Robust stabilizing controllers with robust avoidance properties (for linear systems with nontrivial drift)

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Motivation: Obstacle avoidance & target set stabilization

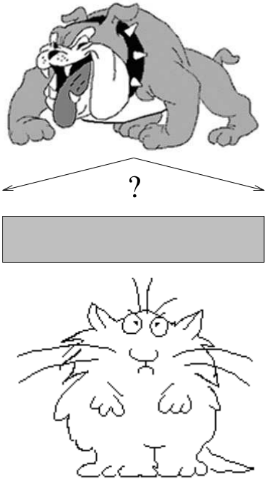

Figure borrowed from: E. D. Sontag, *Nonlinear Feedback Stabilization Revisited*, volume 25 of Progress in Systems and Control Theory, pages 223-262. Birkhäuser, 1999

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Setting:

- (Linear) Dynamical system

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) \in \mathbb{R}^n, \quad (u \in \mathbb{R}^1)$$

- Obstacle: $\mathcal{B}_\delta(\hat{x}) \subset \mathbb{R}^n \setminus \{0\}$
- Target set: $0 \in \mathbb{R}^n$

Problem formulation:

Define $u : \mathbb{R}_{\geq 0} \to \mathbb{R}$ such that

1. $\lim_{t\to\infty} x(t; u(t)) = 0$
2. $x(t; u(t)) \notin \mathcal{B}_\delta(\hat{x})\ \forall\, t \in \mathbb{R}_{\geq 0}$ (and $\delta > 0$)

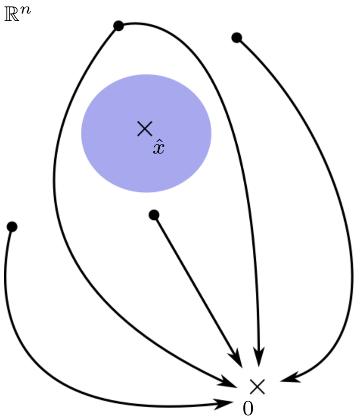
Assume

- (A, B) controllable, i.e.,

$$\forall x_1, x_2 \in \mathbb{R}^n, \forall \varepsilon > 0 \quad \exists\, u : [0, \varepsilon] \to \mathbb{R} :$$
$$x(0; u(t)) = x_1 \ \&\ x(\varepsilon; u(t)) = x_2.$$

However

- It is easy to address 1. & 2. separately. How to ensure 1. & 2. simultaneously?
- How to define a (state dependent) feedback law (i.e., $u(x(t))$ instead of $u(t)$)?



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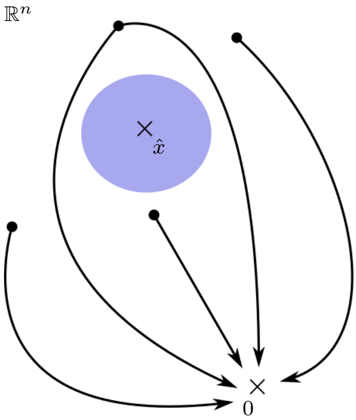

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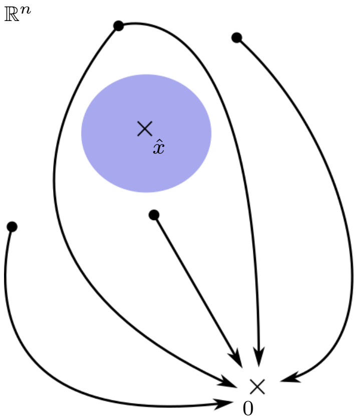
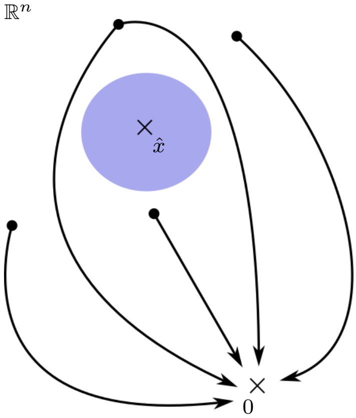


Outline

- Related control settings
- Difficulties in the combined avoidance/stabilization problem
- Hybrid controller design for the combined control problem



Related Settings, Applications and Solutions

Setting:

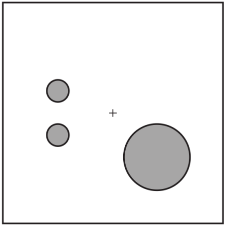
- Obstacle avoidance & target set stabilization; a special case of constrained control

Applications:

