

# Hybrid Dynamical Systems illustrated by case studies

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# Control Augmentation for tracking & obstacle avoidance

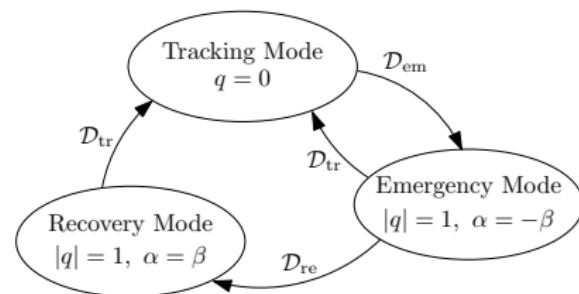
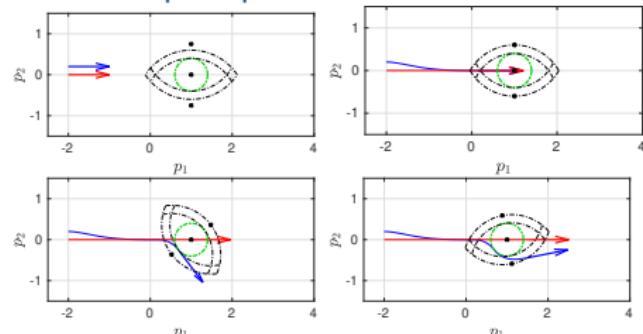
**Setting:** Standard unicycle model

$$\dot{x} = \begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} v \cos(\phi) \\ v \sin(\phi) \\ w \end{bmatrix}, \quad u = \begin{bmatrix} v \\ w \end{bmatrix}$$

with state  $x \in \mathbb{R}^3$  and input  $u \in [-2, 2]^2 \subset \mathbb{R}^2$

Obstacle avoidance + reference tracking:

Closed-loop snapshots:



- reference trajectory, • closed-loop solution, • obstacle

# From discrete + continuous to hybrid dynamical systems

**Continuous** dynamical system

$$\frac{dx(t)}{dt} = f(x(t)), \quad \forall t \in \mathbb{R}_{\geq 0}$$

**Discrete** dynamical system

$$x(k+1) = g(x(k)), \quad \forall k \in \mathbb{Z}_{\geq 0}$$

(possible discrete variables)



**Hybrid** dynamical system

$$\begin{cases} \frac{dx(t, k)}{dt} = f(x(t, k)), & x(t, k) \in \mathcal{C} \subset \mathbb{R}^n \\ x(t, k+1) = g(x(t, k)), & x(t, k) \in \mathcal{D} \subset \mathbb{R}^n \end{cases}$$

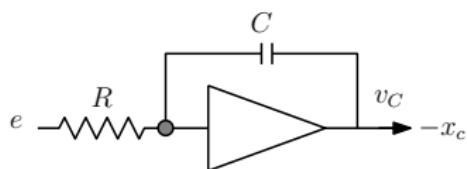
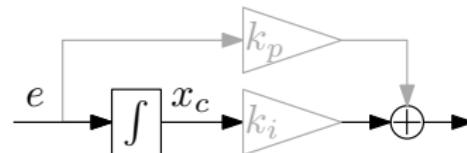
- **Continuous** time domain  $t \in \mathbb{R}_{\geq 0}$  and **Discrete** time domain  $k \in \mathbb{Z}_{\geq 0}$  merged into **Hybrid** time domain  $(t, k) \in \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$
- Solution  $x$  can “flow” if  $\in \mathcal{C}$ , can “jump” if  $\in \mathcal{D}$
- Fundamental stability results now available (Converse Lyapunov theorems, ISS, invariance principle,  $\mathcal{L}_p$  stability) [Teel '04 → '12]

# An analog integrator and its Clegg extension Clegg [1958]

**Proportional Integral (PI) control** comprise an integrator

**Example:** Error feedback

$$u(t) = k_p e(t) + k_i \underbrace{\int_0^t e(\tau) d\tau}_{=x_c}$$



- In an analog integrator, the state information is stored in a capacitor:

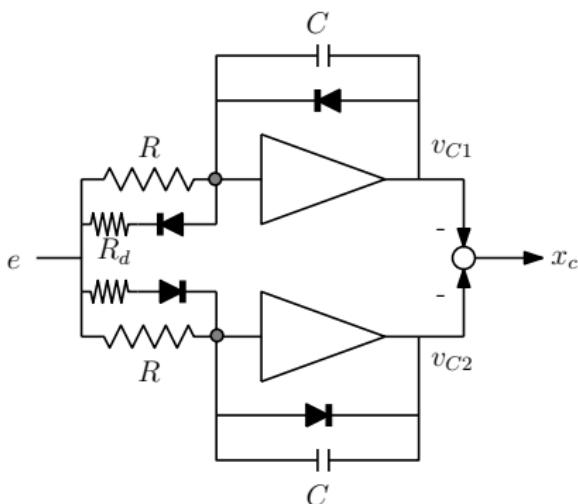
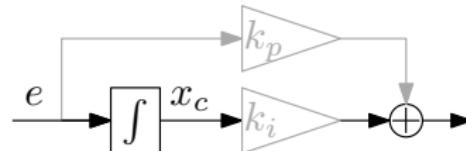
$$\dot{x}_c = (RC)^{-1} e$$

# An analog integrator and its Clegg extension Clegg [1958]

**Proportional Integral (PI) control** comprise an integrator

**Example:** Error feedback

$$u(t) = k_p e(t) + k_i \underbrace{\int_0^t e(\tau) d\tau}_{=x_c}$$



- Clegg's integrator Clegg [1958]:
  - feedback diodes: the **positive** part of  $x_c$  is all and only coming from the **upper** capacitor (and viceversa)
  - input diodes: when  $e \leq 0$  the upper capacitor is reset and the lower one integrates (and viceversa) [ $R_d \ll 1$ ]
- As a consequence  $\Rightarrow e$  and  $x_c$  **never have opposite signs**

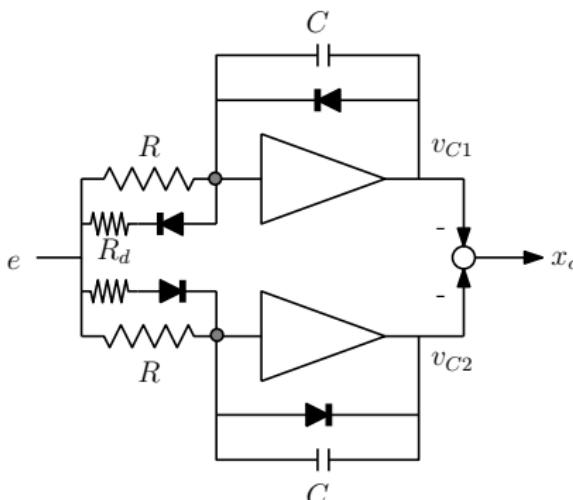
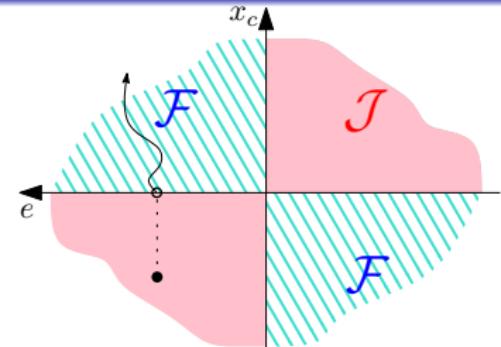
# Hybrid dynamics rule flowing or jumping of solutions

