To stick or to slip: Lyapunov-based reset PID for positioning systems with Coulomb and Stribeck friction

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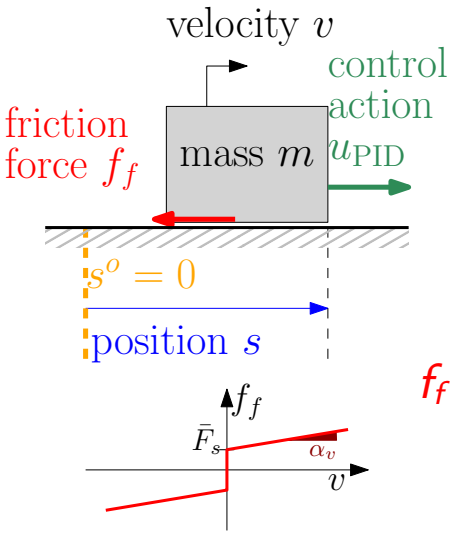
Outline

Coulomb friction and discontinuous right-hand side



$$u_{\text{PID}}(t) := -\bar{k}_p s(t) - \bar{k}_i \int_0^t s(\tau)d\tau - \bar{k}_d \frac{ds(t)}{dt}$$
$$= -\bar{k}_p s(t) - \bar{k}_i e_i(t) - \bar{k}_d v(t),$$

$$f_f(u_{\text{PID}}, v) := \begin{cases} \bar{F}_s \operatorname{sign}(v) + \alpha_v v, & \text{if } v \neq 0 \\ u_{\text{PID}}, & \text{if } v = 0,\ |u_{\text{PID}}| < \bar{F}_s \\ \bar{F}_s \operatorname{sign}(u_{\text{PID}}), & \text{if } v = 0,\ |u_{\text{PID}}| \geq \bar{F}_s \end{cases}$$

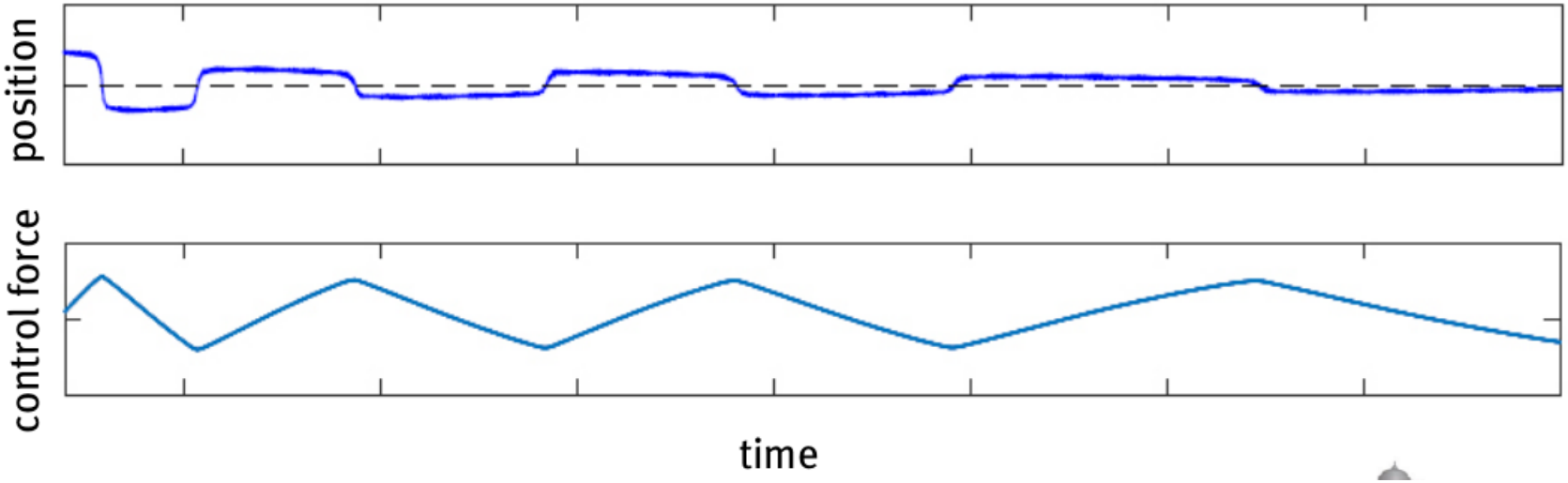
$$m\dot{v} = u_{\text{PID}} - f_f(u_{\text{PID}}, v)$$

