To stick or to slip: Lyapunov-based reset PID for positioning systems with Coulomb and Stribeck friction

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1. Problem description and model

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Coulomb friction and discontinuous right-hand side

\[
\dot{s} = v
\]

\[
\dot{v} = \begin{cases} 
    u - F_s & \text{if } v > 0 \text{ or } (v = 0, u \geq F_s) \\
    0 & \text{if } (v = 0, |u| < F_s) \\
    u + F_s & \text{if } v < 0 \text{ or } (v = 0, u \leq -F_s)
\end{cases}
\]

\[
u = -k_p s - k_v v - k_i e_i,
\]

\[
u_{PID}(t) := -\bar{k}_p s(t) - \bar{k}_i \int_0^t s(\tau)d\tau - \bar{k}_d \frac{ds(t)}{dt} 
\equal{} -\bar{k}_p s(t) - \bar{k}_i e_i(t) - \bar{k}_d v(t),
\]

\[
f_f(u_{PID}, v) := \begin{cases} 
    \bar{F}_s \text{sign}(v) + \alpha_v v, & \text{if } v \neq 0 \\
    u_{PID}, & \text{if } v = 0, \ |u_{PID}| < \bar{F}_s \\
    \bar{F}_s \text{sign}(u_{PID}), & \text{if } v = 0, \ |u_{PID}| \geq \bar{F}_s
\end{cases}
\]

\[
m \dot{v} = u_{PID} - f_f(u_{PID}, v)
\]

\[\triangleright\text{PID action and viscous force combined in } u := \frac{u_{PID} - \alpha_v v}{m} \quad \text{for } v = 0\]

\[\triangleright\text{normalize physical param's } \bar{k}_p, \bar{k}_i, \bar{k}_d, \bar{F}_s \text{ as } (k_p, k_v, k_i) := \left(\frac{\bar{k}_p}{m}, \frac{\bar{k}_d + \alpha_v}{m}, \frac{\bar{k}_i}{m}\right), \quad F_s := \frac{\bar{F}_s}{m}\]

\[
\dot{e}_i = s
\]

\[
\dot{s} = v
\]
The problem is industrially relevant with Coulomb effect.

Industrial High-precision motion control system (electron microscope) experiments:

- Graph showing position, control force, and time over a range.
- Diagram of a motor system with labeled parts (1-9).
- Graph showing friction force $F_f$ against velocity $v$.

Measured friction nonlinearity points.
Experiments show instability with Stribeck effect

- Stribeck effect causes “hunting” instability with PID feedback

Same experimental device shows Stribeck with different ambient, lubrication and wear conditions
Reformulation as a suitable differential inclusion

\[ \dot{e}_i = s \]
\[ \dot{s} = v \]
\[ u = -k_i e_i - k_p s - k_v v, \]
\[ u - \dot{v} = \begin{cases} 
+ F_s & \text{if } v > 0 \\
\text{sat}_{F_s}(u) & \text{if } v = 0 \\
- F_s & \text{if } v < 0 
\end{cases} \]

\[ \dot{e}_i = s \]
\[ \dot{s} = v \]
\[ \dot{v} \in -k_i e_i - k_p s - k_v v - F_s \text{SGN}(v) \]

- Physical model: intuitive, but hard to prove existence of solutions and stability properties with a discontinuous right hand side
- Differential inclusion: existence of solutions and \textit{ad hoc} Lyapunov tools

\textbf{Lemma BASIC (solutions are unique and complete)}

For any initial condition \( z(0) = (e_i(0), s(0), v(0)) \in \mathbb{R}^3 \), the green differential inclusion has a unique solution defined for all \( t \geq 0 \).
The interest in dynamics with friction had its peak in the 1990’s.