- Obstacle avoidance, collision avoidance, safety
- Navigation of mobile robots

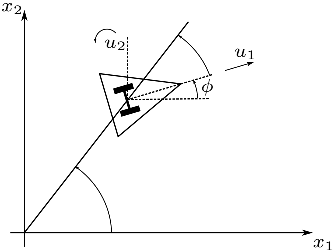
Control Solutions:

- Artificial potential fields and navigation functions
- (Control) Lyapunov functions and (control) barrier functions
- Model predictive control
  - (Motion planning and reference tracking)



Artificial potential fields & navigation functions

Figures borrowed from: K. M. Lynch, F. C. Park, *Modern Robotics: Mechanics, planning, and control*, Cambridge University Press, 2017



Mobile robot (nonholonomic integrator):

$$\begin{aligned} \dot{x}_1 &= u_1 \cos(\phi), \\ \dot{x}_2 &= u_1 \sin(\phi), \\ \dot{\phi} &= u_2. \end{aligned}$$

Simplified mobile robot: $\dot{x} = u$

Artificial potential fields:

- Use gradient to guarantee a decrease with respect to the target set
- Local minima? ($\rightsquigarrow$ Navigation functions)
- Potential fields necessarily have saddle points

(Control) Lyapunov and (control) barrier functions

Nonlinear system: $\dot{x} = f(x, u), \qquad (x \in \mathbb{R}^n, u \in \mathbb{R}^m)$

Obstacle: $\mathcal{D} \subset \mathbb{R}^n$.

Definition (Control Lyapunov function (CLF))

A **continuously differentiable** function $V : \mathbb{R}^n \to \mathbb{R}$ is called **Control Lyapunov function** (CLF) if there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$
$$\forall x \in \mathbb{R}^n \backslash \{0\} \; \exists u \in \mathbb{R}^m \text{ such that } \quad \langle \nabla V(x), f(x, u) \rangle < 0$$

$\rightsquigarrow$ Guarantees global asymptotic stability of the origin

Definition (Control Barrier Function (CBF))

A **continuously differentiable** function $B : \mathbb{R}^n \to \mathbb{R}$ is called **control barrier function** (CBF) if

$$B(x) > 0 \quad \forall\, x \in \mathcal{D} \quad \text{and} \quad B(x) = 0 \quad \forall\, x \in \partial\mathcal{D}$$
$$\forall x \in \mathbb{R}^n \backslash \mathcal{D} \; \exists u \in \mathbb{R}^m \text{ such that } \quad \langle \nabla B(x), f(x, u) \rangle \leq 0$$

$\rightsquigarrow$ Guarantees avoidance of $\mathcal{D}$

Our motivation

Many control approaches in the literature on obstacle avoidance and target set stabilization:

- Consider a given dynamical system $\dot{x} = f(x) + g(x)u$ and obstacles $\mathcal{D}_i \subset \mathbb{R}^n$, $i = 1, 2, \ldots$
- Implicitly assume the existence of functions or implicitly assume feasibility of optimization problems

Then:

- Then obstacle avoidance and target set stabilization for almost all initial conditions is concluded
- The controller design is applied to systems without drift, i.e., $f(x) = 0$, in general

Our setting/motivation:

- Consider controllable/stabilizable linear systems $\dot{x} = Ax + Bu$ and obstacle centroids $\hat{x}_i \in \mathbb{R}^n$, $i = 1, 2, ...$
- Consider a stabilizing controller …

Then:

- Explicitly derive a controller which guarantees robust stabilization of the origin and robust avoidance of a neighborhood around the centroids for all initial conditions.
- Explicitly derive a (maximal) size around the centroids which can be robustly avoided
- …augment the stabilizing controller to additionally ensure avoidance. ($\rightsquigarrow$ Minimally invasive avoidance controller.)

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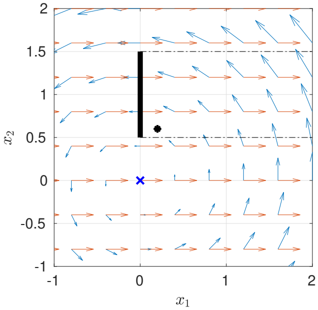
The combined avoidance/stabilization problem: Ex. 1



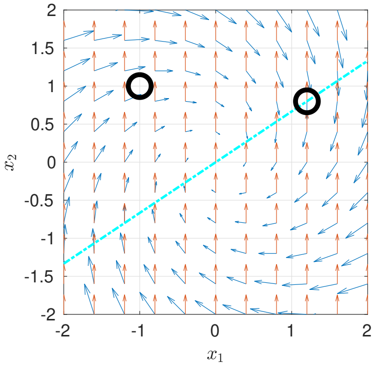
Systems with nontrivial drift

- Consider

$$\dot{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

  - The system is controllable
  - The influence of u is limited (only horizontal) ($\rightsquigarrow$ Behind the obstacle, u can only be used to stall time)

