Linear flux observers for induction motors with quadratic Lyapunov certificates

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Overview

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Main Equations of the Induction Motor LTV model

State space model of the induction motor is:

$$\begin{aligned} \dot{x}(t) =& A(t)x(t) + Bu(t), \\ y(t) =& Cx(t), \end{aligned} \qquad (\spadesuit)$$

where x, u and y are respectively the state, the input and the output vectors defined as:

$$x = \begin{bmatrix} i_{sd} & i_{sq} & \psi_{rd} & \psi_{rq} \end{bmatrix}^T; \; y = \begin{bmatrix} i_{sd} & i_{sq} \end{bmatrix}^T; \; u = \begin{bmatrix} u_{sd} & u_{sq} \end{bmatrix}^T,$$

denoting direct and quadrature stator voltage $u_{s\star}$ and current $i_{s\star}$ and rotor flux $\psi_{r\star}$, and

$$A(t) = \begin{bmatrix} -\gamma & 0 & \alpha\beta & \beta\omega_{re}(t) \\ 0 & -\gamma & -\beta\omega_{re}(t) & \alpha\beta \\ \alpha L_m & 0 & -\alpha & -\omega_{re}(t) \\ 0 & \alpha L_m & \omega_{re}(t) & -\alpha \end{bmatrix}; B = \begin{bmatrix} \frac{1}{\sigma L_s} & 0 \\ 0 & \frac{1}{\sigma L_s} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}; C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}^T.$$

Parameters σ, α, β and γ, are defined as:

$$\sigma = 1 - \frac{L_m^2}{L_s L_r} \in (0,1), \quad \alpha = \frac{R_r}{L_r} > 0, \quad \beta = \frac{L_m}{\sigma L_s L_r} > 0, \quad \gamma = \frac{R_s}{\sigma L_s} + \beta\alpha L_m > 0. \qquad \text{(PARS)}$$

With matrices

$$A(t) = \begin{bmatrix} -\gamma & 0 & \alpha\beta & \beta\omega_{re}(t) \\ 0 & -\gamma & -\beta\omega_{re}(t) & \alpha\beta \\ \alpha L_m & 0 & -\alpha & -\omega_{re}(t) \\ 0 & \alpha L_m & \omega_{re}(t) & -\alpha \end{bmatrix}; B = \begin{bmatrix} \frac{1}{\sigma L_s} & 0 \\ 0 & \frac{1}{\sigma L_s} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}; C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}^T,$$

the linear time-varying (LTV) equations (♠), can be written as (with $J = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$):

$$\dot{x}(t) = \Big(\overbrace{\begin{bmatrix} -\gamma & \alpha\beta \\ \alpha L_m & -\alpha \end{bmatrix}}^{\bar{A}} \otimes I + \overbrace{\begin{bmatrix} 0 & -\beta\omega_{re}(t) \\ 0 & \omega_{re}(t) \end{bmatrix}}^{\Omega(t)} \otimes J \Big) x(t) + \Big(\overbrace{\begin{bmatrix} \frac{1}{\sigma L_s} \\ 0 \end{bmatrix}}^{\bar{B}} \otimes I \Big) u(t),$$

$$y(t) = \Big(\overbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}^{\bar{C}} \otimes I \Big) x(t),$$

The simplified LTI model naturally arising from original LTV model is:

$$\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}u(t), \quad \bar{y}(t) = \bar{C}\bar{x}(t). \qquad \text{(♠reduced)}$$

Lemma

Matrix $\bar{A}$ is Hurwitz and triple $(\bar{A}, \bar{B}, \bar{C})$ is controllable and observable for any value of the physical parameters satisfying (PARS).