- **modeling direction**
  - **Dahl model:**
  - **models by Bliman and Sorine:**
  - **LuGre model:**
  - **Leuven model:**
Set-valued friction and PID control

- use of set-valued mapping for the friction force, and hence differential inclusions

- uncontrolled multi-degree-of-freedom mechanical systems:

- PD controlled 1 d.o.f. system:

- combination of set-valued friction laws and Lyapunov tools:

- stability of compact attractors

- for the same setting (point mass + PID controller), with Coulomb and viscous friction only it was proven that no stick-slip limit cycle (so-called hunting) exist:
  
  
Coulomb-only friction provides an initially simplified setting

- Coulomb friction experience suggests (slow) convergence and stability

State equations with $z = (e_i, s, v)$ are

\[
\begin{bmatrix}
\dot{e}_i \\
\dot{s} \\
\dot{v}
\end{bmatrix} \in 
\begin{bmatrix}
s \\
v \\
v
\end{bmatrix} - k_i e_i - k_p s - k_v v - F_s \text{SGN}(v)
\]

\[
= \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-k_i & -k_p & -k_v
\end{bmatrix} z - 
\begin{bmatrix}
0 \\
0 \\
F_s
\end{bmatrix} \text{SGN}(v)
\]

- Standing assumption about the PID gains is probably necessary for GAS

Assumption LIN

In the absence of friction ($F_s = 0$), the origin is globally asymptotically stable (GAS). Equivalently,

\[k_i > 0, \ k_p > 0, \ k_v k_p > k_i.\]
With Coulomb Friction the largest set of equilibria is GAS

Assumption LIN

In the absence of friction ($F_s = 0$), the origin is globally asymptotically stable (GAS). Equivalently,

$$k_i > 0, \quad k_p > 0, \quad k_v k_p > k_i.$$ 

- For $z = (e_i, s, v)$ and
  
  $$\dot{e}_i = s$$
  
  $$\dot{s} = v$$
  
  $$\dot{v} \in -k_i e_i - k_p s - k_v v - F_s \text{SGN}(v)$$

  the set of equilibria making $\dot{z} = 0$ are $s = v = 0$ and $|e_i| \leq \frac{F_s}{k_i}$.

- Denote the corresponding set (it depends on $k_i!!$)

  $$\mathcal{A} := \left\{ (e_i, s, v) : \ s = 0, \ v = 0, \ e_i \in \left[ -\frac{F_s}{k_i}, \frac{F_s}{k_i} \right] \right\}.$$

Theorem C-GAS (Coulomb-GAS) Bisoffi et al. [2018]

With Coulomb friction, under Assumption LIN, set $\mathcal{A}$ is 1) globally attractive and 2) Lyapunov stable $\Leftrightarrow \exists \beta \in KL$ such that $|z(t)|_{\mathcal{A}} \leq \beta(|z(0)|_{\mathcal{A}}, t), \ \forall t \geq 0.$
Illustration by simulation is informative

\[
f_c = 1 \text{ m/s}^2
\]

\[
(k_v, k_p, k_i) = (6.4, 3, 4)
\]

→ complex conjugate roots

\[
\dot{z} \in \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-k_i & -k_p & -k_v
\end{bmatrix}
\begin{bmatrix}
e_i \\
s \\
v
\end{bmatrix} - \begin{bmatrix}
0 \\
0 \\
f_c
\end{bmatrix} \text{SGN}(v)
\]

\[
(k_v, k_p, k_i) = (1.5, 0.66, 0.08)
\]

→ three distinct real roots
Change of coordinates simplifies $A$

- Apply change of coordinates
  
  \[
  \sigma := -k_i s \\
  \phi := -k_i e_i - k_p s \quad \text{to} \quad \dot{z} := \begin{bmatrix} \dot{e}_i \\ \dot{s} \\ \dot{v} \end{bmatrix} \in \begin{bmatrix} 0 & 1 & 0 \\ 0 & -k_i & -k_p \\ -k_i & -k_p & -k_v \end{bmatrix} z - \begin{bmatrix} 0 \\ F_s \end{bmatrix} \text{SGN}(v)
  \]

\[ v := v \]

... and get dynamics

\[
\dot{x} := \begin{bmatrix} \dot{\sigma} \\ \dot{\phi} \\ \dot{v} \end{bmatrix} \in \begin{bmatrix} -k_i v \\ \sigma - k_p v \\ \phi - k_v v - F_s \text{SGN}(v) \end{bmatrix} = \begin{bmatrix} 0 & 0 & -k_i \\ 1 & 0 & -k_p \\ 0 & 1 & -k_v \end{bmatrix} \begin{bmatrix} \sigma \\ \phi \\ v \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ F_s \end{bmatrix} \text{SGN}(v)
\]

\[ = Ax - b \text{SGN}(v) =: F(x) \]

- Attractor (simpler expression independent of $k_i$)

  \[ A = \{ (\sigma, \phi, v) : |\phi| \leq F_s, \sigma = 0, v = 0 \} \]

