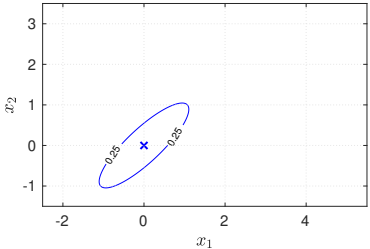
Local stabilization of the origin:

- Closed loop dynamics for $u = Kx$:

$$\dot{x} = Ax + B\,\mathrm{sat}_{[u^-,u^+]}(Kx)$$

- If $A_{cl} = A + BK$ Hurwitz, then 0 is locally asymp. stable.
- Lyapunov function: $V(x) = x^T P x \qquad [A_{cl}^T P + P A_{cl} < 0]$
- Estimate of the basin of attraction: (sublevel set)

$$\{x \in \mathbb{R}^n |\ x^T P x \le c\}, \quad c = \frac{\lambda_{\min}(P)}{\|K\|_2^2} \cdot \min\{\min\{u^-, u^+\}\}$$



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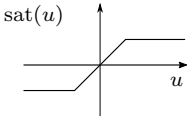

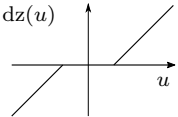

- Subspace of induced equilibria:

$$\Gamma = \{(x_e, u_e) \in \mathbb{R}^n \times \mathbb{R}^m |\ 0 = Ax_e + Bu_e\}$$

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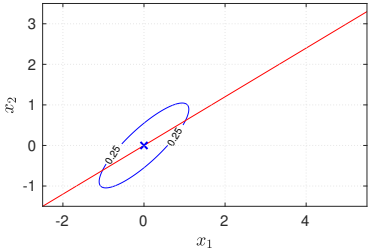

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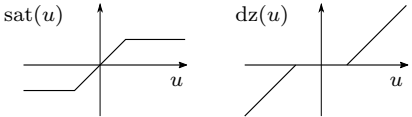

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- Stab. of induced equilibria: $\tilde{x} = x - x_e,\ \tilde{u} = u - u_e$

$$\dot{x} = \dot{\tilde{x}} = A(x - x_e) + B(\mathrm{sat}_{[u^-,u^+]}(u) - u_e)$$
$$= A\tilde{x} + B\,\mathrm{sat}_{[u^- + u_e, u^+ - u_e]}(\tilde{u})$$

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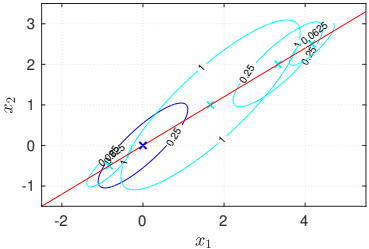

Summary: Stabilizing controller design & Maximization of the basin of attraction

- Saturated linear systems:

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- Estimate of the basin of attraction: (sublevel set)

$$\{x \in \mathbb{R}^n \,|\, (x - x_e)^T P (x - x_e) \leq c\};$$

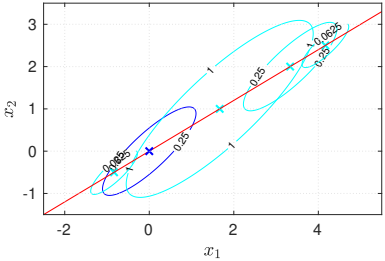
$$c = \frac{\lambda_{\min}(P)}{\|K\|_2^2} \cdot \min\{\min\{u^- + u_e, u^+ - u_e\}\}$$

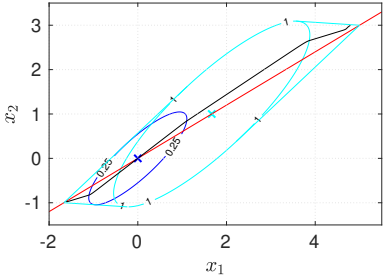
Goal:

- Stabilize $x = 0$
- Maximize the estimate of the basin of attraction

Idea:

- Stabilize reference points x_e
- Pull/Shift x_e to the origin
- Use forward invariance of sublevel sets of Lyapunov functions





Assumptions and definitions

Saturated linear systems:

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- Average saturation range & average saturation center:

$$\bar{u} = \tfrac{1}{2}(u^+ + u^-), \qquad u_\circ = \tfrac{1}{2}(u^+ - u^-)$$

- Assumptions:
  - $\bar{u} = \mathbb{1} \in \mathbb{R}^m$ ($\rightsquigarrow$ Scaling of B)
  - (A, B) is stabilizable

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Unique representation of equilibrium pairs:

- Let $\begin{bmatrix} A^\perp \\ B^\perp \end{bmatrix}$ represent the kernel of $\begin{bmatrix} A & B \end{bmatrix}$
- Then: $\Gamma = \{(A^\perp\delta, B^\perp\delta) |\ \delta \in \mathbb{R}^q\}$

$$x_e(\delta) = A^\perp\delta, \quad u_e(\delta) = B^\perp\delta, \quad \delta \in \mathbb{R}^q$$