Observer design and main result

Let us consider the following LTV observer dynamics:

$$\dot{\hat{x}} = \left[\bar{A} \otimes I + \Omega(t) \otimes J\right] \hat{x} + \left(\bar{B} \otimes I\right) u + \left[\bar{L} \otimes I + \begin{bmatrix} 0 \\ \rho\, \omega_{re}(t) \end{bmatrix} \otimes J\right] (y - \hat{y}). \qquad (\heartsuit)$$

Theorem (Main)

Consider any constant gain $\bar{L}$ such that $\bar{A} - \bar{L}\bar{C}$ is Hurwitz and any pair of positive definite matrices $\bar{P} = \left[\begin{smallmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{smallmatrix}\right]$, $\bar{Q}$ such that:

$$\mathrm{He}\left(\bar{P}(\bar{A} - \bar{L}\bar{C})\right) = \left(\bar{P}(\bar{A} - \bar{L}\bar{C})\right) + \left(\bar{P}(\bar{A} - \bar{L}\bar{C})\right)^T \leq -\bar{Q} < 0. \qquad (\clubsuit)$$

Then, selecting $\rho = \frac{\beta p_{11} - p_{12}}{p_{22}}$, and denoting the estimation error as $e = x - \hat{x}$, the following quadratic Lyapunov conditions hold:

$$V(e) = \frac{1}{2} e^T \left(\bar{P} \otimes I\right) e = e^T \begin{bmatrix} p_{11} & 0 & p_{12} & 0 \\ 0 & p_{11} & 0 & p_{12} \\ p_{12} & 0 & p_{22} & 0 \\ 0 & p_{12} & 0 & p_{22} \end{bmatrix} e \text{ is positive definite}$$

$$\dot{V}(e) = \langle \nabla V(e), \dot{e} \rangle = -e^T \left(\bar{Q} \otimes I\right) e,$$

along all solutions to ($\spadesuit$), ($\heartsuit$) for any time-varying $t \mapsto \omega_{re}(t)$.

Interpretations of Theorem **Main**

We can design the gain $\bar{L}$ for the 2×2 LTI model (♠reduced) and then we obtain the same features for the 4×4 LTV model (♠).

Clearly, the **error variables** e **depend on** ω_{re} and exhibit a peculiar time-varying transient, but the **upper bound on** $V(e)$ **is a simple exponential function**.

Tight upper bound by solving the convex optimization with α_ℓ being the spectral abscissa of $\bar{A} - \bar{L}\bar{C}$ (namely $\alpha_\ell = -\max_i \left(\mathrm{Re}\{\lambda_i(\bar{A} - \bar{L}\bar{C})\}\right)$),

$$\min_{k, \bar{P}=\bar{P}^T} k, \quad \text{subject to:}$$

$$\mathrm{He}\left(\bar{P}(\bar{A} - \bar{L}\bar{C})\right) \leq -2\alpha_\ell \bar{P}, \qquad (= -\bar{Q})$$

$$I \leq \bar{P} \leq kI,$$

to obtain the lifted bound:

$$|e(t)| \leq \sqrt{k}\, \mathrm{e}^{-\alpha_\ell t} |e(0)|, \quad \forall t \geq 0$$

This result follows from Theorem 2 applied with $\bar{Q} = 2\alpha_\ell \bar{P}$.

Observer gain $\bar{L}$ selection for $\mathcal{L}_2$ gain minimization

Gain $\bar{L}$ can be chosen to **minimize** the $\mathcal{L}_2$ gain between a disturbance d (acting on the current measurement) and the estimation error e.

Theorem **Main** can be applied with $\rho = 0$, which leads to the following error dynamics:

$$\dot{e} = \left((\bar{A} - \bar{L}\bar{C}) \otimes I + \Omega(t) \otimes J\right) e + \left(\bar{L} \otimes I\right) d. \qquad \text{(ERR)}$$

An upper bound μ on the $\mathcal{L}_2$ gain from d to e for dynamics (ERR) can be minimized by solving the LMI formulation of the Bounded Real Lemma [1][1]

$$\min_{\mu,\bar{P},\bar{X}} \mu, \quad \text{subject to:}$$

$$\bar{P} = \begin{bmatrix} p_{11} & \beta p_{11} \\ \beta p_{11} & p_{22} \end{bmatrix} > 0,$$

$$\mathrm{He} \begin{bmatrix} \bar{P}\bar{A} - \bar{X}\bar{C} & -\bar{X} & 0 \\ 0 & -\frac{\mu}{2} I & 0 \\ I & 0 & -\frac{\mu}{2} I \end{bmatrix} < 0,$$

$$\mathrm{He}\left(\bar{P}\bar{A} - \bar{X}\bar{C}\right) \leq -2\alpha_{\mathrm{des}} \bar{P},$$

and then selecting $\bar{L} = \bar{P}^{-1}\bar{X}$, where $\alpha_{\mathrm{des}} > 0$ is any desired convergence rate

[1] [1] G. Dullerud, F. Paganini, *ACourse in Robust Control Theory.* Springer, 2000.