- Distance to attractor

  \[ |x|_A^2 := \left( \inf_{y \in A} |x - y| \right)^2 = \sigma^2 + v^2 + dz_{F_s}(\phi)^2 \]
Lyapunov-like function is discontinuous!!

\[
V(x) := \left[ \begin{array}{c} \sigma \\ v \end{array} \right]^T \left[ \begin{array}{cc} \frac{k_v}{k_i} & -1 \\ -1 & k_p \end{array} \right] \left[ \begin{array}{c} \sigma \\ v \end{array} \right] + \min_{f \in F_s \text{ SGN}(v)} |\phi - f|^2
\]

\[
= \min_{f \in F_s \text{ SGN}(v)} \left[ \begin{array}{c} \sigma \\ \phi - f \end{array} \right]^T \left[ \begin{array}{ccc} \frac{k_v}{k_i} & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & k_p \end{array} \right] \left[ \begin{array}{c} \sigma \\ \phi - f \end{array} \right] = \min_{f \in F_s \text{ SGN}(v)} \left[ \begin{array}{c} \sigma \\ \phi - f \end{array} \right]^T P \left[ \begin{array}{c} \sigma \\ \phi - f \end{array} \right]
\]

- Immediate to check:
  - \( V(x) = 0 \text{ if and only if } x \in \mathcal{A} \)
  - \( V \) is not continuous
  for \( \{(\sigma_i, \phi_i, v_i)\}_{i=0}^{+\infty} = \{(0, 0, (\frac{1}{2})^i)\}_{i=0}^{+\infty} \), \( V \) converges to \( F_s^2 \) but \( V(0) = 0 \)
Properties of the Lyapunov-like function $V$

The Lyapunov-like function $V$ is:

1. **lower semicontinuous (lsc)**
   \[ V(\bar{x}) \leq \lim_{x \to \bar{x}} V(x), \quad \forall \bar{x} \in \mathbb{R}^3 \]  
   (Regularity)

2. **lower bounded**: There exist $c_1, c_2 > 0$ such that
   \[ c_1|x|_A^2 \leq V(x) \leq c_2|x|_A^2 + 2F_s^2 \quad \forall x \in \mathbb{R}^3 \]  
   (Sandwich)

3. **decreasing along trajectories**: $\exists c > 0$: for each solution $x = (\sigma, \phi, v)$,
   \[ \forall t_2 \geq t_1 \geq 0, \quad V(x(t_2)) - V(x(t_1)) \leq -c \int_{t_1}^{t_2} v(t)^2 dt. \]  
   (Flow)

• Proof of Theorem C-GAS given in Bisoffi et al. [2018] using:
  • auxiliary function and state partition for **stability**
  • Integral invariance principle of E.P. Ryan (1999) for **attractivity**
A closer look at the slow transients reveals promising ideas

- Solutions show long stick phases in the band $\mathcal{E}_{\text{stick}} := \{ x \in \mathbb{R}^3 : v = 0, |\phi| \leq F_s \}$

- Lyapunov function suggests reversing the sign of $\phi$ (reset to $-\phi$) when $\phi v \leq 0$

  \[
  V(x) := \begin{bmatrix} \sigma \\ v \end{bmatrix}^T \begin{bmatrix} k_v \\ k_i \\ -1 \end{bmatrix} \begin{bmatrix} \sigma \\ v \end{bmatrix} + \min_{f \in F_s \text{ SGN}(v)} |\phi - f|^2
  \]

- Solutions would then jump across the band $\mathcal{E}_{\text{stick}}$

- Time-regularized solutions (with timer $\tau$) imposes dwell time $t_{k+1} - t_k \geq \delta$
Reset PID Control Design Improves Coulomb Transient

- Overall state involves \( x = (\sigma, \phi, v) \) and \( \tau \in [0, 2\delta] \), that is \( (x, \tau) \in \mathbb{R}^3 \times [0, 2\delta] \)

- Hybrid closed loop with reset PID (no knowledge of \( F_s \) required)

\[
\begin{aligned}
\dot{x} &\in F(x) : = 
\begin{bmatrix}
0 & 0 & -k_i \\
1 & 0 & -k_p \\
0 & 1 & -k_v \\
\end{bmatrix}
\begin{bmatrix}
\sigma \\
\phi \\
v \\
\end{bmatrix} - 
\begin{bmatrix}
0 \\
0 \\
F_s \\
\end{bmatrix}
\text{SGN}(v), \\
\dot{\tau} & = 1 - dz(\tau/\delta) \\
x^+ & = g(x) : = 
\begin{bmatrix}
\sigma & -\alpha \phi & v \\
\end{bmatrix}^\top, \\
\tau^+ & = 0
\end{aligned}
\]

\( (x, \tau) \in C : = \mathbb{R}^3 \times [0, 2\delta] \setminus D \)

\( C \) and \( D \) are the flow and jump sets.