The combined avoidance/stabilization problem: Ex. 2



The location of the obstacle:

- Consider

$$\dot{x} = \begin{bmatrix} -1 & \frac{3}{2} \\ -\frac{3}{2} & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

  (The system is controllable)
- Subspace of induced equilibria: ($B \in \mathbb{R}^n$)

$$\mathcal{E} = \{y \in \mathbb{R}^n : 0 = Ay + B\nu,\ \nu \in \mathbb{R}\}$$

- Obstacle $\mathcal{D}$ with $\mathcal{D} \cap \mathcal{E} = 0$
  - Use the natural drift Ax to 'leave the obstacle behind' and use Bu to avoid the obstacle
- Obstacle $\mathcal{D}$ with $\mathcal{D} \cap \mathcal{E} \neq 0$
  - Use u to destabilize a point $\hat{x} \in \mathcal{D} \cap \mathcal{E}$ to avoid the obstacle

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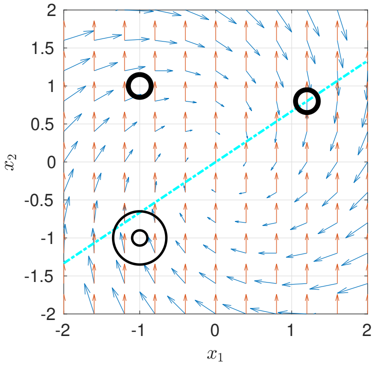

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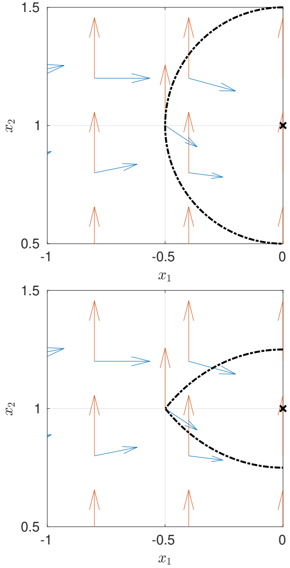
The combined avoidance/stabilization problem: Ex. 3



The shape of the obstacle

- Consider again ($\dot{x} = Ax + Bu$)

$$\dot{x} = \begin{bmatrix} -1 & \frac{3}{2} \\ -\frac{3}{2} & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

- Consider an obstacle $\mathcal{D} \subset \mathbb{R}^n$ with a smooth boundary
  - ⇝ There exists a point $x \in \partial\mathcal{D}$ such that
    - ★ B and the tangent $T(x)$ of $\partial\mathcal{D}$ are linearly dependent
    - ★ Ax points inside $\mathcal{D}$

Problem formulation & hybrid controller framework

Setting:

$$\dot{x} = Ax + Bu, \qquad A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^n$$

Obstacles: $p \in \mathbb{N}$

$$\hat{x}_i \in \mathbb{R}^n \setminus \{0\}, i = 1, \dots, p \quad \Rightarrow \hat{\mathcal{X}} := \bigcup_{i=1}^{p} \{\hat{x}_i\}$$

Problem

Consider the linear system and a **robustly stabilizing feedback law**

$$u_s = K_s x.$$

For given $\varepsilon_2 > \varepsilon_1 > 0$, construct an avoidance (safety) controller $\gamma(x)$ such that

(i) *the origin $x = 0$ is robustly globally asymptotically stable*

(ii) $\gamma(x)$ *satisfies*

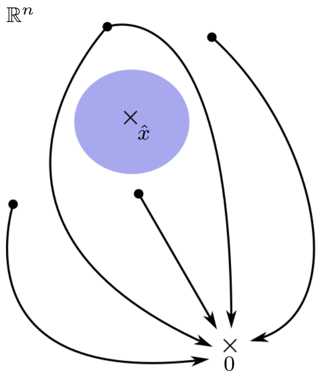
$$\gamma(x) = K_s x \quad \forall x \in \mathbb{R}^n \setminus \mathcal{B}_{\varepsilon_2}(\hat{\mathcal{X}})$$