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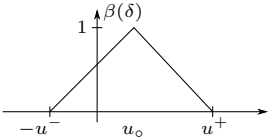
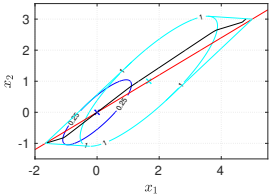
Maximal sublevel sets: $\beta : \mathbb{R}^q \to \mathbb{R}$

- $\beta(\delta) = \min\{\min\{u^- + u_e(\delta), u^+ - u_e(\delta)\}\}$
  $= \min\{\min\{u^- + B^\perp\delta, u^+ - B^\perp\delta\}\}$
- $\beta(\delta) \in [0, 1] \quad \forall\, \delta \in \Delta := \{\delta | -u^- \le B^\perp\delta \le u^+\}$





Symmetric stabilizer & basin of attraction

Assumption (Symmetric Stabilizer)

Consider $\dot{x} = Ax + B\,\mathrm{sat}_{[u^-,u^+]}(u)$

(A, B) stabilizable, $\overline{u} = \mathbb{1}$ and let $\alpha \in \mathbb{R}_{>0}$.

Consider the LMI:

$$\max_{Q,W,Y,U} \quad \log\det(Q)$$

subject to $\quad U > 0$ diagonal, $\quad Q = Q^\top > 0$

$$\mathrm{He}\begin{bmatrix} (A+\alpha I)Q + BW & -BU \\ W+Y & -U \end{bmatrix} < 0$$

$$\begin{bmatrix} 1 & Y_{[k]} \\ Y_{[k]}^\top & Q \end{bmatrix} \geq 0, \quad k = 1, \ldots, m.$$

$\rightsquigarrow$ Solution: $\quad Q, W, Y, U.$

Define:

$$K = WQ^{-1}, \quad P = Q^{-1}$$

Proposition (Symmetric Stabilizer)

- *The feedback law*

$$u = k(x,\delta) := u_e(\delta) + K(x - x_e(\delta))$$

  locally exponentially stabilizes $x_e(\delta)$

- *Local Lyapunov function:*

$$V_\delta(x) = |x - x_e(\delta)|_P^2 = (x - x_e(\delta))^T P (x - x_e(\delta))$$

$$\tfrac{d}{dt} V_\delta(x(t)) \leq -2\alpha V_\delta(x(t)) \quad (\rightsquigarrow \textit{exponential stability})$$

- *Estimate of the basin of attraction:*

$$\mathcal{E}_\delta(P) = \{x \in \mathbb{R}^n \mid |x - x_e(\delta)|_P \leq \beta(\delta)\}$$

Notation: $\quad \mathrm{He}\, M = M + M^\top$

Recall: $\quad \beta(\delta) = \min\{\min\{u^- + u_e(\delta), u^+ - u_e(\delta)\}\}$

Optimization-based shifted stabilizer

- Consider: Induced equilibrium pair

$$x_e(\delta) = A^{\perp}\delta, \qquad u_e(\delta) = B^{\perp}\delta$$

for $\delta \in \Delta = \{\delta | -u^- \leq u_e(\delta) \leq u^+\}$

- Estimate of the basin of attraction (BA) of $x_e(\delta)$:

$$\mathcal{E}_\delta(P) = \{x \in \mathbb{R}^n : \ |x - x_e(\delta)|_P \leq \beta(\delta)\}$$

- Union of all the estimates of the BA:

$$\mathcal{R} = \bigcup_{\delta \in \mathrm{int}(\Delta)} \mathcal{E}_\delta(P), \qquad \overline{\mathcal{R}} = \text{closure of } \mathcal{R}$$

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- For $x \in \overline{\mathcal{R}}$ consider:

$$\begin{aligned} \delta^\star(x) \in \operatorname*{argmin}_{\delta \in \Delta} \ & \delta^T \delta \\ \text{subject to } \ & |x - x_e(\delta)|_P \leq \beta(\delta) \end{aligned} \qquad (\spadesuit)$$

$\rightsquigarrow$ optimal equilibrium pair

$(x_e^\star, u_e^\star) = (x_e(\delta^\star(x)), u_e(\delta^\star(x)))$ with $x \in \mathcal{E}_{\delta^\star(x)}(P)$

- Corresponding feedback law:

$$u = k(x, \delta^\star(x)) := u_e(\delta^\star(x)) + K(x - x_e(\delta^\star(x)))$$

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Lemma (Optimization problem properties)