Explicit selections of observer gain $\bar{L}$ and certificate $\bar{P}$

A few relevant explicit selections of $\bar{L}$ can be given for Theorem **Main**:

1 (**Open-loop observer**) Selection:

$$\bar{L} = \begin{bmatrix} 0 & 0 \end{bmatrix}^T, \quad \bar{P} = \begin{bmatrix} \sigma L_s L_r & 0 \\ 0 & 1 \end{bmatrix} > 0, \quad \bar{Q} = 2\begin{bmatrix} \gamma\sigma L_s L_r & -\alpha L_m \\ -\alpha L_m & \alpha \end{bmatrix} > 0, \quad \text{(Lzero)}$$

is such that $\bar{A} - \bar{L}\bar{C}$ is Hurwitz and $\bar{P}$, $\bar{Q}$ satisfy (♣).

2 (**Speed of convergence** α) Selection:

$$\bar{L} = \begin{bmatrix} \alpha - \gamma & \alpha L_m \end{bmatrix}^T, \quad \bar{P} = \begin{bmatrix} \frac{1}{\alpha\beta} & 0 \\ 0 & \frac{\beta}{\alpha} \end{bmatrix} > 0, \quad \bar{Q} = \begin{bmatrix} \frac{2}{\beta} & -1 \\ -1 & 2\beta \end{bmatrix} > 0,$$

is such that $\bar{A} - \bar{L}\bar{C}$ is Hurwitz and assigns both eigenvalues of $\bar{A} - \bar{L}\bar{C}$ at $-\alpha$. Moreover selections $\bar{P}$, $\bar{Q}$ satisfy (♣).

3 (**Arbitrary speed of convergence** $(\alpha + \eta)$). Given any scalar $\eta > 0$, selection:

$$\bar{L} = \begin{bmatrix} \alpha - \gamma + 2\eta \\ \alpha L_m + \frac{\eta}{\beta}\left(1 + 2\frac{\eta}{\alpha}\right) \end{bmatrix}, \quad \bar{P} = \begin{bmatrix} \frac{\eta}{\alpha}\left(1 + 2\frac{\eta}{\alpha}\right) & -\frac{\beta}{\alpha}\eta \\ -\frac{\beta}{\alpha}\eta & \beta^2 \end{bmatrix}, \quad \bar{Q} = 2(\alpha+\eta)\bar{P} \quad \text{(Lspeed)}$$

is such that $\bar{A} - \bar{L}\bar{C}$ is Hurwitz and $\bar{P}$, $\bar{Q}$ satisfy (♣).

Reduced order observer

Given any gain $\bar{L}$ and any matrices $\bar{P} = \left[\begin{smallmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{smallmatrix}\right] > 0$, $\bar{Q} = \left[\begin{smallmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{smallmatrix}\right] > 0$ satisfying Theorem **Main**, we introduce the reduced order observer [2, Lemma 3.1][2]:

$$\dot{\hat{\phi}} = A_\psi(t) \begin{bmatrix} y \\ \hat{\psi} \end{bmatrix} + p_{22}^{-1} p_{12} \left(A_i(t) \begin{bmatrix} y \\ \hat{\psi} \end{bmatrix} + \frac{1}{\sigma L_s} u \right) \qquad (\heartsuit\text{red})$$

$$\hat{\psi} = \hat{\phi} - p_{22}^{-1} p_{12} y, \qquad A(t) = \begin{bmatrix} \frac{A_i(t)}{A_\psi(t)} \end{bmatrix} = \begin{bmatrix} \frac{-\gamma I \quad \alpha\beta I - \beta\omega_{re}(t) J}{\alpha L_m I \quad -\alpha I + \omega_{re}(t) J} \end{bmatrix}$$