$\Longrightarrow$ *Robust Semiglobal Preservation*

(iii) *the closed loop solution $x(\cdot;\gamma)$ satisfies*

$$x(t;\gamma) \notin \mathcal{B}_{\varepsilon_1}(\hat{\mathcal{X}}), \forall t \in \mathbb{R}_{\geq 0}, \forall x_0 \notin \mathcal{B}_{\varepsilon_2}(\hat{\mathcal{X}})$$

$\Longrightarrow$ *Robust Semiglobal $\hat{x}$-avoidance*



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Hybrid controller design

Given: Controller selection

$$\gamma(x,q) = u_q(x),\ q \in \underbrace{\{-p,\dots,-1,0,1,\dots,p\}}_{\mathcal{Q}}.$$

Orchestrate the controller selection through the flow map:

$$\dot{\xi} = \begin{bmatrix} \dot{x} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} Ax + B\gamma(x,q) \\ 0 \end{bmatrix}, \qquad \xi \in \mathcal{C}$$

and the jump map

$$\xi^+ = \begin{bmatrix} x^+ \\ q^+ \end{bmatrix} \in \begin{bmatrix} x \\ \{i \in \mathbb{N} | \xi \in \mathcal{D}_i\} \end{bmatrix}, \qquad \xi \in \mathcal{D}$$

where

- $\mathcal{D} = \bigcup_{i=1}^{p} \mathcal{D}_i \subset \mathbb{R}^n \times \mathcal{Q}$ (Jump set)
- $\mathcal{C} \subset \mathbb{R}^n \times \{1,\dots,p\}$ (Flow set)

Obstacle avoidance ($\hat{x} \notin \mathcal{E}$): Problem formulation and assumptions

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Just one Obstacle:

$$\hat{x} \in \mathbb{R}^n \backslash \mathcal{E}$$

Set of induced equilibria:

$$\mathcal{E} = \{y \in \mathbb{R}^n : 0 = Ay + B\nu,\ \nu \in \mathbb{R}\}$$

Basic Assumption

(a) Matrix $A_s := A + BK_s$ is Hurwitz. ✓

(b) $|B| = 1$. ✓

(c) The norm $x \mapsto |x|^2$ is contractive under the stabilizer $u_s = K_s x$ (i.e., $V(x) = x^T x$ is a Lyapunov function.) ✓

Discussion:

(a) (A, B) stabilizable (Controllability is not necessary)

(b) Coordinate transformation: $B_\circ = B/|B|$, $u_\circ = |B|u$.

(c) Lyapunov function: $V_\circ(x) = x^T S_\circ^T S_\circ x$. Coordinate transformation: $x_\circ = S_\circ x$.

Design of feedback γ: The **wipeout property**

Remark

- For each $x \in \mathcal{B}_\eta(\hat{x})$ we have

$$\dot{H}(x) = \langle \nabla H(x), Ax + Bu \rangle \geq 0, \quad \forall u \in \mathbb{R}$$

- For each $\bar{\eta} < \eta$, there exists $\underline{h} > 0$ such that

$$\langle \nabla H(x), Ax + Bu \rangle \geq \underline{h}, \quad \forall u \in \mathbb{R}, \forall x \in \mathcal{B}_{\bar{\eta}}(\hat{x})$$

Remark

- A solution $x(\cdot)$ such that $x(t) \in \mathcal{B}_{\bar{\eta}}(\hat{x})$ $\forall t \in [0, T]$, satisfies

$$\langle w_{\hat{x}}, x(T) - x(0) \rangle \geq T \frac{\underline{h}}{|\nabla H(x)|}.$$

- A solution $x(\cdot)$ such that $x(t) \in \mathcal{B}_{\eta}(\hat{x})$ $\forall t \in [0, T]$, satisfies

$$\langle w_{\hat{x}}, x(t_2) - x(t_1) \rangle \geq 0 \qquad \forall 0 \leq t_1 \leq t_2 \leq T.$$

- Distance to induced equilibria:

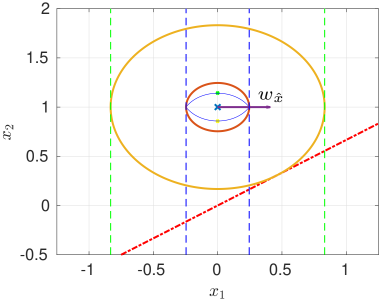
$$\eta^2 := \min_{y \in \{y \mid \exists u, Ay + Bu = 0\}} |\hat{x} - y|^2$$

- Linear “wipeout” function/direction:

$$H(x) = \hat{x}^T A_B^T x = \hat{x}^T A^T (I - BB^T) x$$

$$w_{\hat{x}} = \frac{\nabla H(x)}{|\nabla H(x)|} = \frac{A_B \hat{x}}{\sqrt{\hat{x}^T A_B^T A_B \hat{x}}}$$