## Hybrid Clegg integrator:

$$\dot{x}_c = (RC)^{-1}e, \quad \text{allowed when } x_c e \geq 0,$$

$$x_c^+ = 0, \quad \text{allowed when } x_c e \leq 0,$$

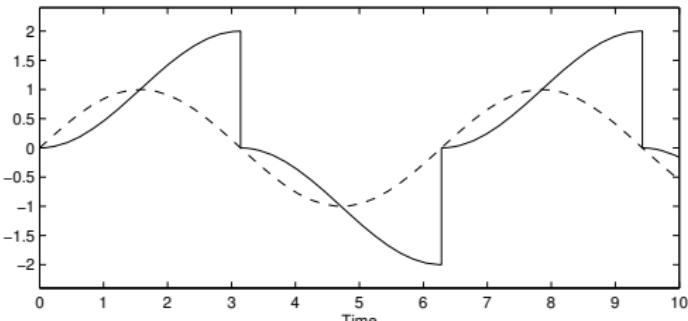
- Flow set  $\mathcal{F}$ : where  $x_c$  may flow (1st eq'n)
- Jump set  $\mathcal{J}$ : where  $x_c$  may jump (2nd eq'n)



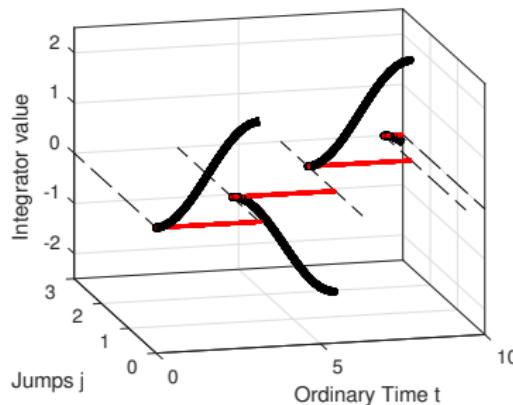
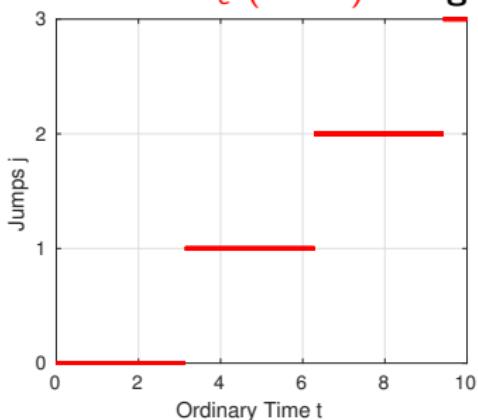
- Clegg's integrator [Clegg \[1958\]](#):
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- As a consequence  $\Rightarrow e$  and  $x_c$  **never have opposite signs**

# Example: Clegg response to a sine input $e(t, j) = \sin(t)$

- Solid: projection of  $x_c(t, j)$  on continuous time axis  $t$
- Dash: projection of  $e(t, j)$  on continuous time axis  $t$



- Domain  $\text{dom } x_c$  (in red) and graph (**bold black**) of the solution  $x_c$



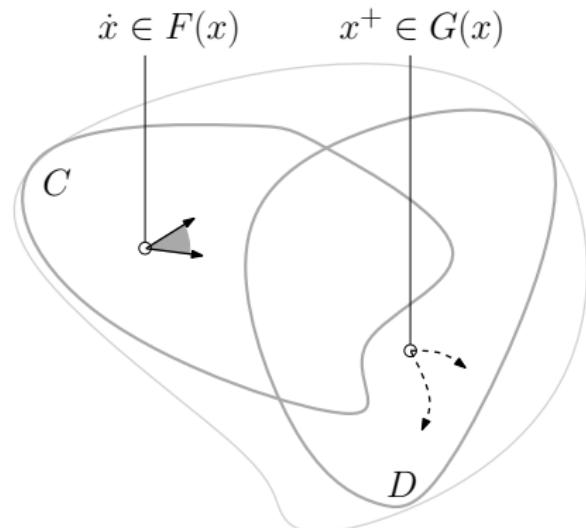
# Hybrid dynamical systems review: dynamics

Key works: Goebel et al. [2009, 2012], Teel et al. [2013], Prieur et al. [2013]

$$\mathcal{H} = (\mathcal{C}, \mathcal{D}, F, G)$$

- $n \in \mathbb{N}$  (state dimension)
- $\mathcal{C} \subseteq \mathbb{R}^n$  (flow set)
- $\mathcal{D} \subseteq \mathbb{R}^n$  (jump set)
- $F : \mathcal{C} \Rightarrow \mathbb{R}^n$  (flow map)
- $G : \mathcal{D} \Rightarrow \mathbb{R}^n$  (jump map)

$$\mathcal{H} : \begin{cases} \dot{x} \in F(x), & x \in \mathcal{C} \\ x^+ \in G(x), & x \in \mathcal{D} \end{cases}$$



# Hybrid dynamical systems review: continuous dynamics

Key works: Goebel et al. [2009, 2012], Teel et al. [2013], Prieur et al. [2013]

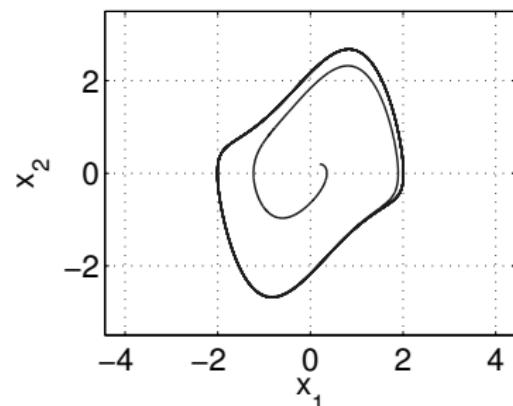
$$\mathcal{H} = (\mathcal{C}, \mathcal{D}, F, G)$$

- $n \in \mathbb{N}$  (state dimension)
- $\mathcal{C} \subseteq \mathbb{R}^n$  (flow set)
- $\mathcal{D} \subseteq \mathbb{R}^n$  (jump set)
- $F : \mathcal{C} \Rightarrow \mathbb{R}^n$  (flow map)
- $G : \mathcal{D} \Rightarrow \mathbb{R}^n$  (jump map)

$$\mathcal{H} : \begin{cases} \dot{x} \in F(x), & x \in \mathcal{C} \\ x^+ \in G(x), & x \in \mathcal{D} \end{cases}$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 + x_2(1 - x_1^2) \end{cases}$$

Van der Pol



# Hybrid dynamical systems review: discrete dynamics

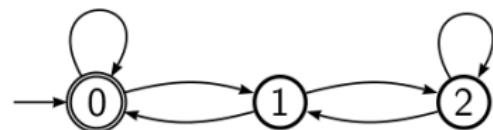
Key works: Goebel et al. [2009, 2012], Teel et al. [2013], Prieur et al. [2013]

$$\mathcal{H} = (\mathcal{C}, \mathcal{D}, \mathcal{F}, \mathcal{G})$$

- $n \in \mathbb{N}$  (state dimension)
- $\mathcal{C} \subseteq \mathbb{R}^n$  (flow set)
- $\mathcal{D} \subseteq \mathbb{R}^n$  (jump set)
- $\mathcal{F} : \mathcal{C} \rightarrow \mathbb{R}^n$  flow map)
- $\mathcal{G} : \mathcal{D} \rightarrow \mathbb{R}^n$  (jump map)

$$\mathcal{H} : \begin{cases} \dot{x} \in \mathcal{F}(x), & x \in \mathcal{C} \\ x^+ \in \mathcal{G}(x), & x \in \mathcal{D} \end{cases}$$

$$x^+ \in \begin{cases} \{0, 1\} & \text{if } x = 0 \\ \{0, 2\} & \text{if } x = 1 \\ \{1, 2\} & \text{if } x = 2 \end{cases}$$