- Explanation of the jump set \( D \):
  - \( \phi \sigma \leq 0 \) so the solution is overshoting
  - \( \phi v \leq 0 \) so the Lyapunov function does not increase

- Parameter \( \alpha \in [0, 1] \) tunes robustness \((\alpha = 0)\) vs performance \((\alpha = 1)\)
A closer look at the slow transients (recall)

- Solutions show long stick phases in the band \( \mathcal{E}_{\text{stick}} := \{ x \in \mathbb{R}^3 : v = 0, |\phi| \leq F_s \} \)

- Lyapunov function suggests reversing the sign of \( \phi \) (reset to \( -\alpha \phi \)) when \( \phi v \leq 0 \)

\[
V(x) := \begin{bmatrix} \sigma \\ v \end{bmatrix}^T \begin{bmatrix} k_v \\ k_i \\ -1 \\ -1 \\ k_p \end{bmatrix} \begin{bmatrix} \sigma \\ v \end{bmatrix} + \min_{f \in F_s SGN(v)} |\phi - f|^2
\]

- Solutions would then jump across the band \( \mathcal{E}_{\text{stick}} \)

- Time-regularized solutions (with timer \( \tau \)) imposes dwell time \( t_{k+1} - t_k \geq \delta \)
Reset PID (with $\alpha = 1$) successfully jumps across $E_{\text{stick}}$

- Lyapunov decrease, and decrease of $|x|_A$ suggests exponential convergence

- Bad solutions sequence from $x_k(0) = (\epsilon_k, 0, 0)$ satisfy:
  $$|x_k(t)|_A = |x_k(0)|_A = \epsilon_k \text{ for all } t \leq T_k,$$
  with $\lim_{k \to \infty} \epsilon_k = 0$ and $\lim_{k \to \infty} T_k = +\infty$.
  thus disproving exponential convergence

- However exponential convergence seems to often occur
The same Lyapunov function helps in the reset context

Recall the Lyapunov-like function:

\[ V(x) := \begin{bmatrix} \sigma \\ \nu \end{bmatrix}^T \begin{bmatrix} k_v \\ -1 \\ k_p \end{bmatrix} \begin{bmatrix} \sigma \\ \nu \end{bmatrix} + \min_{f \in F_s \text{SGN(} \nu \text{)}} |\phi - f|^2 \]

### Properties of \( V \) carry over from non-hybrid case

Function \( V \) is **lower semicontinuous** and there exist \( c_1, c_2, c > 0 \) such that:

\begin{align*}
    c_1 |x|_A^2 &\leq V(x) \leq c_2 |x|_A^2 + 2F_s^2 & \forall x \in \mathbb{R}^3 \\
    V(x(t_2,j)) - V(x(t_1,j)) &\leq -c \int_{t_1}^{t_2} \nu(t,j)^2 dt, & \forall \text{ solution } (x, \tau) \\
    V(g(x)) - V(x) &\leq 0, & \forall (x, \tau) \in D
\end{align*}

**Theorem RC-GAS (Reset-Coulomb-GAS)** Beerens et al. [2019]

With Coulomb friction, under Assumption LIN, set \( A \) is \( KL \)-GAS.