- Visualization:



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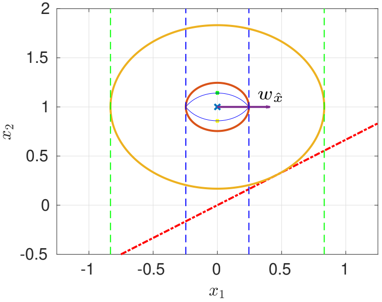


Design of feedback γ: The **avoidance shell** $\mathcal{S}(\delta)$

- Design parameters of the shell $\mathcal{S}(\delta)$:

$$\delta \in \mathbb{R}_{>0}$$
$$\mu \in (0,2)$$

- Definitions: $(q \in \{1,-1\})$

$$\delta_\mu := \delta\left(\frac{1}{\mu} - \frac{\mu}{4}\right)$$
$$\mathcal{O}_q := \mathcal{B}_{\left(\frac{\mu\delta}{2}+\delta_\mu\right)}(\hat{x} - q\delta_\mu B)$$
$$\mathcal{S}(\delta) := \mathcal{O}_1 \bigcap \mathcal{O}_{-1}$$

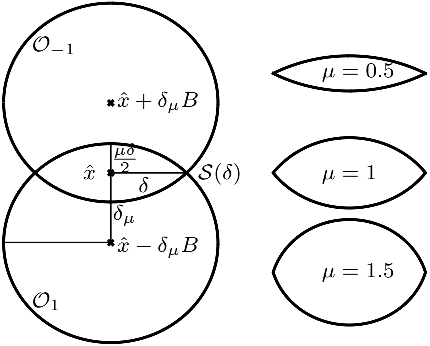
- Hysteresis parameter: $h \in (0,1)$

$$\mathcal{O}_{h,q} = \mathcal{B}_{h\frac{\mu\delta}{2}+\delta_\mu}(\hat{x} - q\delta_\mu B)$$
$$\mathcal{S}_h(\delta) = \mathcal{O}_{h,1} \cap \mathcal{O}_{h,-1}$$
$$\mathcal{\$}_q = \mathcal{S}(\delta) \cap$$
$$\{x \in \mathbb{R}^n : \; qB^T(x - \hat{x}) \geq 0\}$$

- Repulsive Avoidance law: $u_a(x, q, k_r)$

- Robust "above" avoidance $(q = 1)$
- Robust "below" avoidance $(q = -1)$

Design of feedback γ: The **avoidance shell** $\mathcal{S}(\delta)$



- Design parameters of the shell $\mathcal{S}(\delta)$:

$$\delta \in \mathbb{R}_{>0}$$
$$\mu \in (0,2)$$

- Definitions: $(q \in \{1,-1\})$

$$\delta_\mu := \delta\left(\frac{1}{\mu} - \frac{\mu}{4}\right)$$
$$\mathcal{O}_q := \mathcal{B}_{\left(\frac{\mu\delta}{2}+\delta_\mu\right)}(\hat{x} - q\delta_\mu B)$$
$$\mathcal{S}(\delta) := \mathcal{O}_1 \bigcap \mathcal{O}_{-1}$$

- Hysteresis parameter: $h \in (0,1)$

$$\mathcal{O}_{h,q} = \mathcal{B}_{h\frac{\mu\delta}{2}+\delta_\mu}(\hat{x} - q\delta_\mu B)$$
$$\mathcal{S}_h(\delta) = \mathcal{O}_{h,1} \cap \mathcal{O}_{h,-1}$$
$$\$_q = \mathcal{S}(\delta)\cap$$
$$\{x \in \mathbb{R}^n : \; qB^T(x - \hat{x}) \geq 0\}$$

- Repulsive Avoidance law: $u_a(x,q,k_r)$

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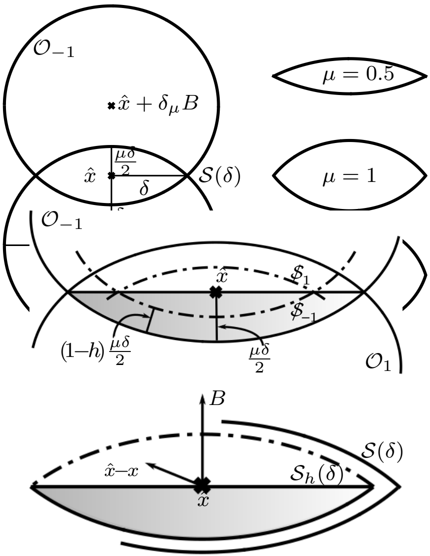
Design of feedback γ: The **avoidance shell** $\mathcal{S}(\delta)$