1. *For all $x \in \overline{\mathcal{R}}$, ($\spadesuit$) is convex and for all $x \in \mathcal{R}$, the interior of the feasible set is nonempty;*
2. *The feasible set $F : \overline{\mathcal{R}} \rightrightarrows \Delta$ is continuous*

$$F(x) = \{\delta \in \Delta | \ |x - x_e(\delta)|_P \leq \beta(\delta)\}$$

3. *$\delta^\star(x) \in \Delta$ is unique for all $x \in \overline{\mathcal{R}}$; and*
4. *$\delta^\star(\cdot) : \mathrm{int}(\mathcal{R}) \to \mathrm{int}(\Delta)$ is Lipschitz continuous.*
5. *$\delta^\star(x)$ satisfies*

$$\delta^\star(x) = 0, \qquad \forall x \in \mathcal{E}_0(P),$$

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$\rightsquigarrow$ The feedback law $u = k(x, \delta^\star(x))$ is Lipschitz continuous

Stabilization with scheduled shifted coordinates

Saturated linear systems:

$$\dot{x} = Ax + B\,\mathrm{sat}_{[u^-,u^+]}(u), \qquad u \in [-u^-, u^+]$$

For $x \in \mathrm{int}(\mathcal{R})$ consider:

$$\delta^\star(x) = \underset{\delta \in \mathrm{int}(\Delta)}{\mathrm{argmin}}\ \delta^T\delta \qquad (\spadesuit)$$

$$\text{subject to } \ |x - x_e(\delta)|_P \leq \beta(\delta)$$

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$$u = k(x, \delta^\star(x)) := u_e(\delta^\star(x)) + K(x - x_e(\delta^\star(x)))$$

Theorem

The control law

- *is Lipschitz continuous for all* $x \in \mathrm{int}(\mathcal{R})$*;*
- *locally preserves performance around the origin;*
- *asymptotically stabilizes the origin* $x = 0$ *with basin of attraction containing* $\mathrm{int}(\mathcal{R})$*.*

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Potential Challenge:

- ($\spadesuit$) needs to be solved in real time

Lemma (Recall from the earlier result)

5. $\delta^\star(x)$ *satisfies*

$$\delta^\star(x) = 0, \qquad \forall x \in \mathcal{E}_0(P),$$

$$|x - x_e(\delta^\star(x))|_P = \beta(\delta^\star(x)), \qquad \forall\, x \in \overline{\mathcal{R}} \backslash \mathcal{E}_0(P)$$

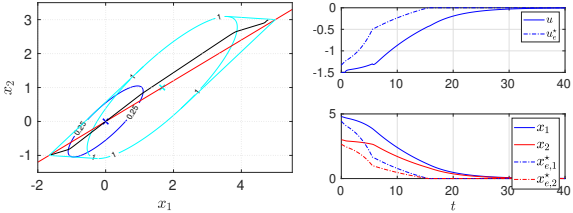
- For $\dim(\delta) = 1$, $\delta^\star$ can be calculated (semi) explicitly through

$$|x + A^\perp \delta^\star|_P^2 = \beta(\delta^\star)^2$$

- For $\dim(\delta) > 1$ use a sample-and-hold update of $\delta^\star$

  $\rightsquigarrow$ Next slides

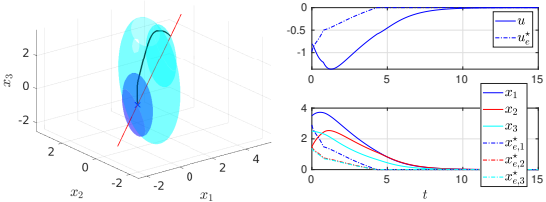
Numerical illustration (for $\dim(\delta) = 1$)



Linear system:

$$\dot{x} = \begin{bmatrix} 0.6 & -0.5 \\ 0.3 & 1.0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 3 \end{bmatrix} u$$

Saturation levels: $u^- = 1.5$, $u^+ = 0.5$



Linear system:

$$\dot{x} = Ax + Bu = \begin{bmatrix} 0.6 & -0.8 & 0.3 \\ 0.8 & 0.6 & 0.5 \\ 1.0 & 0.3 & -1.0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} u$$

Saturation levels: $u^- = 1.5$, $u^+ = 0.5$

A hybrid systems (or sample-and-hold) solution (for $\dim(\delta) > 1$)

Augmented state:

$$\xi = \begin{bmatrix} x \\ \chi \\ \delta \\ \tau \end{bmatrix} \in \overline{\Xi} \qquad \begin{array}{l} \leftarrow \text{state of the plant} \\ \leftarrow \text{sample-and-hold version of } x \\ \leftarrow \text{defines induced equilibrium pair} \\ \leftarrow \text{trigger to update } \delta \ (\tau \in [\underline{\tau}, \overline{\tau}]) \end{array}$$

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$$\xi = \begin{bmatrix} x \\ \chi \\ \delta \\ \tau \end{bmatrix} \in \overline{\Xi}$$

$\leftarrow$ state of the plant
$\leftarrow$ sample-and-hold version of x
$\leftarrow$ defines induced equilibrium pair
$\leftarrow$ trigger to update δ ($\tau \in [\underline{\tau}, \overline{\tau}]$)