- **Stability proof**: same as before using extra (Jump) condition
- **Global attractivity proof**: uses meagre-limsup hybrid invariance principle
Experimental response confirms transient improvement

Beerens et al. [2019]
Without resets, alternative hybrid automaton model

Bisoffi et al. [2019]

- Extended state \( \bar{x} \) includes timer \( \bar{\tau} \) and logic variable \( \bar{q} \) such that \( \bar{q}\bar{v} \geq 0 \)
  \[ \bar{x} := (\bar{\sigma}, \bar{\phi}, \bar{v}, \bar{q}, \bar{\tau}) \in \bar{\Xi} := \{\bar{x} \in \mathbb{R}^3 \times \{-1, 0, 1\} \times [0, 2\delta] \mid \bar{q}\bar{v} \geq 0\}, \]

- Hybrid automaton \( \mathcal{H}_\delta \) (Coulomb, no resets) – semiglobally correct
  \[ \mathcal{H}_\delta : \begin{cases} \dot{\bar{x}} = \bar{f}(\bar{x}), & \bar{x} \in \bar{C} \\ \bar{x}^+ = \bar{g}(\bar{x}), & \bar{x} \in \bar{D} \end{cases} \]
  \[ \bar{C} := \bar{C}_{\text{slip}} \cup \bar{C}_{\text{stick}} \]
  \[ \bar{D} := \bar{D}_{-1} \cup \bar{D}_0 \cup \bar{D}_1 \]

- Smooth Lyapunov function certifies GAS of \( \bar{A} := \{\bar{x} : \bar{\sigma} = \bar{v} = 0, \bar{\phi} \in F_s \text{SGN}(\bar{q})\} \)
  \[ \bar{V}(\bar{x}) := \begin{bmatrix} \bar{\sigma} \\ \bar{v} \end{bmatrix}^T \begin{bmatrix} k_v & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \bar{\sigma} \\ \bar{v} \end{bmatrix} + |\bar{q}|(\bar{\phi} - \bar{q}F_s)^2 + (1 - |\bar{q}|)dz^2_{F_s}(\bar{\phi}). \]
Alternative proof of Theorem C-GAS uses function $\tilde{V}$
Bisoffi et al. [2019]

- Same exact evolution for $V$ (along original sol’ns $x$) and $\tilde{V}$ (along sol’ns $\tilde{x}$ to $H_\delta$)

Properties of smooth $\tilde{V}$ are convenient
- Function $\tilde{V}$ is smooth and there exist $\alpha_1, \alpha_2 \in K_\infty$, $c > 0$ such that:
  \[
  \alpha_1(|\tilde{x}|_{\mathcal{A}}) \leq \tilde{V}(\tilde{x}) \leq \alpha_2(|\tilde{x}|_{\mathcal{A}}) \quad \forall \tilde{x} \in \bar{\Xi} \quad \text{(Sandwich)}
  \]
  \[
  \langle \nabla \tilde{V}(\tilde{x}), \bar{f}(\tilde{x}) \rangle = -cv^2, \quad \forall \tilde{x} \in C_{\text{slip}} \cup C_{\text{stick}} \quad \text{(Flow)}
  \]
  \[
  \tilde{V}(\bar{g}_i(\tilde{x})) - \tilde{V}(\tilde{x}) \leq 0, \quad \forall \tilde{x} \in D_i, i \in \{1, -1, 0\} \quad \text{(Jump)}
  \]

- Alternative proof of Theorem C-GAS given in Bisoffi et al. [2019] using:
  - auxiliary function and state partition for stability
  - proof of attractivity using the following arguments
    - Original solutions $x$ are uniformly bounded
    - Solutions $\tilde{x}$ of hybrid automaton $H_\delta$ semiglobally reproduces any original solution $x$ evolving in a compact set $\mathcal{K}(\delta)$ such that $\lim_{\delta \to 0} \mathcal{K}(\delta) = \mathbb{R}^3$
      - Smooth Lyapunov function $\tilde{V}$ certifies global attractivity for $H_\delta$
      - Attract. for $H_\delta \Rightarrow$ semiglobal Attract. $\Rightarrow$ Attract. of the original system
  - Interesting connections with (bi)simulation concepts found in computer science
With resets, extended automaton includes extra variable

- Extended state \( \bar{x} := (\bar{\sigma}, \bar{\phi}, \bar{v}, \bar{q}, \bar{\tau}, \bar{a}) \in \bar{\Xi} \) with logic variable \( \bar{a} \) such that \( \bar{a}\bar{v} \geq 0 \)

\[
\bar{\Xi} := \{ \bar{x} \in \mathbb{R}^3 \times \{-1, 0, 1\} \times [0, 2\delta] \times \{-1, 1\} \mid \bar{q}\bar{v} \geq 0, \bar{a}\bar{\phi} \geq 0, \bar{a}\bar{v} \geq 0 \},
\]