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- Repulsive Avoidance law: $u_a(x,q,k_r)$

- Robust “above” avoidance $(q=1)$
- Robust “below” avoidance $(q=-1)$

Nominal and Robust Avoidance and Stabilization Theorems

Theorem (**Nominal** avoidance+GAS theorem)

For any robustness gain $k_r \geq 0$ the closed-loop enjoys,

- *(**Nominal** shell avoidance) For any initial condition outside the* outer p *shells, all* **nominal** *solutions remain outside the* inner p *shells*
- *(**Nominal** GAS) The origin is UGAS for the* **nominal** *dynamics*

Theorem (**Robust-in-the-small** avoidance+GAS theorem)

*For any robustness gain $k_r > 0$ there exists a (**small enough**) positive definite continuous perturbation $\sigma(\cdot)$ such that the closed-loop enjoys,*

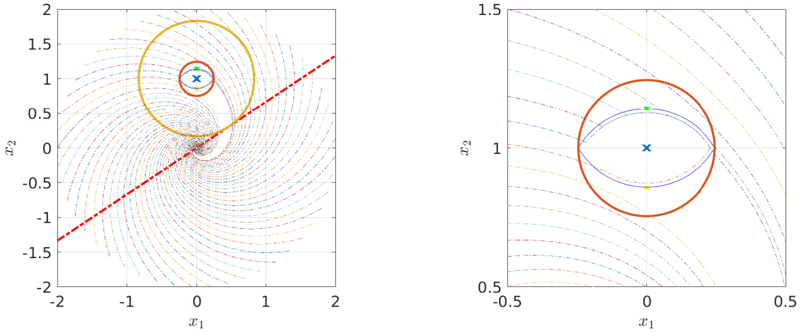
- *(**S-Robust** shell avoidance) For any initial condition outside the* outer p *shells, all* **S-perturbed** *solutions remain outside the* inner p *shells*
- *(**S-robust** GAS) The origin is UGAS for the* **S-perturbed** *dynamics*

Theorem (**Robust-in-the-large** avoidance theorem – No GAS!)

*For any non-negative definite perturbation $\sigma(\cdot)$ there exists a (**large enough**) robustness gain k_r such that the closed-loop enjoys,*

- *(**L-Robust** shell avoidance) For any initial condition outside the* outer p *shells, all* **L-perturbed** *solutions remain outside the* inner p *shells*
- *(**L-robust** GAS) Cannot be guaranteed with a large σ unless extra assumptions hold on the stabilizer*

Numerical example



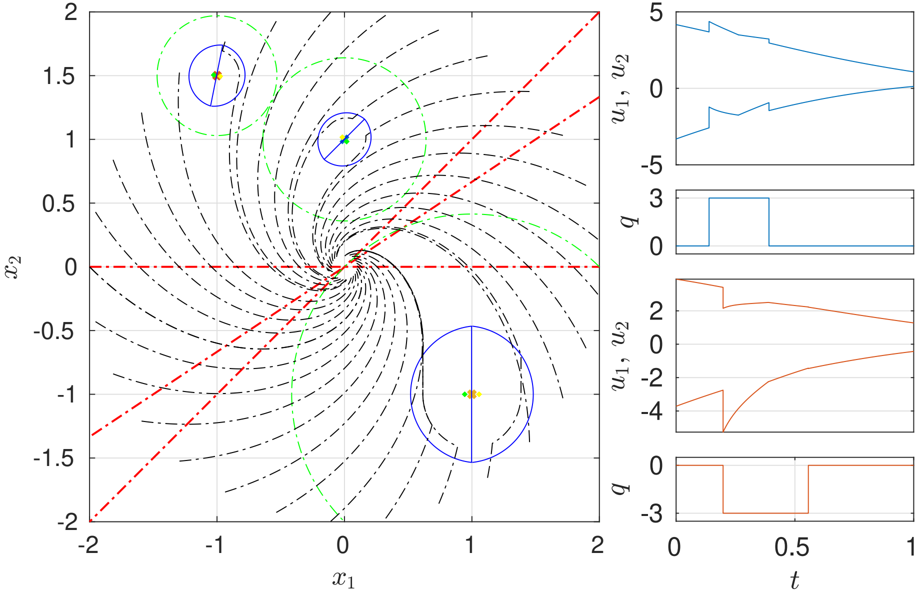
System parameters:

$$\dot{x} = \begin{bmatrix} -1.0 & 1.5 \\ -1.5 & -1.0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad \hat{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \qquad \begin{aligned} \sigma(A) &= \{-1 + 1.5i, -1 - 1.5i\} \\ \sigma(A + A^T) &= \{-2, -2\} \end{aligned}$$