Proposition (Reduced Order Observer)

If matrices $(\bar{P}, \bar{Q})$, and gain $\bar{L}$ satisfy (♣), then the flux estimation error $e_\psi = \psi - \hat{\psi}$ satisfies the following quadratic Lyapunov conditions:

$$V_\psi(e_\psi) = \frac{1}{2} e_\psi{}^T (p_{22} \otimes I) e_\psi \quad \textit{is positive definite}$$

$$\dot{V}_\psi(e_\psi) = -\alpha \left(1 - \beta p_{22}^{-1} p_{12}\right) V_\psi(e_\psi),$$

along dynamics (♠), (♡red).

[2] [2] G. Besancon, *Remarks on nonlinear adaptive observer design.* Systems & control letters, 41(4), pp 271-280, 2000.

Comparison with works presented in the literature

With selection (Lspeed), if $\eta = \beta\alpha c \implies p_{22}^{-1} p_{12} = -c$,
$\implies$ the reduced observer coincides with [3, Equation (3.33)][3]:

$$\dot{\hat{\phi}} = \Big(\begin{bmatrix} -\alpha(1+\beta c) & \alpha L_m + c(\gamma - \alpha - \alpha\beta) \end{bmatrix} \otimes I + \\ + \begin{bmatrix} \omega_{re}(t)(1+\beta c) & c\omega_{re}(t)(1+\beta c) \end{bmatrix} \otimes J \Big) \begin{bmatrix} \hat{\phi} \\ i \end{bmatrix} - \frac{c}{\sigma L_s} u,$$
$$\hat{\psi} = \hat{\phi} + ci.$$

With selection (Lzero), $\implies p_{22}^{-1} p_{12} = 0$,
$\implies$ the reduced-order observer coincides with observer [3, equation (3.8)][3]:

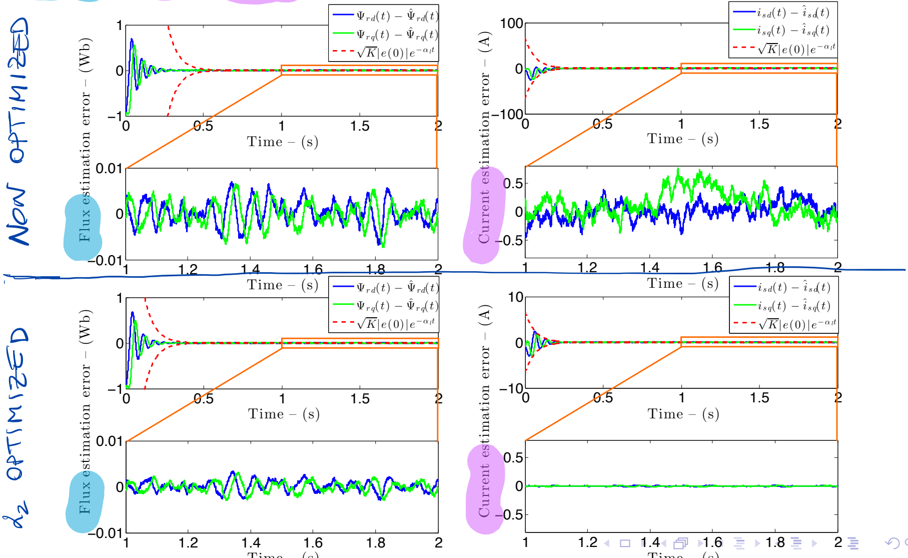
$$\dot{\hat{\phi}} = \Big(\begin{bmatrix} -\alpha & \alpha L_m \end{bmatrix} \otimes I + \begin{bmatrix} \omega_{re}(t) & 0 \end{bmatrix} \otimes J \Big) \begin{bmatrix} \hat{\phi} \\ i \end{bmatrix},$$
$$\hat{\psi} = \hat{\phi},$$

[3] [3] Riccardo Marino, Patrizio Tomei, and Cristiano M Verrelli. *Induction motor control design.* Springer, 2010.