- Hybrid automaton for overshooting solutions, wherein \( \bar{\phi}\bar{v} \geq 0 \), then \( \bar{a} = \text{sign}(\phi) \)

\[
\mathcal{H}_\delta : \begin{cases} 
\dot{\bar{x}} = \bar{f}(\bar{x}), & \bar{x} \in \bar{C} \\
\bar{x}^+ = \bar{g}(\bar{x}), & \bar{x} \in \bar{D}
\end{cases}
\]

\( \bar{C} := \bar{C}_{\text{slip}} \cup \bar{C}_{\text{stick}} \)

\( \bar{D} := D_{-1} \cup D_0 \cup D_1 \)

\( \bar{C}_{\text{slip}} := \{ \bar{x} \in \bar{\Xi} : |\bar{q}| = 1 \} \)

\( \bar{C}_{\text{stick}} := \{ \bar{x} \in \bar{\Xi} : \bar{q} = 0, \bar{v} = 0, \bar{a}\bar{\phi} \leq F_s \} \)

\( D_1 := \{ \bar{x} \in \bar{\Xi} : \bar{q} = 0, \bar{v} = 0, \bar{a}\bar{\phi} \geq F_s, \bar{\tau} \in [\delta, 2\delta] \} \)

\( D_{-1} := \{ \bar{x} \in \bar{\Xi} : \bar{q} = 0, \bar{v} = 0, \bar{a}\bar{\phi} \geq F_s, \bar{\tau} \in [\delta, 2\delta] \} \)

\( D_0 := \{ \bar{x} \in \bar{\Xi} : |\bar{q}| = 1, \bar{v} = 0, \bar{a}\bar{\phi} \leq F_s \}, \)
Homogeneous automaton explains exponential convergence

- State transformation provides homogeneous hybrid dynamics

\[ \hat{x} := (\dot{\sigma}, \dot{\phi}, \dot{v}, \ddot{q}, \ddot{\tau}, \dddot{a}) \quad \mapsto \quad \hat{\hat{x}} := (\dot{\hat{\sigma}}, \dot{\hat{\phi}}, \dot{\hat{v}}, \ddot{\hat{q}}, \ddot{\hat{\tau}}, \dddot{\hat{a}}) = (\sigma, \phi - \ddot{a} F_s, \dot{v}, q, \tau, \dddot{a}). \]

- With \( \alpha = 1 \), denoting \( \hat{x}_0 = (\hat{\sigma}, \hat{\phi}, \hat{v}) \), we get partial homogeneity in \( \hat{x}_0 \)

\[
\begin{align*}
\hat{x}_0 &= A_F(\hat{q}, \hat{a}) \hat{x}_0, \quad \hat{x} \in \hat{C} \\
\hat{x}_0^+ &= A_J(\hat{q}, \hat{a}) \hat{x}_0, \quad \hat{x} \in \hat{D}
\end{align*}
\]

\[ \hat{C} := \hat{C}_{\text{slip}} \cup \hat{C}_{\text{stick}} \]

\[ \hat{D} := \hat{D}_{-1} \cup \hat{D}_0 \cup \hat{D}_1 \]

- Exploiting \( \dddot{a} = \text{sign}(\dddot{\phi}) \), with \( \alpha = 1 \) we can prove \( \exists M > 0, \mu > 0 \) satisfying

\[ |(\sigma, \phi - \text{sign}(\phi) F_s, v)| \leq M e^{-\mu t} |\sigma_0| \]

for all solutions starting at stick-to-slip transition \( \hat{x}_0 = (\sigma_0, 0, 0) \).
Streibeck model includes extra nonlinearity

- New velocity weakening function $\psi$ (previously zero):

$$
\begin{bmatrix}
\dot{\sigma} \\
\dot{\phi} \\
\dot{v}
\end{bmatrix}
=
\begin{bmatrix}
0 & 0 & -k_i \\
1 & 0 & -k_p \\
0 & 1 & -k_v
\end{bmatrix}
\begin{bmatrix}
\sigma \\
\phi \\
v
\end{bmatrix}
-
\begin{bmatrix}
0 \\
0 \\
F_s
\end{bmatrix}
(SGN(v) + \psi(v)),
$$

Assumption STRIB

Assumption LIN holds ($k_i > 0$, $k_p > 0$, $k_v k_p > k_i$). Moreover, the velocity weakening function $\psi$ is globally Lipschitz and satisfies