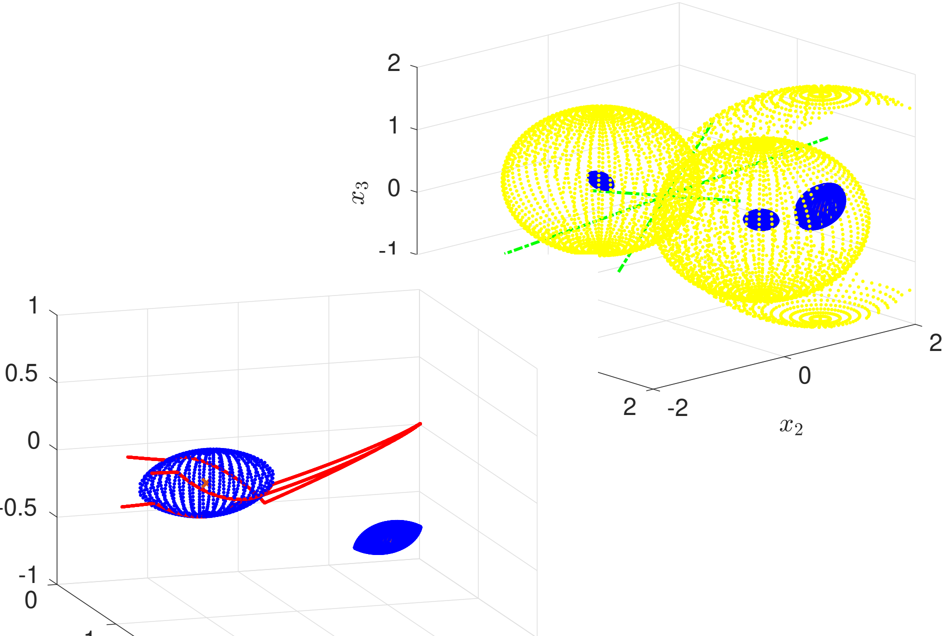
$$u_s = 0, \qquad \mu = 1.15, \qquad \eta = 0.8321, \qquad \zeta = 1.8028, \qquad \delta^* = 0.2455$$

Multiple Obstacles and multiple inputs

Use one input for avoidance and the other inputs for the wipeout property



The construction is independent of the state dimension $x \in \mathbb{R}^n$



Obstacle avoidance ($\hat{x} \in \mathcal{E}$): Problem formulation and assumptions

Setting:

- (Linear) Dynamical system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) \in \mathbb{R}^n, \quad (u \in \mathbb{R}^1)$$

- (A, B) controllable (stabilizability is not enough)

Subspace of induced equilibria: $(B \in \mathbb{R}^n)$

- $\mathcal{E} = \{y \in \mathbb{R}^n | 0 = Ay + B\nu, \ \nu \in \mathbb{R}\}$
- (W.l.o.g.) $\mathcal{E} = \mathrm{span}(A^{-1}B)$

Remember:

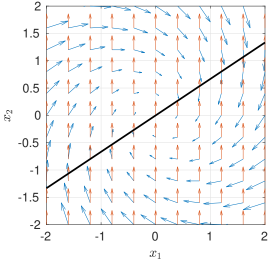
Let $\hat{x} \in \mathcal{E}$ and $0 = A\hat{x} + B\nu_{\hat{x}}$. Then

$$\begin{aligned} \dot{z} &= \overset{\cdot}{\overbrace{x - \hat{x}}} = A(x - \hat{x}) + B(u - \nu_{\hat{x}}) \\ &= Az + Bv \end{aligned}$$

where $z = x - \hat{x}$ and $v = u - \nu_{\hat{x}}$.