Domain of interest:

$$\Xi := \mathcal{E}_\delta(P) \times \mathcal{E}_\delta(P) \times \mathrm{int}(\Delta) \times [0, \overline{\tau}]$$

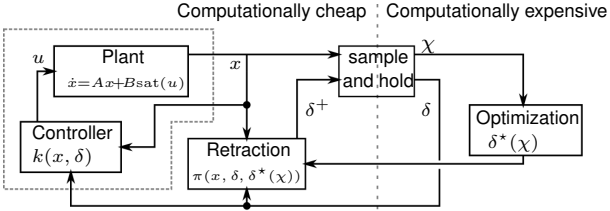
Flow set and jump set:

$$\mathcal{C} = \overline{\Xi}, \qquad \mathcal{D} = \mathcal{E}_\delta(P) \times \mathcal{E}_\delta(P) \times \Delta \times [\underline{\tau}, \overline{\tau}],$$

Flow map and jump map:

$$\dot{\xi} = \begin{bmatrix} \dot{x} \\ \dot{\chi} \\ \dot{\delta} \\ \dot{\tau} \end{bmatrix} = F(\xi) := \begin{bmatrix} Ax + B\,\mathrm{sat}(k(x,\delta)) \\ 0 \\ 0 \\ 1 \end{bmatrix}, \ \xi \in \mathcal{C}$$

$$\xi^+ = \begin{bmatrix} x^+ \\ \chi^+ \\ \delta^+ \\ \tau^+ \end{bmatrix} = G(\xi) := \begin{bmatrix} x \\ x \\ \pi(x, \delta, \delta^\star(\chi)) \\ 0 \end{bmatrix}, \ \xi \in \mathcal{D}$$



Main idea:

- Update δ (i.e., shifted eq x_e) at discretely (non-periodic)
- Update δ when solution of optimization problem becomes available
- Retraction: Convex combination of δ and $\delta^\star(\chi)$ so that

$$x \in \mathcal{E}_{\delta^+}(P), \qquad [\delta^+ = \pi(x, \delta, \delta^\star(\chi))]$$

A hybrid systems (or sample-and-hold) solution (for $\dim(\delta) > 1$)

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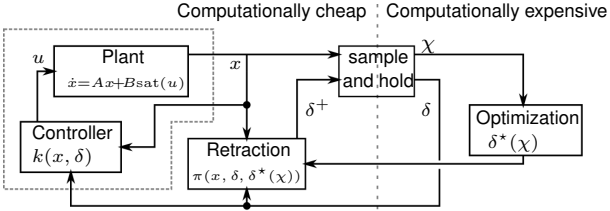

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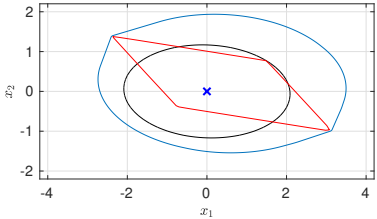
Extension:

- Update δ based on suboptimal solution of opt. problem

$\rightsquigarrow$ Interplay between continuous time system and iterative (discrete time) optimization algorithm

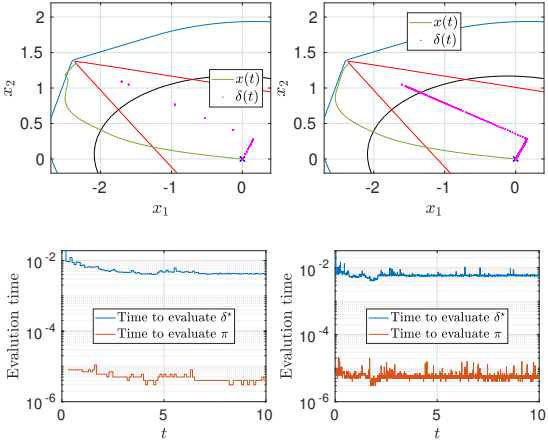
Numerical Example



Example $\dot{x} = Ax + B\,\mathrm{sat}_{[u^-,u^+]}(u)$

$$A = \begin{bmatrix} 0.6 & -0.8 \\ 0.8 & 0.6 \end{bmatrix}, \qquad B = \begin{bmatrix} 1.2045 & 1.4183 \\ 0.1259 & 1.3739 \end{bmatrix}$$

$$u^- = \frac{2}{3}\begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad u^+ = \frac{2}{3}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$



$\underline{\tau} = 0.1, \quad \overline{\tau} = 0.1$ $\qquad$ $\underline{\tau} = 0.01, \quad \overline{\tau} = 0.1$

Extensions: Anti-windup controller design

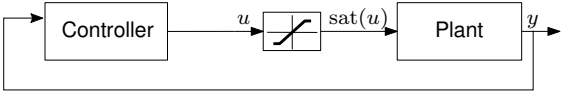
Setting:

- Plant dynamics

$$\dot{x}_p = A_p x_p + B_p \operatorname{sat}_{[u^-,u^+]}(u)$$
$$y = C_p x_p + D_p \operatorname{sat}_{[u^-,u^+]}(u)$$

- Linear dynamic output feedback:

$$\dot{x}_c = A_c x_c + B_c y$$
$$u = C_c x_c + D_c y$$



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- Linear dynamic output feedback:

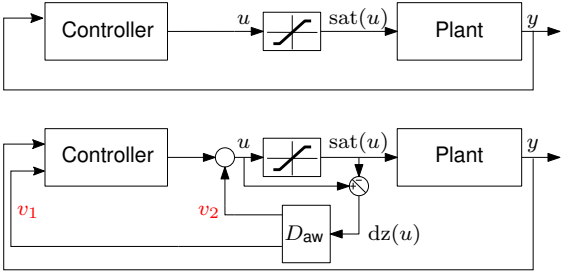
$$\dot{x}_c = A_c x_c + B_c y + v_1$$
$$u = C_c x_c + D_c y + v_2$$

- Static anti-windup gain D_{aw}:

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} D_{aw,1}(u - \operatorname{sat}_{[u^-,u^+]}(u)) \\ D_{aw,2}(u - \operatorname{sat}_{[u^-,u^+]}(u)) \end{bmatrix}$$
$$= D_{aw}(u - \operatorname{sat}_{[u^-,u^+]}(u))$$

- Overall static anti-windup closed loop: $x = [x_p^\top \; x_c^\top]^\top$

$$\dot{x} = A_{cl} x + (B_{cl,q} + B_{cl,v} D_{aw}) \operatorname{dz}_{[u^-,u^+]}(u)$$
$$u = C_{cl} x + (D_{cl,q} + D_{cl,v} D_{aw}) \operatorname{dz}_{[u^-,u^+]}(u)$$



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$$= D_{aw}(u - \operatorname{sat}_{[u^-,u^+]}(u)) + \eta$$

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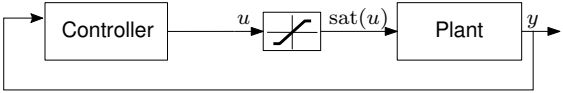

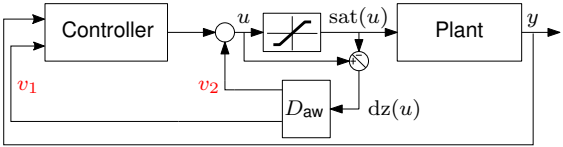

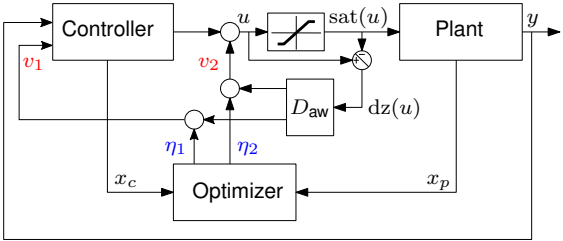

Anti-windup gain selection

Overall static anti-windup closed loop: $x = [x_p^\top \; x_c^\top]^\top$

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For $\alpha > 0$ consider the semidefinite program (SDP)

$$\min_{Q,Y,U,X} \quad \log\det(Q) \text{ bsubject to}$$

$$Q = Q^\top > 0, \quad U > 0 \text{ diagonal},$$

$$\text{He}\begin{bmatrix} (A_{cl}+\alpha I)Q & B_{cl,q}U + B_{cl,v}X \\ C_{cl,u}Q - Y & D_{cl,q}U + D_{cl,v}X - U \end{bmatrix} < 0$$

$$\begin{bmatrix} 1 & Y_{[k]} \\ Y_{[k]}^\top & Q \end{bmatrix} \geq 0, \quad k = 1,\ldots,n_u.$$

Assumption (♡)

The linear closed loop is exponentially stable, $B_p \neq 0$ and the average saturation range satisfies $\bar{u} = \frac{1}{2}(u^+ + u^-) = \mathbb{1} \in \mathbb{R}^m$. ◇

Proposition

Consider the overall system, define

$$\alpha > 0 \qquad \textit{and} \qquad c = \min\{\min\{u^-, u^+\}\}.$$

Let Assumption (♡) be satisfied and assume that (SDP) is feasible. Then, selecting the static anti-windup gain as

$$D_{aw} = XU^{-1},$$

the algebraic loop is well posed and the set

$$\{x \in \mathbb{R}^n : |x|_{Q^{-1}} := \sqrt{x^\top Q^{-1} x} \leq c\}$$

is contained in the basin of attraction of the origin. Moreover, $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$,

$$V(x) = x^\top Q^{-1} x$$

is a Lyapunov function satisfying

$$\langle \nabla V(x), \dot{x} \rangle \leq -2\alpha V(x) \qquad \forall x \in \{x \in \mathbb{R}^n : |x|_{Q^{-1}} \leq c\}.$$