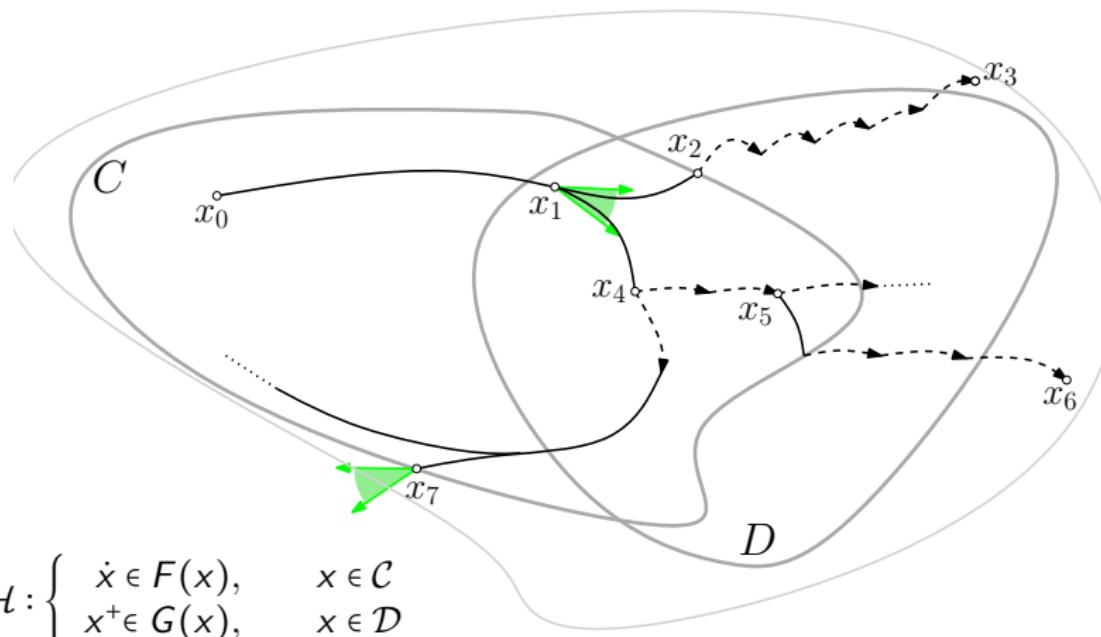


A possible sequence of states from  $x_0 = 0$  is:

$$(0 \cdot 1 \cdot 2 \cdot 1)^i \quad i \in \mathbb{N}$$

# Hybrid dynamical systems review: trajectories

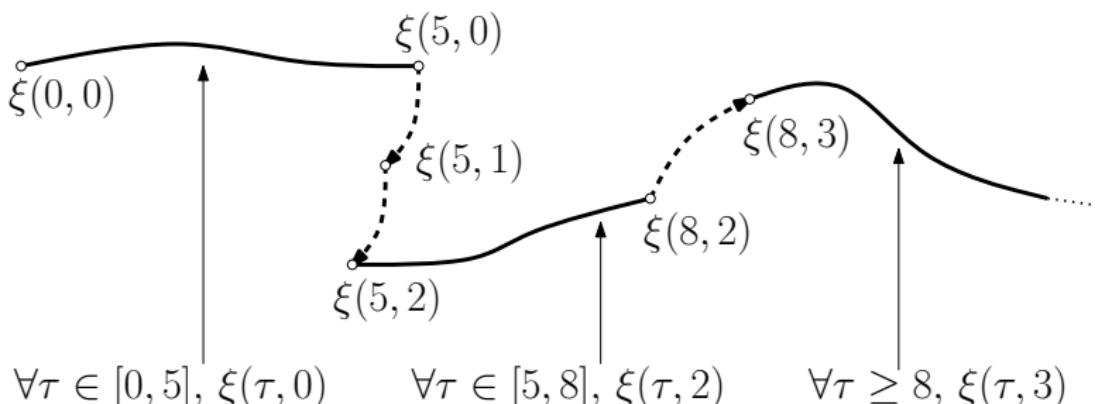
Key works: Goebel et al. [2009, 2012], Teel et al. [2013], Prieur et al. [2013]



# Hybrid dynamical systems review: hybrid time

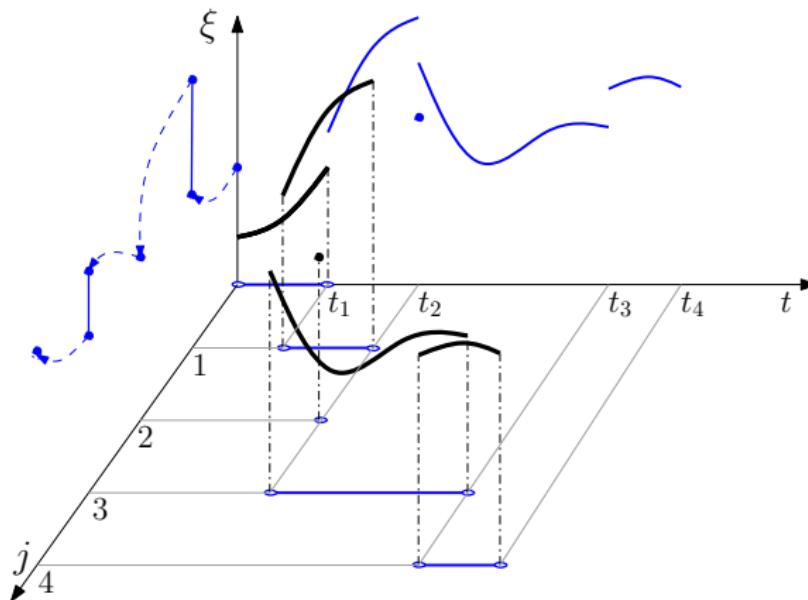
The motion of the state is parameterized by two parameters:

- $t \in \mathbb{R}_{\geq 0}$ , takes into account the elapse of time during the continuous motion of the state;
- $j \in \mathbb{Z}_{\geq 0}$ , takes into account the number of jumps during the discrete motion of the state.



# Hybrid dynamical systems review: solution

- Formally, a solution satisfies the **flow dynamics** when flowing and satisfies the **jump dynamics** when jumping



# Hybrid dynamical systems review: Lyapunov theorem

**Th'm** Teel et al. [2013] Given Euclidean norm  $|x| = \sqrt{x^T x}$  and system

$$\mathcal{H} : \begin{cases} \dot{x} = f(x), & x \in \mathcal{C} \\ x^+ = g(x), & x \in \mathcal{D}, \end{cases}$$

assume that function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  satisfies for some scalars  $c_1, c_2$  positive and  $c_3$  positive:

$$c_1|x|^2 \leq V(x) \leq c_2|x|^2, \quad \forall x \in \mathcal{C} \cup \mathcal{D} \cup G(\mathcal{D})$$

$$\langle \nabla V(x), f(x) \rangle \leq -c_3|x|^2, \quad \forall x \in \mathcal{C},$$

$$V(g(x)) - V(x) \leq -c_3|x|^2, \quad \forall x \in \mathcal{D},$$

then the origin is uniformly globally exponentially stable (UGES) for  $\mathcal{H}$ , namely there exist  $K, \lambda > 0$  such that all solutions satisfy

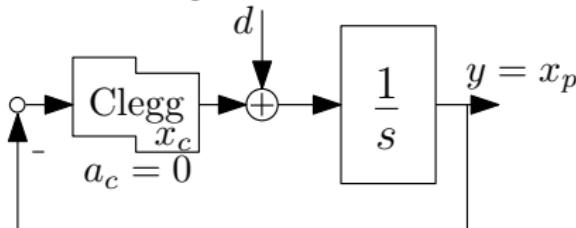
$$|\xi(t, j)| \leq K e^{\lambda(t+j)} |\xi(0, 0)|, \quad \forall (t, j) \in \text{dom } \xi$$

Note: Lyapunov conditions comprise **flow** and **jump** conditions.