- $|\psi(v)| \leq F_s$
- $v \psi(v) \geq 0$ for all $v$
- it is linear in a small enough interval around zero (namely, for some $\varepsilon_v$, $|v| \leq \varepsilon_v \Rightarrow \psi(v) = L_2 v$).
Stribeck “hunting instability” needs a different solution

- Reset PID solution solving Coulomb is not successful for Stribeck hunting effect
A two-stage reset PID proposed in Beerens et al. [2020]

- Add boolean state $b \in \{-1, 1\}$ such that $bv\sigma \geq 0$:
  1) $b = 1$ in the overshooting phase $v\sigma \geq 0$
  2) $b = -1$ in the approaching phase $v\sigma \leq 0$

- Ensure that the integral action $e_i$ points in the direction of position error $s$
  This corresponds to imposing $\phi\sigma \geq \frac{k_p}{k_i}\sigma^2$

- Overall state $\xi := (x, b) := (\sigma, \phi, v, b)$ evolves in $\Xi$, where
  $$\Xi := \{(x, b) \in \mathbb{R}^3 \times \{-1, 1\} : bv\sigma \geq 0, \sigma\phi \geq \frac{k_p}{k_i}\sigma^2, bv\phi \geq 0\}.$$  

- Jumps at zero-crossing of $\sigma$ and $v$, wherein state $b$ alternates between $-1$ and 1
  $$\begin{bmatrix} \sigma^+ \\ \phi^+ \\ v^+ \\ b^+ \end{bmatrix} = g_{\sigma}(\xi) := \begin{bmatrix} \frac{\sigma}{v} \\ -\phi \\ -v \\ -b \end{bmatrix}, \quad \xi \in \mathcal{D}_{\sigma} := \{\xi \in \Xi : \sigma = 0, b = 1\}$$
  $$\begin{bmatrix} \sigma^+ \\ \phi^+ \\ v^+ \\ b^+ \end{bmatrix} = g_{v}(\xi) := \begin{bmatrix} \frac{k_p}{k_i}\sigma \\ \frac{\sigma}{\phi} \\ \frac{v}{\phi} \\ -b \end{bmatrix}, \quad \xi \in \mathcal{D}_{v} := \{\xi \in \Xi : v = 0, b = -1\}.$$  

- Can prove that $\phi$ is never zero along sol’ns, so $\mathcal{D}_{\sigma}$ and $\mathcal{D}_v$ robustly implemented as
  $$\mathcal{D}_{\sigma}^r := \{\xi : \sigma\phi \leq 0, b = 1\}, \quad \mathcal{D}_v^r := \{\xi : v\phi \geq 0, b = -1\}.$$
The reset closed-loop eliminates the hunting instability

Theorem STRI-GAS (Reset-Stribeck-GAS) Beerens et al. [2020]:

Under Assumption STRIB, the compact set

\[ A_e := A \times \{-1, 1\} = \{\xi \in \Xi: \sigma = v = 0, |\phi| \leq F_s\}. \]

is $\mathcal{KL}$-globally asymptotically stable.

- The proof of Theorem STRI-GAS requires using the hybrid automaton trick (a new automaton, a new “smooth” Lyapunov function).
Stribeck hybrid automaton is more sophisticated
Beerens et al. [2020]

- Extended state $\bar{\xi}$ includes timer $\bar{\tau}$ and logic variable $\bar{q}$ such that $\bar{q}\bar{v} \geq 0$

  $\bar{\xi} := (\bar{\sigma}, \bar{\phi}, \bar{v}, \bar{b}, \bar{q}, \bar{\tau}) \in \bar{\Xi} := \{ \bar{\xi} | \bar{q}\bar{v} \geq 0, \bar{b}\bar{q}\bar{\sigma} \geq 0, \bar{\sigma}\bar{\phi} \geq \frac{k_p}{k_i} \bar{\sigma}^2, \bar{b}\bar{q}\bar{\phi} \geq 0 \}$.