Anti-windup augmentation (Construction of η)

Plant controller dynamics:

$$\dot{x}_p = A_p x_p + B_p u \qquad \dot{x}_c = A_c x_c + B_c y + \eta_1$$
$$y = C_p x_p + D_p u \qquad u = C_c x_c + D_c y + \eta_2$$

Induced equilibria for fixed η:

$$\begin{bmatrix} A_p & B_p \\ C_p & D_p \end{bmatrix} \begin{bmatrix} x_{p_e} \\ u_e \end{bmatrix} = \begin{bmatrix} 0 \\ y_e \end{bmatrix}$$
$$\begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \begin{bmatrix} x_{c_e} \\ y_e \end{bmatrix} = \begin{bmatrix} 0 \\ u_e \end{bmatrix} + \begin{bmatrix} -\eta_1 \\ -\eta_2 \end{bmatrix}$$

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i.e.,

$$\underbrace{\begin{bmatrix} A_p & 0 & 0 & B_p & 0 & 0 \\ 0 & A_c & B_c & 0 & I & 0 \\ C_p & 0 & -I & D_p & 0 & 0 \\ 0 & C_c & D_c & -I & 0 & I \end{bmatrix}}_{M} \begin{bmatrix} x_{p_e} \\ x_{c_e} \\ y_e \\ u_e \\ \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Kernel of M (i.e., $M \cdot M^\perp = 0$):

$$M^\perp = \left[M_{x_p}^{\perp\top}, M_{x_c}^{\perp\top}, M_y^{\perp\top}, M_u^{\perp\top}, M_{\eta_1}^{\perp\top}, M_{\eta_2}^{\perp\top} \right]^\top$$

Representation of 'equilibria':

$$x_e(\delta) = \begin{bmatrix} x_{p_e}(\delta) \\ x_{c_e}(\delta) \end{bmatrix} = \begin{bmatrix} M_{x_p}^\perp \delta \\ M_{x_c}^\perp \delta \end{bmatrix} \qquad \begin{aligned} y_e(\delta) &= M_y^\perp \delta, \\ u_e(\delta) &= M_u^\perp \delta, \end{aligned}$$

$$\eta(\delta) = \begin{bmatrix} \eta_1(\delta) \\ \eta_2(\delta) \end{bmatrix} = \begin{bmatrix} M_{\eta_1}^\perp \delta \\ M_{\eta_2}^\perp \delta \end{bmatrix}$$

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i.e.,

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δ-optimization:

$$\delta^{\star}(x) := \underset{\delta \in \Delta}{\operatorname{argmin}} \, |\delta|^2$$

subject to $|x - x_e(\delta)|_{Q^{-1}} \leq \beta(\delta)$.

where

$$\Delta = \{\delta \in \mathbb{R}^q | \; -u^- \leq M_u^{\perp}\delta \leq u^+\}$$
$$\beta(\delta) = \min\{\min\{u^- + u_e(\delta), u^+ - u_e(\delta)\}\}$$

Static anti-windup controller (state feedback!!):

$$v_1 = D_{aw,1}(u(x) - \mathrm{sat}_{[u^-,u^+]}(u(x))) + \eta_1(\delta^{\star}(x))$$
$$v_2 = D_{aw,2}(u(x) - \mathrm{sat}_{[u^-,u^+]}(u(x))) + \eta_2(\delta^{\star}(x))$$

Anti-windup correction: Main result

Representation of 'equilibria':

$$x_e(\delta) = \begin{bmatrix} x_{p_e}(\delta) \\ x_{c_e}(\delta) \end{bmatrix} = \begin{bmatrix} M_{x_p}^{\perp}\delta \\ M_{x_c}^{\perp}\delta \end{bmatrix} \qquad \begin{aligned} y_e(\delta) &= M_y^{\perp}\delta, \\ u_e(\delta) &= M_u^{\perp}\delta, \end{aligned}$$

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Recall

$$\mathcal{E}_\delta(Q^{-1}) = \left\{x \in \mathbb{R}^n : |x - x_e(\delta)|_{Q^{-1}} \leq \beta(\delta)\right\}$$

Overall closed-loop plant-controller pair:

$$\dot{x} = A_{cl}x + (B_{cl,q} + B_{cl,v}D_{aw})\,\mathrm{dz}_{[u^-,u^+]}(u) + B_{cl,\eta}\eta(\delta)$$

$$u = C_{cl}x + (D_{cl,q} + D_{cl,v}D_{aw})\,\mathrm{dz}_{[u^-,u^+]}(u) + D_{cl,\eta}\eta(\delta)$$

Theorem

Consider the overall closed-loop plant-controller pair. Then, the algebraic loop is well posed and the origin $x_e = 0$ is asymptotically stable with basin of attraction containing the set

$$\mathcal{R} := \bigcup_{\delta\in\operatorname{int}(\Delta)} \mathcal{E}_\delta(Q^{-1})$$