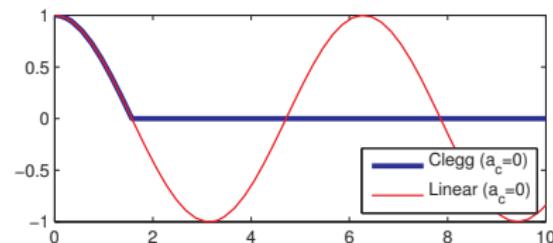
Note: UGES is characterized in terms of hybrid time  $(t, j)$

# Example 1: Clegg connected to an integrator plant

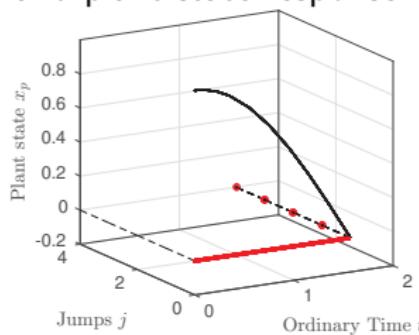
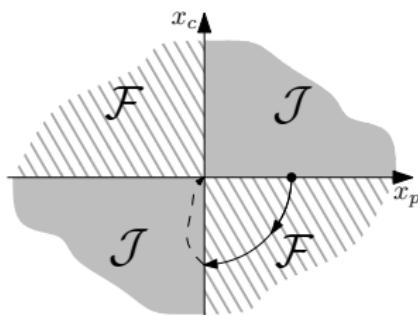
- Block diagram:



- Output response overcomes linear control limitations Beker et al. [2001]

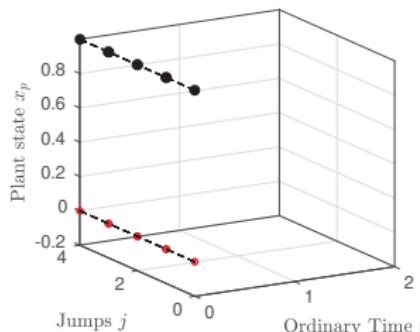
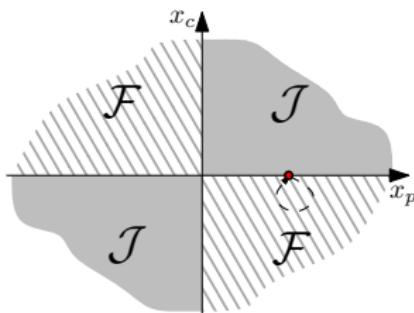


- Hybrid solution on the phase-plane and plant state response

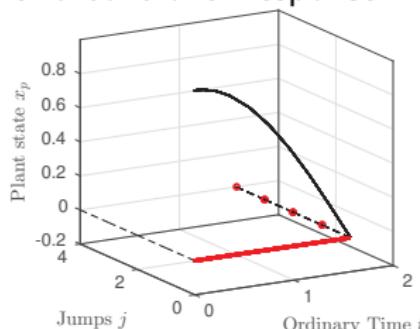
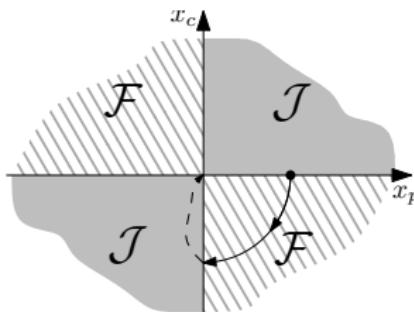


# Example 1: there exists another **bad** solution!

- A **bad** nonconverging “discrete” solution

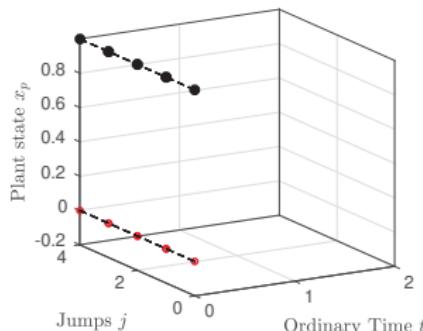
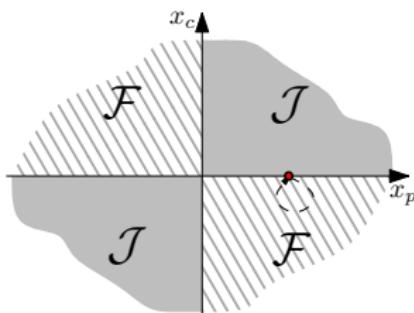


- Hybrid solution on the phase-plane and controller response



# Space or time regularization to eliminate **bad** solutions

- A **bad** nonconverging “discrete” solution



- Two options to avoid multiple instantaneous resets (jumps)

Inhibit jumps for  $\rho$  continuous time after each jump

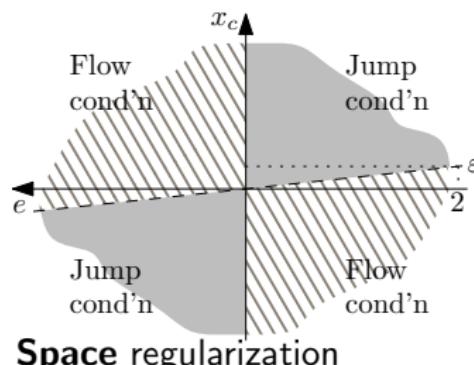
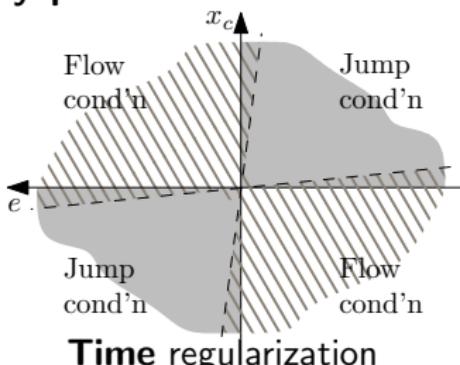
**Time** regularization

Ensure that jumps map to the interior of  $\mathcal{F}$ , away from  $\mathcal{J}$

**Space** regularization

# Space or time regularization to eliminate **bad** solutions

- **Lyapunov conditions** must be enforced on suitable sets



- Two options to avoid multiple instantaneous resets (jumps)

Inhibit jumps for  $\rho$  continuous time after each jump

**Time** regularization

Ensure that jumps map to the interior of  $\mathcal{F}$ , away from  $\mathcal{J}$

**Space** regularization

# Time Regularization can induce overflowing in the set $\mathcal{J}$

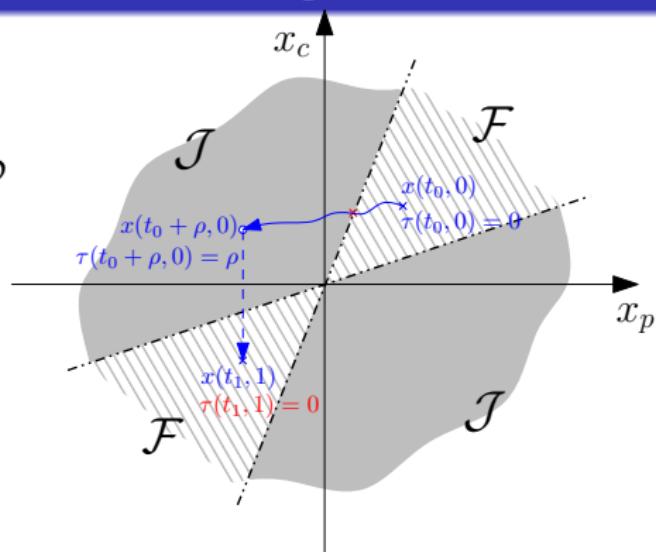
$$\mathcal{H} \left\{ \begin{array}{lcl} \dot{x} & = & Ax + Bd \\ \dot{\tau} & = & 1 \\ x^+ & = & Gx \\ \tau^+ & = & 0 \\ z & = & C_z x + D_{zd} d \end{array} \right. \quad \begin{array}{l} (x, \tau) \in \mathcal{C} \\ (x, \tau) \in \mathcal{D} \end{array}$$

$$\mathcal{C} = \{(x, \tau) : x \in \mathcal{F} \text{ or } \tau \in [0, \rho]\}$$

$$\mathcal{D} = \{(x, \tau) : x \in \mathcal{J} \text{ and } \tau \geq \rho\}$$

$$\mathcal{F} = \{x \in \mathbb{R}^n : x^\top M x \leq 0\}$$

$$\mathcal{J} = \{x \in \mathbb{R}^n : x^\top M x \geq 0\}$$



- ☺ Persistent flowing of solutions
- ☺ Overflow in the set  $\mathcal{J}$  ( $t_1 \geq t_0 + \rho$ )

**Theorem:** Nešić et al. [2011] From partial homogeneity it follows that

$$\text{LAS} \Leftrightarrow \text{GES} \Leftrightarrow \mathcal{L}_p \text{ stable from } d \Leftrightarrow \text{ISS from } d$$

# Performance analysis result: $V(x) = x^T Px$ quadratic

Nešić et al. [2008], Fichera et al. [2016]