Example:

$$\dot{x} = \begin{bmatrix} -1 & \frac{3}{2} \\ -\frac{3}{2} & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

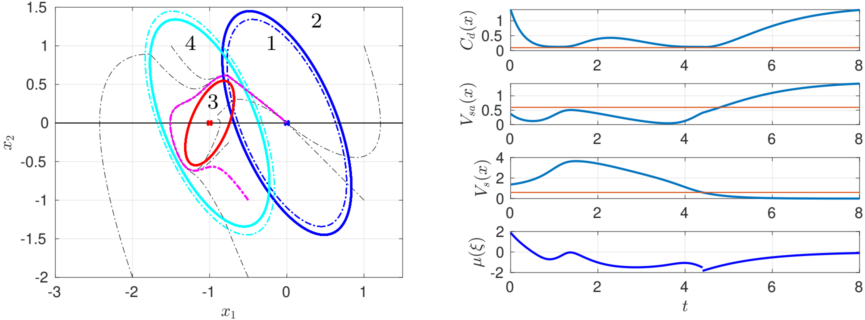


Obstacle centroid such that

$$\hat{x} \in \mathcal{E}$$

Target set $0 \in \mathbb{R}^n$

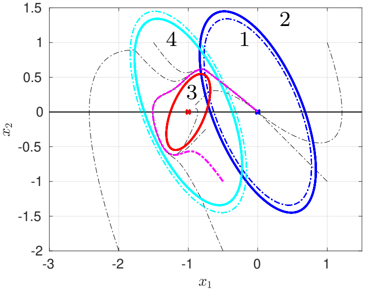
Intuitive controller design



Controller design:

- In 1 & 2: $\gamma(\xi) = u_s(x) = K_s(x)$ (asymptotically stabilize 0)
- In 3: $\gamma(\xi) = u_d(x) = K_d(x - \hat{x}) + \nu_{\hat{x}}$ (completely destabilize $\hat{x}$)
- In 4: $\gamma(\xi) = (1 - \lambda(x))u_{sa}(x) + \lambda(x)u_d(x), \quad (\lambda(x) \in [0, 1])$ (stay away from $\hat{x}$)
- (Dashed lines: avoid Zeno behavior)

Obstacle avoidance & target set stabilization ($\hat{x} \in \mathcal{E}$)



Controller design:

- In 1 & 2: $\gamma(\xi) = u_s(x) = K_s(x)$
- In 3: $\gamma(\xi) = u_d(x) = K_d(x - \hat{x}) + \nu_{\hat{x}}$
- In 4:
  $\gamma(\xi) = (1 - \lambda(x))u_{sa}(x) + \lambda(x)u_d(x)$,
  $(\lambda(x) \in [0, 1])$
- (Dashed lines: avoid Zeno behavior)

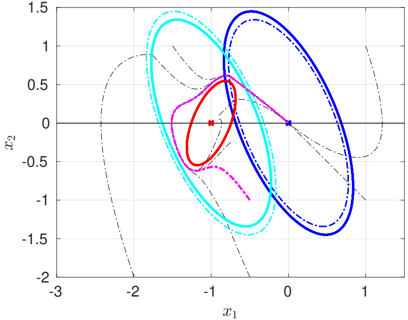
Assumptions:

- $\dot{x} = Ax + Bu$, $x \in \mathbb{R}^n$, $u \in R^1$
- (A, B) controllable
  ((A, B) stabilizable is not enough)
- $\hat{x} \in \mathcal{E} = \mathrm{span}(A^{-1}B)$

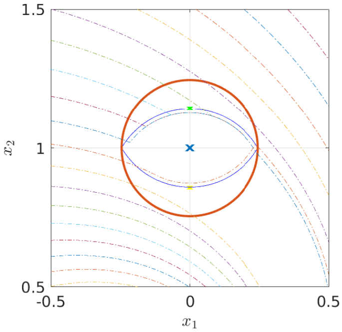
Results:

- $\forall\ n \geq 2$: Obstacle avoidance
- $n = 2$: Obstacle avoidance & global asymptotic stability
- $\forall\ n > 2$ odd: No global asymptotic stability
- $\forall\ n \geq 4$ even: Obstacle avoidance & maybe global asymptotic stability
  (We cannot exclude the existence of periodic orbits)

Conclusion & discussion



$\hat{x} \in \mathcal{E}$



$\hat{x} \notin \mathcal{E}$

Linear system: $\dot{x} = Ax + Bu, \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m$

Assumption:

- (A, B) controllable
- (A, B) stabilizable

If $m > 1$:

- $\hat{x} \in \{y | 0 = Ay + Bu, u \in \mathbb{R}^m\} \backslash \{0\}$
- $\hat{x} \in \mathbb{R}^n \backslash \{0\}$

Closed-loop properties:

- Only applicable if $n \geq 2$ is even
- Guarantees only for $n = 2$
- Independent of $n \in \mathbb{N}$, $n \geq 2$

How to enlarge the avoidance neighborhood? Nonlinear dynamical systems?