Anti-windup controller design (Driverless Segway)

Segway dynamics (inverted pendulum):

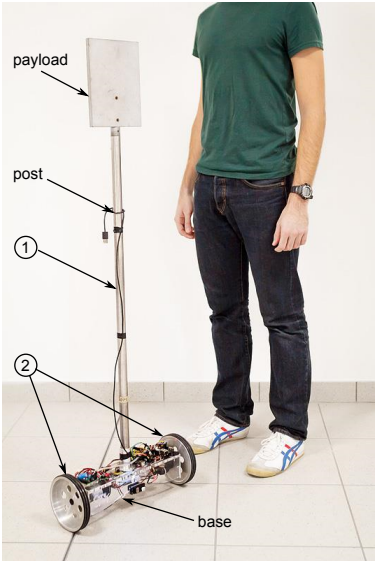
$$\ddot{s} = \frac{j_1\dot{\vartheta}^2 - m_b g l\cos(\vartheta)}{j_1 j_2 - (m_b l r\cos(\vartheta))^2} m_b l r^2 \sin(\vartheta) - v_f \dot{s} + \frac{j_1 + m_b l r\cos(\vartheta)}{j_1 j_2 - (m_b l r\cos(\vartheta))^2} r K_u u$$

$$\ddot{\vartheta} = \frac{g j_2 - m_b l r^2\cos(\vartheta)\dot{\vartheta}^2}{j_1 j_2 - (m_b l r\cos(\vartheta))^2} m_b l \sin(\vartheta) - \frac{j_2 + m_b l r\cos(\vartheta)}{j_1 j_2 - (m_b l r\cos(\vartheta))^2} K_u u$$

- s: position; ϑ: angle;

Plant state:

$$x_p = \left[\, x_{p1}, x_{p2}, x_{p3}, x_{p4} \,\right]^\top = \left[\, s \quad \vartheta \quad \dot{s} \quad \dot{\vartheta} \,\right]^\top$$



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$$\ddot{\vartheta} = \frac{g j_2 - m_b l r^2\cos(\vartheta)\dot{\vartheta}^2}{j_1 j_2 - (m_b l r\cos(\vartheta))^2} m_b l \sin(\vartheta) - \frac{j_2 + m_b l r\cos(\vartheta)}{j_1 j_2 - (m_b l r\cos(\vartheta))^2} K_u u$$

- s: position; ϑ: angle;

Plant state:

$$x_p = \left[\, x_{p1}, x_{p2}, x_{p3}, x_{p4} \,\right]^\top = \left[\, s \quad \vartheta \quad \dot{s} \quad \dot{\vartheta} \,\right]^\top$$

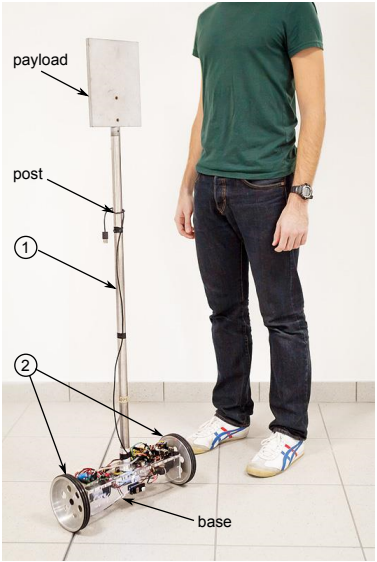
PID controller:

$$u = k_1 s + k_2 \vartheta + k_3 \dot{s} + k_4 \dot{\vartheta} + k_5 \int s + k_6 \int \vartheta$$

i.e.,

$$\dot{x}_c = k_5 s + k_6 \vartheta$$

$$u = x_c + k_1 s + k_2 \vartheta + k_3 \dot{s} + k_4 \dot{\vartheta}$$



Anti-windup controller design (Driverless Segway)

Segway dynamics (inverted pendulum):

$$\ddot{s} = \frac{j_1\dot{\vartheta}^2 - m_b g l\cos(\vartheta)}{j_1 j_2 - (m_b l r\cos(\vartheta))^2} m_b l r^2 \sin(\vartheta) - v_f \dot{s} + \frac{j_1 + m_b l r\cos(\vartheta)}{j_1 j_2 - (m_b l r\cos(\vartheta))^2} r K_u u$$

$$\ddot{\vartheta} = \frac{g j_2 - m_b l r^2\cos(\vartheta)\dot{\vartheta}^2}{j_1 j_2 - (m_b l r\cos(\vartheta))^2} m_b l \sin(\vartheta) - \frac{j_2 + m_b l r\cos(\vartheta)}{j_1 j_2 - (m_b l r\cos(\vartheta))^2} K_u u$$

- s: position; ϑ: angle;