**Theorem:** Consider system  $\mathcal{H}$ . If there exist  $P = P^T > 0$ , non-negative  $\tau_F, \tau_R \in \mathbb{R}_{\geq 0}$  and positive  $\bar{\gamma}$ , s.t.

$$(Flow) \quad \begin{pmatrix} A^T P + PA - \tau_F M & PB & C_z^T \\ B^T P & -\bar{\gamma} I & D_{zd}^T \\ C_z & D_{zd} & -\bar{\gamma} I \end{pmatrix} < 0,$$

$$(Jump) \quad G^T P G - P + \tau_R M \leq 0,$$

Then, by virtue of  $V(x) = x^T Px$ , for any  $\gamma$  satisfying

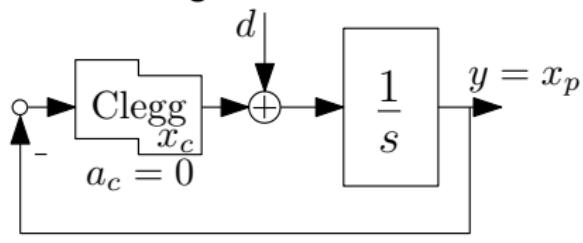
$$\gamma \geq \bar{\gamma}, \quad \gamma > \sqrt{2|D_{zd}|},$$

there exists  $\bar{\rho} > 0$  such that for any  $\rho \in (0, \bar{\rho})$ :

- ① the set  $\mathcal{A} = \{(x, \tau) : x = 0\}$  is globally exponentially stable for the hybrid system  $\mathcal{H}$  with  $d = 0$ ;
- ② the  $t$ - $\mathcal{L}_2$  gain from  $d$  to  $z$  is  $\leq \gamma$ , for all  $d \in t\text{-}\mathcal{L}_2$ .

# Example 1: Clegg connected to an integrator

- Block diagram:

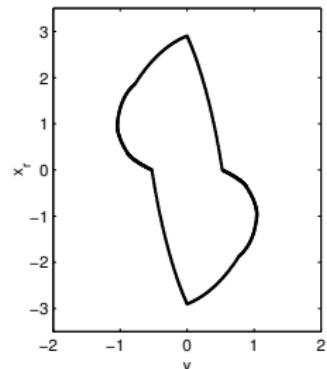
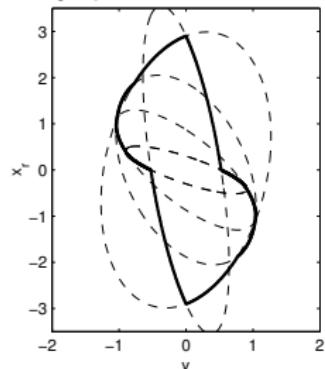
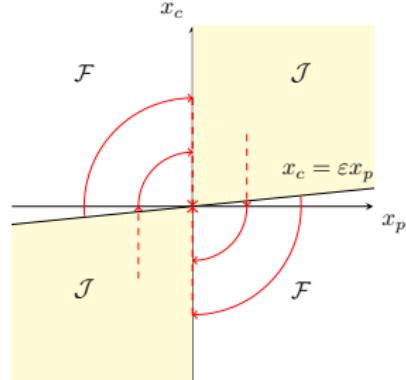


- Gain  $\gamma_{dy}$  estimates ( $N = \#$  of sectors)

$N$	2	4	8	50
gain $\gamma_{dy}$	2.834	1.377	0.914	0.87

- A lower bound:  $\sqrt{\frac{\pi}{8}} \approx 0.626$

- Lyapunov func'n level sets for  $N = 4$



- Quadratic Lyapunov functions are unsuitable [Zaccarian et al. \[2011\]](#)

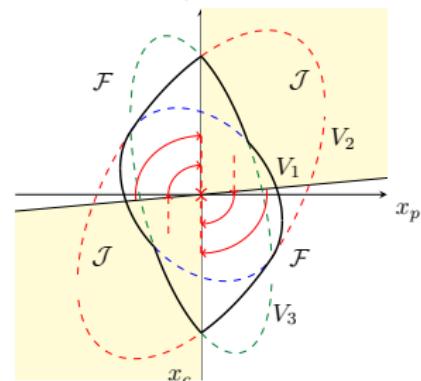
- $P_1, \dots, P_4$  cover 2nd/4th quadrants
- $P_0$  covers 1st/3rd quadrants

# Convex Lyapunov function for Example 1

Della Rossa et al. [2019]

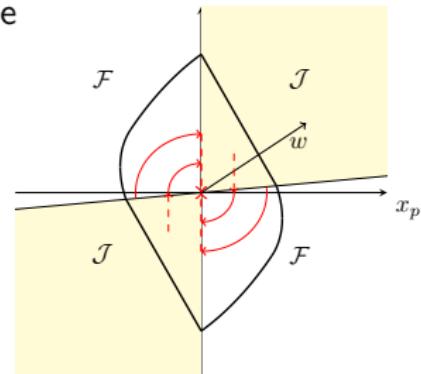
- Mid of quadratics provides another nonconvex Lyapunov function certifying GES for the previous example

$$\begin{aligned} V_{\text{mid}}(x) &= \text{mid}\{V_1, V_2, V_3\} \\ &:= \max\{\min\{V_1, V_2\}, \\ &\quad \min\{V_2, V_3\}, \min\{V_1, V_3\}\} \end{aligned}$$



- May twist it into a convex Lyapunov certificate

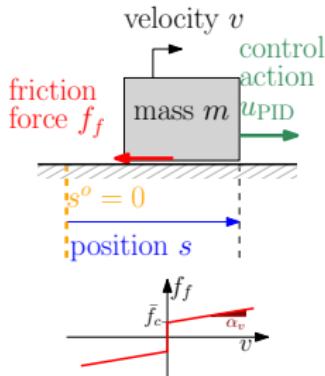
$$V_{\text{conv}}(x) = \begin{cases} V_{\text{mid}}(x), & \text{if } x \in \mathcal{F} \\ \langle w, x \rangle^2 & \text{if } x \in \mathcal{J} \end{cases}$$



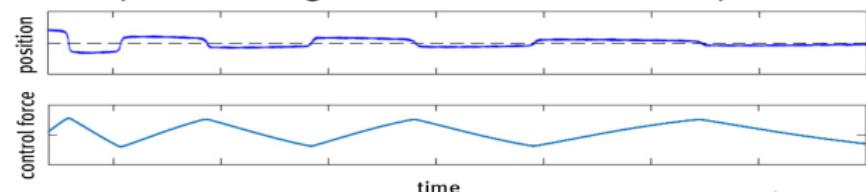
# Reset PID control to compensate Coulomb friction

Bisoffi et al. [2018], Beerens et al. [2019]

- Coulomb friction causes slow transients with PID feedback



- Manipulation stage of an electron microscope



Discontinuous Lyapunov function:

$$V(x) := \begin{bmatrix} \sigma \\ v \end{bmatrix}^T \begin{bmatrix} \frac{k_v}{k_i} & -1 \\ -1 & k_p \end{bmatrix} \begin{bmatrix} \sigma \\ v \end{bmatrix} + \min_{f \in f_c} \text{SGN}(v) |\phi - f|^2$$