- Hybrid automaton $\mathcal{H}_\delta$ (Stribeck, resets) – semiglobally correct

  $\mathcal{H}_\delta : \begin{cases} \dot{\bar{\xi}} = \bar{f}(\bar{\xi}), & \bar{\xi} \in \bar{\mathcal{C}} \\ \bar{\xi}^+ = \bar{g}(\bar{\xi}), & \bar{\xi} \in \bar{\mathcal{D}} \end{cases}$

  $\bar{\mathcal{C}} := \mathcal{C}_{\text{slip}} \cup \mathcal{C}_{\text{stick}}$

  $\bar{\mathcal{D}} := \mathcal{D}_- \cup \mathcal{D}_0 \cup \mathcal{D}_1 \cup \mathcal{D}_\sigma \cup \mathcal{D}_v$

- Lipschitz Lyapunov function shows GAS of $\bar{\mathcal{A}}_e := \{ \bar{\xi} | \bar{\sigma} = \bar{\nu} = 0, \bar{\phi} \in F_s \text{SGN}(\bar{b}\bar{q}) \}$

  $\bar{V}_e(\bar{\xi}) := \begin{bmatrix} \bar{\sigma} \\ \bar{\nu} \end{bmatrix}^\top \begin{bmatrix} \frac{k_v}{k_i} & -1 \\ -1 & k_p \end{bmatrix} \begin{bmatrix} \bar{\sigma} \\ \bar{\nu} \end{bmatrix} + |\bar{q}|(\bar{\phi} - \bar{b}\bar{q}F_s)^2 + (1 - |\bar{q}|)dz^2_{F_s}(\bar{\phi})$

  $+ 2\frac{k_p}{k_i} F_s(\bar{b}\bar{q}\bar{\sigma} + (1 - |\bar{q}|)|\bar{\sigma}|)$
Proof of Theorem STRI-GAS uses function $\bar{V}_e$

- Function $\bar{V}_e$ is not smooth but Lipschitz $\Rightarrow$ can use Clarke nonsmooth tools

**Properties of Lipschitz $\bar{V}_e$ are convenient**

Function $\bar{V}_e$ is **Lipschitz** and there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $c > 0$ such that:

$$
\alpha_1(|\bar{\xi}|_{\bar{A}_e}) \leq \bar{V}_e(\bar{\xi}) \leq \alpha_2(|\bar{\xi}|_{\bar{A}_e}) \quad \forall \bar{\xi} \in \bar{\Xi} \quad \text{(Sandwich)}
$$

$$
\bar{V}_e^\circ(\bar{\xi}) := \max_{\nu \in \partial \bar{V}_e(\bar{\xi})} \langle \nu, \bar{f}(\bar{\xi}) \rangle \leq -c\bar{v}^2, \quad \forall \bar{x} \in C_{\text{slip}} \cup C_{\text{stick}} \quad \text{(Flow)}
$$

$$
\bar{V}_e(\bar{g}_i(\bar{\xi})) - \bar{V}_e(\bar{\xi}) \leq 0, \quad \forall \bar{\xi} \in \mathcal{D}_i, i \in \{1, -1, 0, \sigma, \nu\} \quad \text{(Jump)}
$$

- Proof of Theorem STRI-GAS given in Beerens et al. [2020] using:
  - proof of **uniform global attractiveness (UGA)** using the following arguments
    - Original solutions $x$ are uniformly bounded (not as trivial as with Coulomb)
    - Solutions $\bar{\xi}$ of hybrid automaton $\mathcal{H}_\delta$ semiglobally reproduces any original solution $\xi$ evolving in a compact set $\mathcal{K}(\delta)$ such that $\lim_{\delta \to 0} \mathcal{K}(\delta) = \mathbb{R}^3$
  - Lipschitz Lyapunov function $\bar{V}_e$ certifies UGA for $\mathcal{H}_\delta$
  - UGA for $\mathcal{H}_\delta$ $\Rightarrow$ semiglobal UA $\Rightarrow$ UGA of the original system
  - UGA and strong forward invariance of $\bar{A}_e$ implies **stability**.

- Interesting connections with (bi)simulation concepts found in computer science
Experimental response confirms GAS recovery

Beerens et al. [2020]
Wrap up and acknowledgements

Conclusions:

• Differential inclusion model for PID controlling sliding mass with Coulomb/Striбeck friction effects
• Coulomb: Lyapunov-based proof of Global Asymptotic (not exponential) Stability
• Coulomb: Reset PID improves transient response (exponential convergence)
• Striбeck: Reset PID resolves “hunting” instability
• The presented results in a recently published vision article (IFAC Annual Reviews in Control) Bisoffi et al. [2020]

Future Work:

• Combine resets for exponential convergence and Striбeck
• Address the case of asymmetric friction laws

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