Plant state:

$$x_p = \left[\, x_{p1}, x_{p2}, x_{p3}, x_{p4} \,\right]^\top = \left[\, s \quad \vartheta \quad \dot{s} \quad \dot{\vartheta} \,\right]^\top$$

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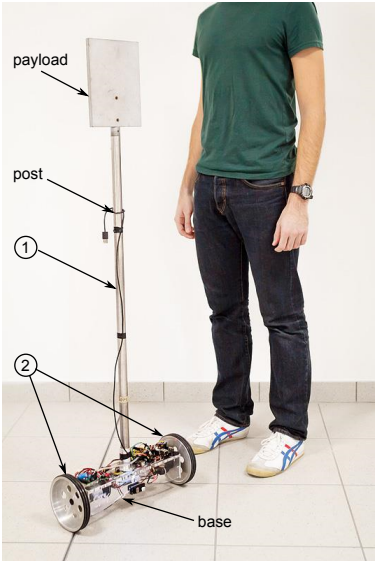
Linearized plant-controller pair:

$$\dot{x}_p = A_p x_p + B_p u \qquad \dot{x}_c = A_c x_c + B_c y$$

$$y = I x_p \qquad u = C_c x_c + D_c y$$



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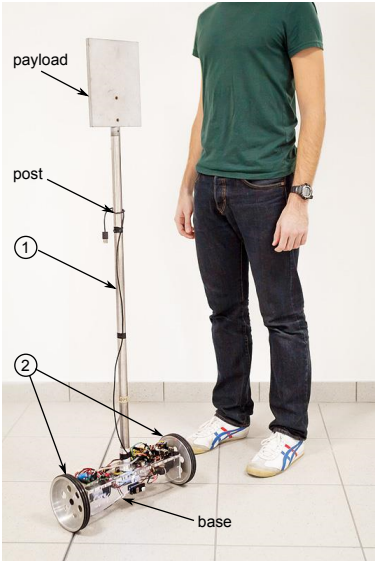
Linearized plant dynamics:

$$A_p = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -2.16 & -0.1 & 0 \\ 0 & 11.01 & 0 & 0 \end{bmatrix}, \quad B_p = \begin{bmatrix} 0 \\ 0 \\ 5.87 \\ -7.20 \end{bmatrix}$$

Controller selection & anti-windup gain:

$$A_c = 0 \qquad B_c = \left[\, 1 \quad 10 \quad 0 \quad 0 \,\right]$$
$$C_c = I \qquad D_c = \left[\, 1.70 \quad 12.90 \quad 1.95 \quad 3.83 \,\right]$$

Saturation limits: $u^- = u^+ = 1$.



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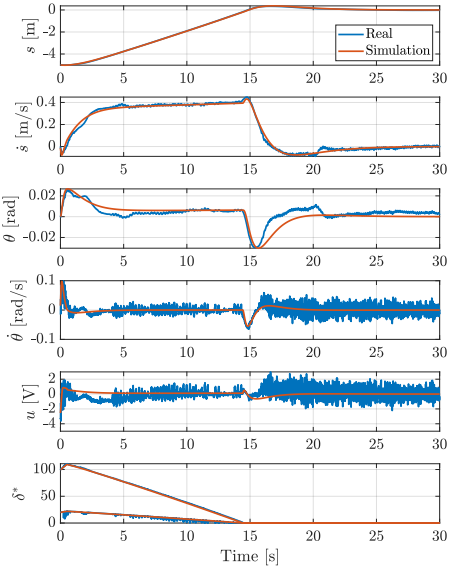

Anti-windup controller design (Driverless Segway)

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Concluding remarks

Setting:

- Saturated linear systems:

$$\dot{x} = Ax + B\,\mathrm{sat}_{[u^-,u^+]}(u), \quad (x \in \mathbb{R}^n,\ u \in \mathbb{R}^m,\ u^-, u^+ \in \mathbb{R}^m_{>0})$$

Summary:

- Increase the estimate of the basin of attraction of the origin by stabilizing shifted reference points.

Observation:

- Estimate of the basin of attraction is unbounded if A is singular
- (Example: stabilization of the inverted pendulum on a cart; segway)

Related controller design techniques:

- Reference governors

Extensions:

- Antiwindup controller designs
  - Design an output feedback
- Discrete-time systems

References:

P. Braun, G. Giordano, C. M. Kellett and L. Zaccarian. *An Asymmetric Stabilizer Based on Scheduling Shifted Coordinates for Single-Input Linear Systems With Asymmetric Saturation*, in IEEE Control Systems Letters, vol. 6, pp. 746-751, 2022

P. Braun, M. Jungers and L. Zaccarian. *Stabilization using shifted equilibria for saturated discrete-time linear systems*, in Proc. of 12th IFAC Symposium on Nonlinear Control Systems, 6 pages, 2023

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P. Braun, M. Saveriano and L. Zaccarian. *Static anti-windup with shifted equilibria on the example of a Segway-like vehicle*, (in preparation)