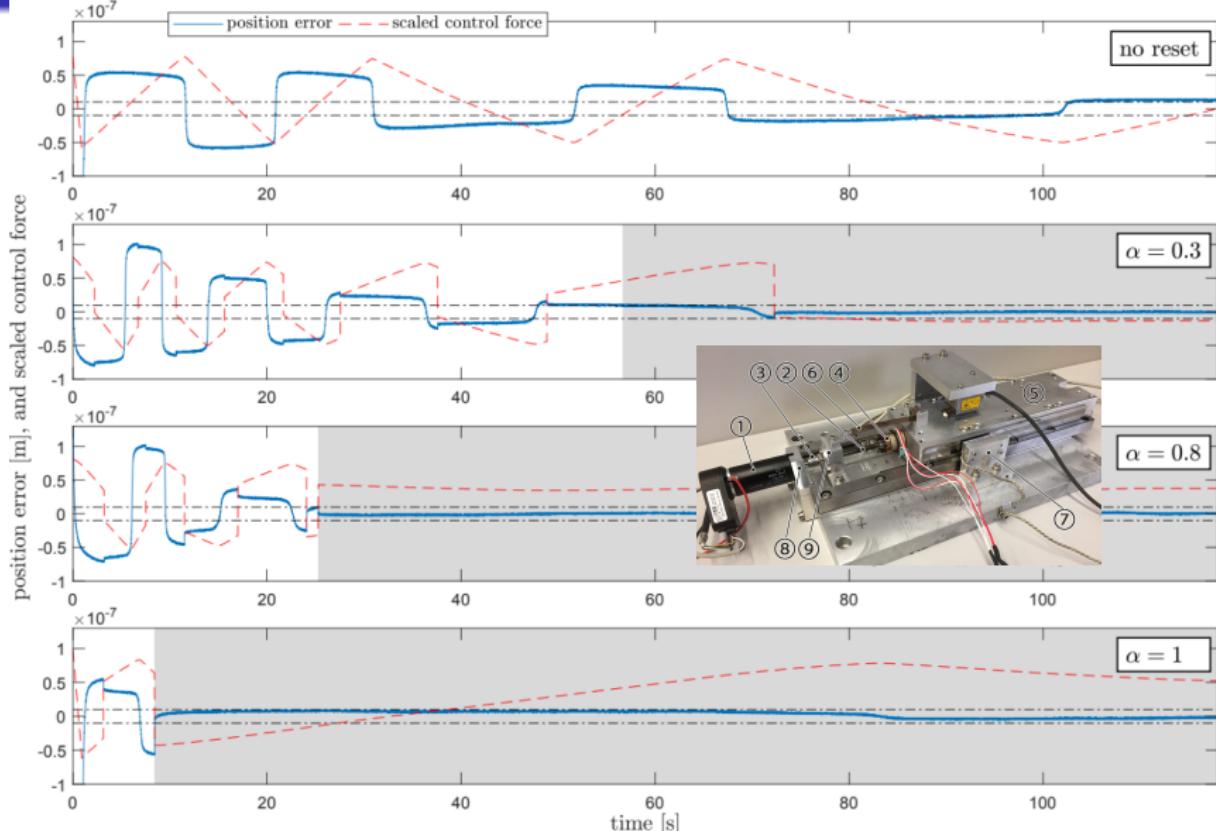
- Hybrid closed loop with reset PID (no knowledge of  $f_c$  required)

$$\dot{x} \in \begin{bmatrix} 0 & 0 & -k_i \\ 1 & 0 & -k_p \\ 0 & 1 & -k_v \end{bmatrix} \begin{bmatrix} \sigma \\ \phi \\ v \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ f_c \end{bmatrix} \text{SGN}(v), \quad x \in \mathcal{C} := \overline{\mathbb{R}^3 \setminus \mathcal{D}},$$

$$x^+ = \begin{bmatrix} \sigma & -\alpha\phi & v \end{bmatrix}^\top, \quad x \in \mathcal{D} := \{x \in \mathbb{R}^3 \mid \phi\sigma \leq 0, \phi v \leq 0, |\phi\sigma| \geq \varepsilon\},$$

# Experimental response shows transient improvement

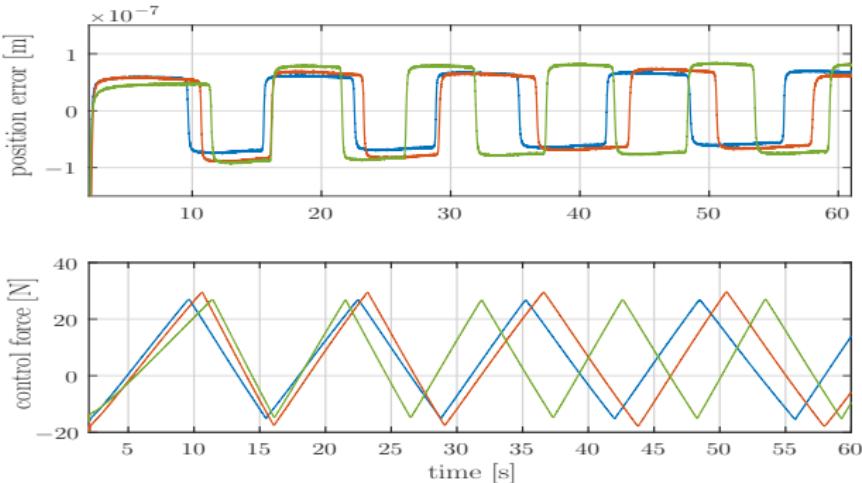
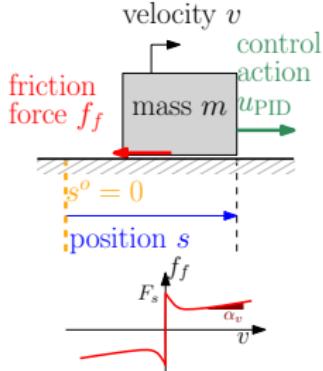
Bisoffi et al. [2018], Beerens et al. [2019]



# Reset PID control to compensate destabilizing Stribeck effect

Bisoffi et al. [2019], Beerens et al. [2021]

- Stribeck effect causes “hunting” instability with PID feedback

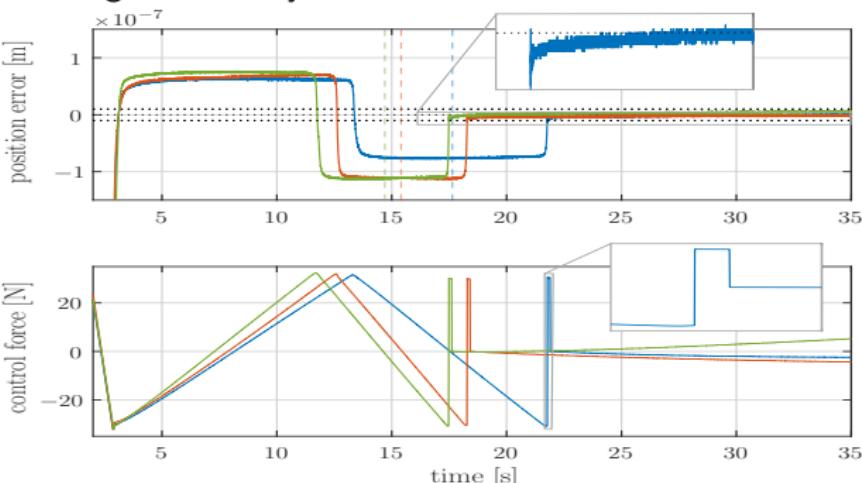
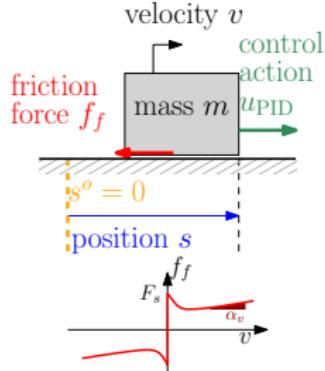


- Reset controller with extra logical state  $h \in \{-1, 1\}$ , jumps from
 
$$\mathcal{D}_\sigma := \{x : \sigma = 0, h = 1\}, \quad \mathcal{D}_v := \{x : v = 0, \sigma\phi \geq \frac{k_p}{k_i}\sigma^2, h = -1\}$$
- Flow is constrained within  $\mathcal{C} := \{x : hv\sigma \geq 0, \sigma\phi \geq \frac{k_p}{k_i}\sigma^2\}$
- Stability proof uses semiglobal dwell time and bisimulation of the dynamics with a rather convoluted hybrid Lyapunov function

# Reset PID control to compensate destabilizing Stribeck

Bisoffi et al. [2019], Beerens et al. [2021]