Simulations show improved results with $\mathcal{L}_2$ optimized gains

Parameters corresponding to a 0.75 KW induction motor

Flux (left) and Current (right) error with (Lspeed) (top) and with $\mathcal{L}_2$ optimal (bottom)

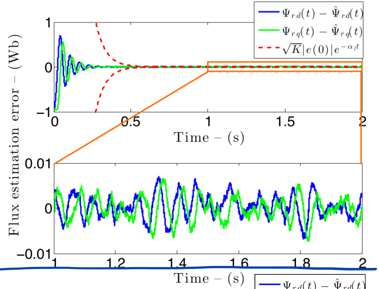


Reduced-order solution: deteriorated flux estimation e_ψ

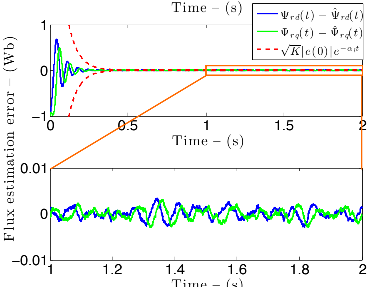
Parameters corresponding to a 0.75 KW induction motor

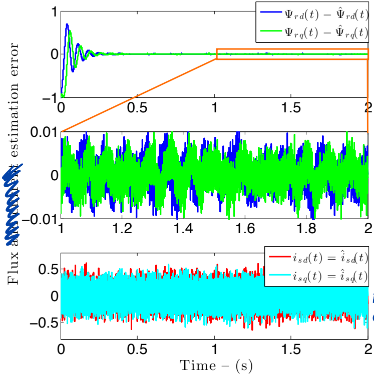
Left: same as before, **Right**: flux estimation error with reduced observer

NON OPTIMIZED



OPTIMIZED λ_2





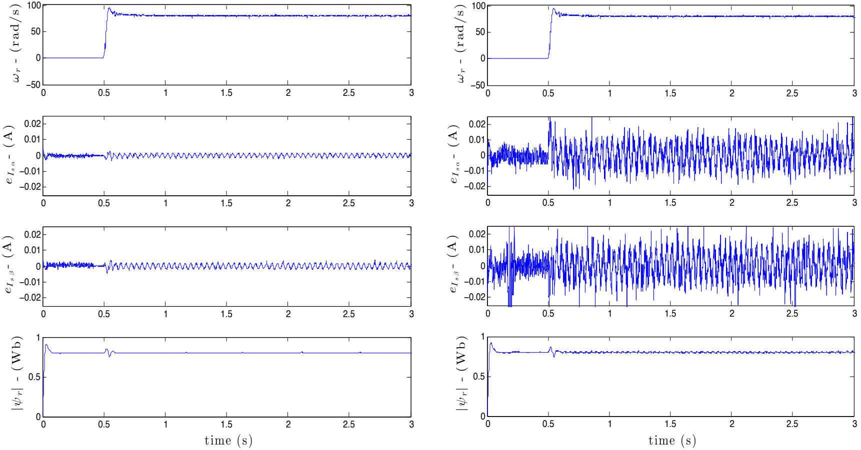
RED

NOISE AFFECTING OUTPUTS

Experimental results confirm advantage of $\mathcal{L}_2$ optimization

Field Oriented Control law on an experimental 0.75 KW induction motor

$\mathcal{L}_2$ optimal gain selection (left) vs explicit (Lspeed) selection (right) for the same α_{des}



Conclusions and perspectives

Conclusions

- Compact representation of the LTV dynamics of the IM
- Full-order Luenberger observer for the IM rotor flux estimation featuring
  - arbitrary global uniform exponential bounds on the estimation error, regardless of the rotor speed
  - Optimal observer gains selection by $\mathcal{L}_2$ optimization
- Reduced-order observer covers existing results as special cases
- Simulation and experimental tests show the effectiveness of the proposed approach.

Future Work

- Follow the same approach for the dual control problem
- Can Kronecker-based "liftings" lead to novel ideas in motor control/estimation?