- Stribeck effect causes “hunting” instability with PID feedback

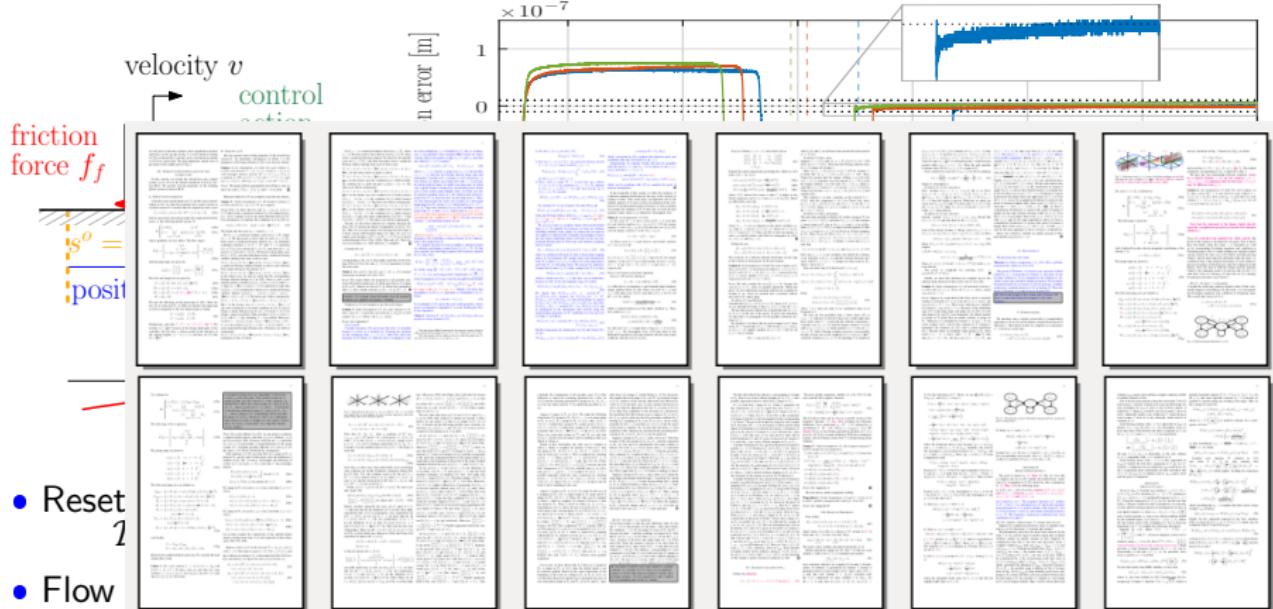


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# Proof of stability is a nightmare 12 pages blurb

Bisoffi et al. [2019], Beerens et al. [2021]

- Stribeck effect causes “hunting” instability with PID feedback



# Impulsive or jump dynamics: impacts

Polyhedral region

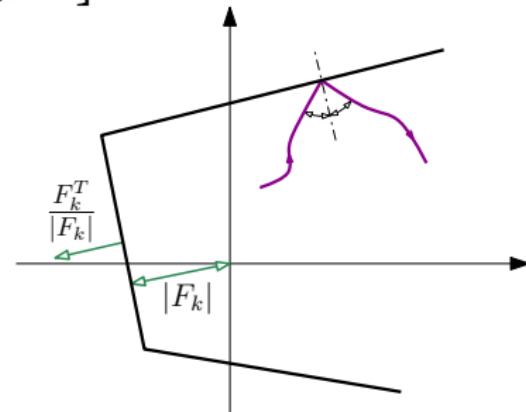
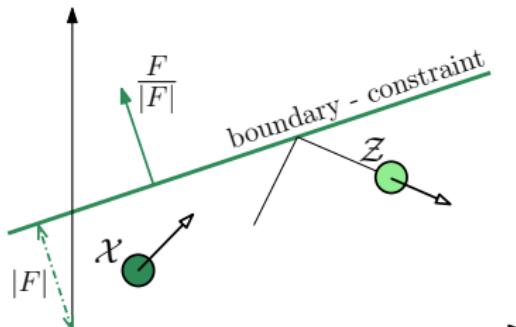
$$\mathcal{F} := \{s \in \mathbb{R}^4 \mid \forall q \in Q, \langle F_q, s_p - s_o \rangle \leq 1\}$$

Dynamic boundary

$$\mathcal{J} := \{s \in \mathcal{F} \mid \exists q \in Q, \langle F_q, s_p - s_o \rangle = 1, \langle F_q, s_v \rangle \geq 0\}$$

Reset at impacts (impulsive phenomena)

$$s^+ = \begin{bmatrix} s_p \\ M(F_q)s_v \end{bmatrix}$$



# Continuous or flow dynamics: free motion

Reference mass dynamics

$$\mathcal{Z} : \begin{cases} \dot{z}_p = z_v \\ \dot{z}_v = \alpha \\ y = Cz \quad (= z_p) \end{cases}$$

Controlled mass dynamics

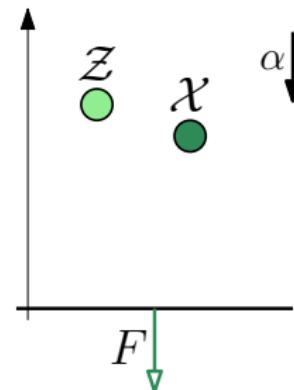
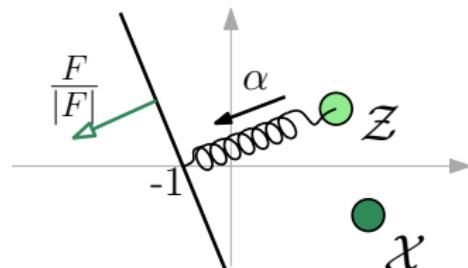
$$\mathcal{X} : \dot{x} = Ax + Bu_c$$

Observer dynamics

$$\hat{\mathcal{X}} : \dot{\hat{x}} = A\hat{x} + u_o$$

$\alpha \in \mathbb{R}^n$  measured,

$$A = \left[ \begin{array}{c|c} 0 & I \\ \hline 0 & 0 \end{array} \right], \quad B = \left[ \begin{array}{c} 0 \\ I \end{array} \right],$$



# Linear feedback does not guarantee stability/convergence

Formulation: asymptotic stability of

$$\mathcal{A}_o = \{(x, z) \mid x = z\}$$

Linear feedback:

$$\begin{aligned} u_c &= K(x - z) + \alpha & \Rightarrow \dot{e} &= (A + BK)e \\ u_o &= L(x_p - z_p) + B\alpha & \Rightarrow \dot{e} &= (A + LC)e \end{aligned}$$

Choose  $K$  and  $L$  s.t.

$$\begin{aligned} V &= (x - z)^T P (x - z) \\ &=: \|x - z\|_P^2 \\ \dot{V} &< -\gamma V, \gamma > 0 \end{aligned}$$

$\Rightarrow \mathcal{A}_o$  asymptotically stable  
(without impacts).

Impacts: stability, convergence!

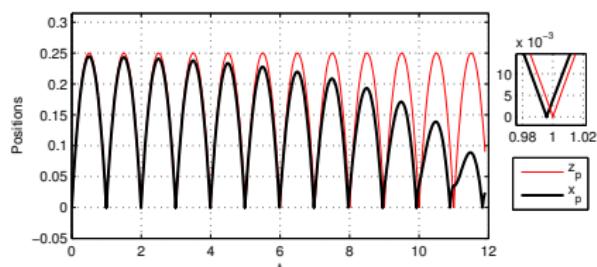
# Bouncing ball tracking is a simple example of instability

Initial conditions and gains:

$$z_0 = [0 \ v]^T$$

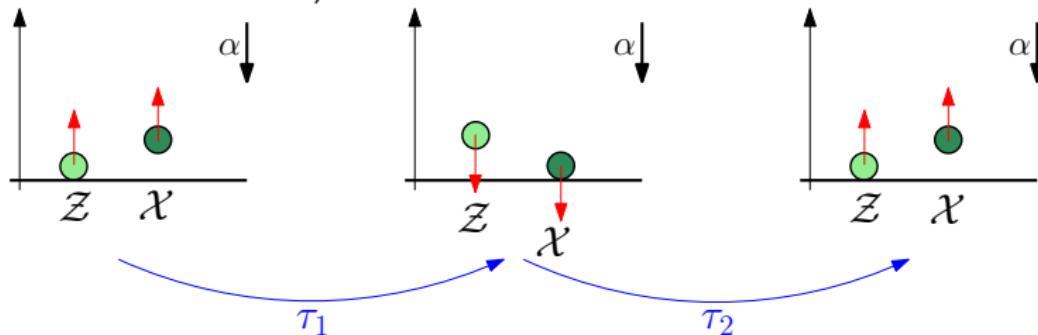
$$x_0 = z_0 + \varepsilon$$

$$K = [-4 \ -4]$$



$e = x - z$  at the  $k$ th impact of  $\mathcal{Z}$  is given approximately by

$$\left( \begin{bmatrix} -1 & 0 \\ (8+2\frac{\alpha}{v}) & -1 \end{bmatrix} e^{\begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix} \frac{2v}{\alpha}} \right)^k \varepsilon, \quad \Rightarrow \quad \text{Unstable for } \frac{v}{\alpha} \leq 0.613.$$



# Bouncing ball tracking is a simple example of instability

Initial conditions and gains:

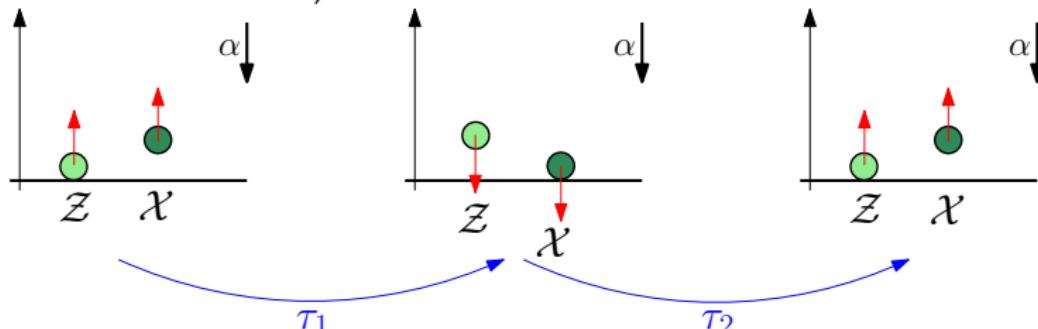
$$z_0 = [0 \ v]^T$$

$$x_0 = z_0 + \varepsilon$$

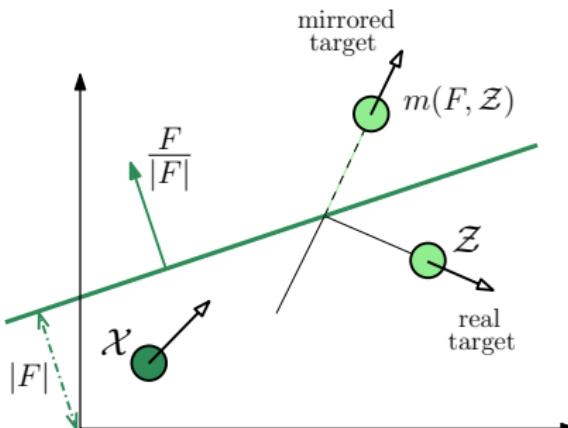
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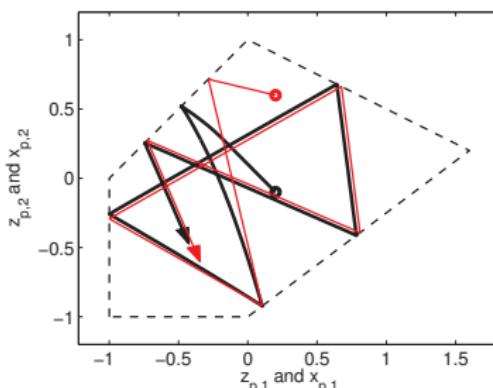
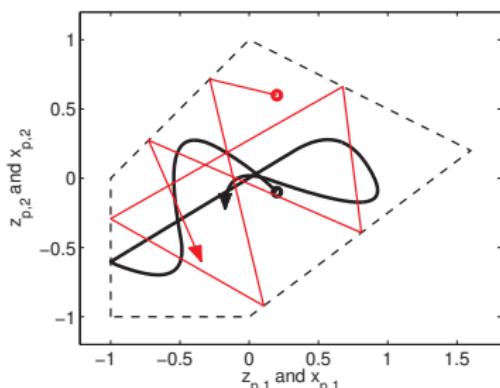
# A possible solution: mirroring through the boundary



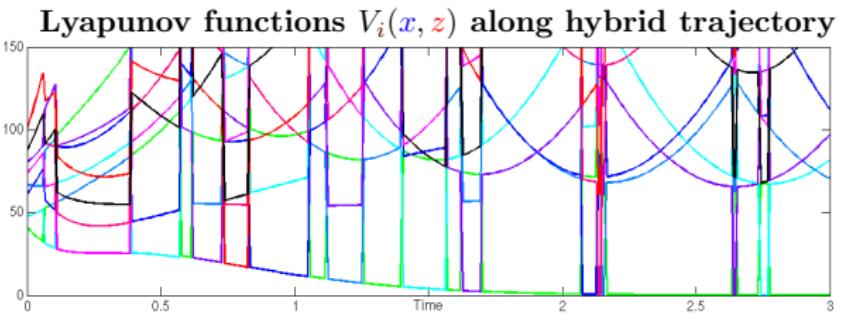
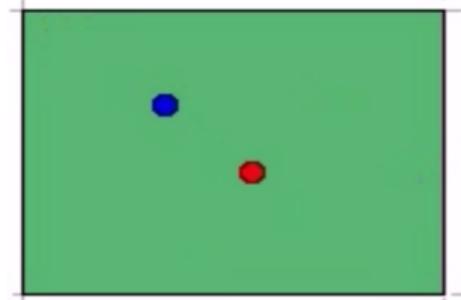
Track/Observe a **mirrored target**:

$$m_q(z) : \mathcal{Q} \times \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$$

- $q = 0$  real target  $\mathcal{Z}$ ;
- $q = i$  mirrored target  $\mathcal{Z}$  through  $F_i$ ;



# Rectangular Billiard: ball $x$ tracks billiard ball $z$



Linear Feedback  
Hybrid Feedback

Idea: Follow the “closest” ball among all the target balls  $z$  mirrored by the walls

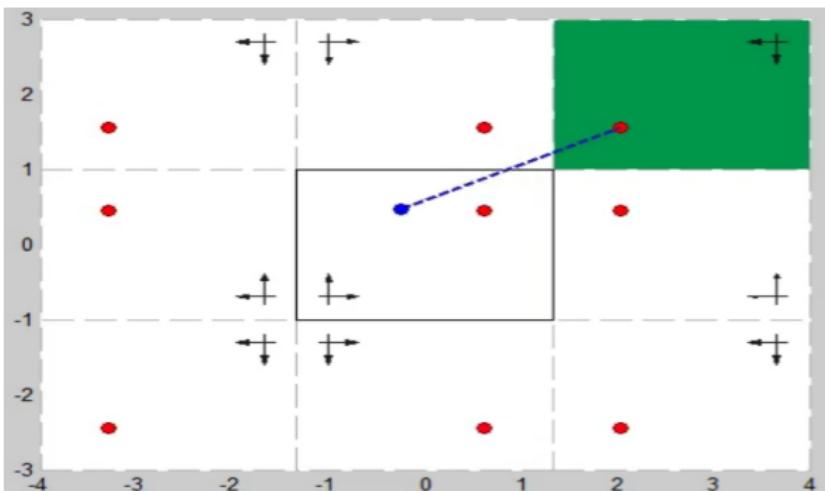
Use Hybrid Lyapunov function

$$V(\mathbf{x}, z) = \min_{i \in \{0, \dots, 9\}} \underbrace{\|\mathbf{x} - m_i(z)\|_P}_{V_i(\mathbf{x}, z)}$$

where

$$m_0(z) = z \text{ (real ball)}$$

$$m_k(z) = M_k z + c_k, k = 1, \dots, 8 \quad (\text{mirrored balls})$$



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