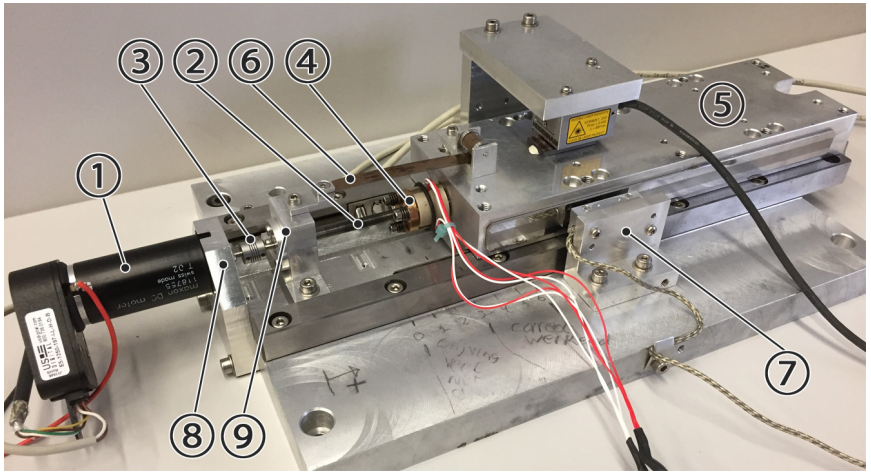
- ▷ PID action and viscous force combined in $u := \frac{u_{\text{PID}} - \alpha_v v}{m}$ $\quad u_{\text{PID}} = m\,u$ for $v = 0$
- ▷ normalize physical param's $\bar{k}_p$, $\bar{k}_i$, $\bar{k}_d$, $\bar{F}_s$ as $(k_p, k_v, k_i) := \left(\frac{\bar{k}_p}{m}, \frac{\bar{k}_d + \alpha_v}{m}, \frac{\bar{k}_i}{m}\right)$, $F_s := \frac{\bar{F}_s}{m}$

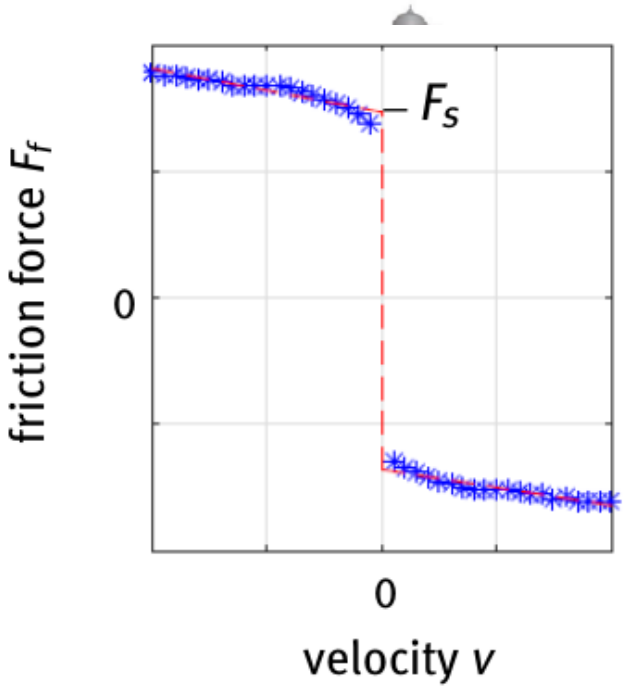
$$\dot{e}_i = s$$
$$\dot{s} = v$$
$$\dot{v} = \begin{cases} u - F_s & \text{if } v > 0 \text{ or } (v = 0, u \geq F_s) \\ 0 & \text{if } (v = 0, |u| < F_s) \\ u + F_s & \text{if } v < 0 \text{ or } (v = 0, u \leq -F_s) \end{cases}$$
$$u = -k_p s - k_v v - k_i e_i,$$

The problem is industrially relevant with Coulomb effect

Industrial High-precision motion control system (electron microscope) experiments:



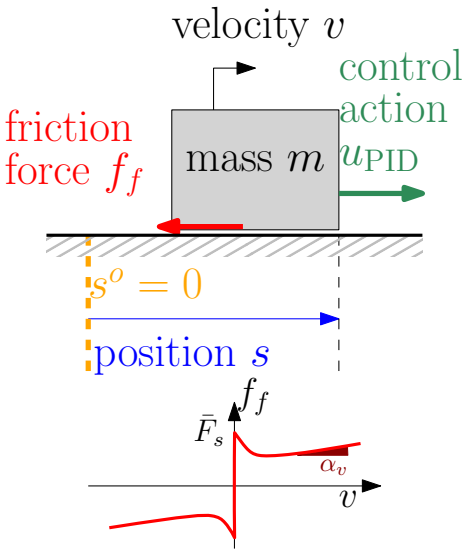


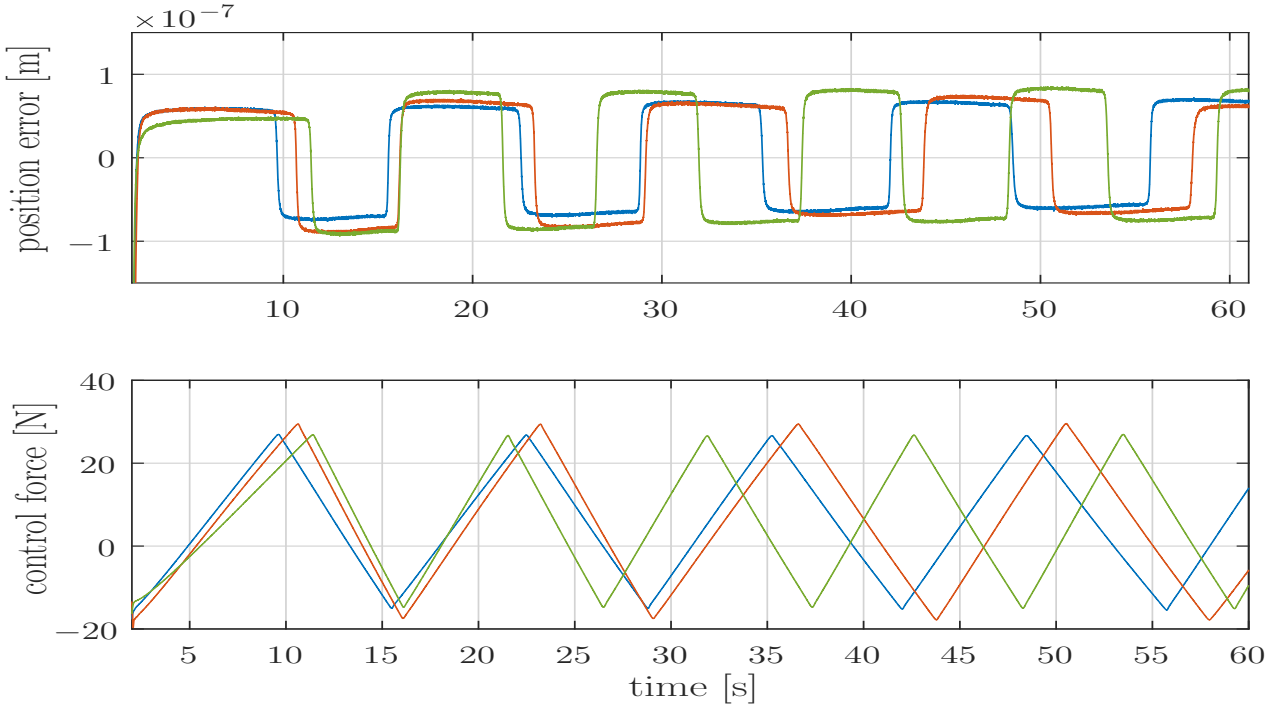


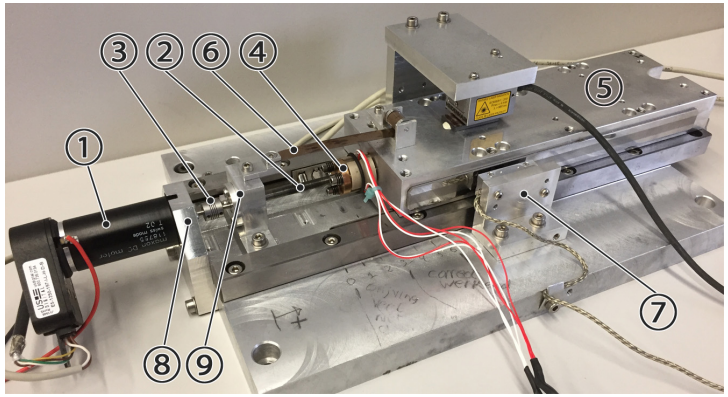
Measured friction nonlinearity points

Experiments show instability with Stribeck effect

- Stribeck effect causes "hunting" instability with PID feedback







Same experimental device shows Stribeck with different ambient, lubrication and wear conditions

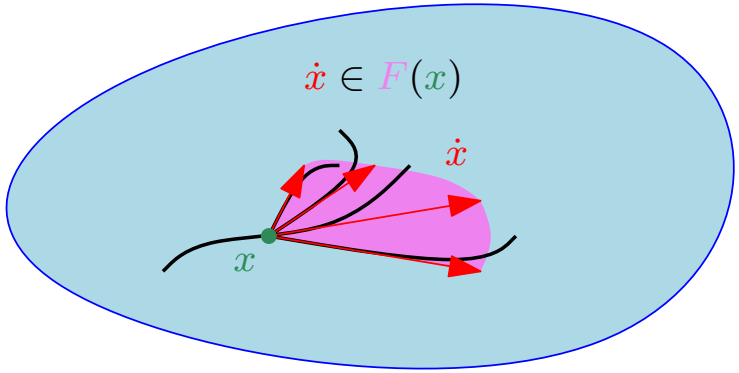
Reformulation as a suitable differential inclusion

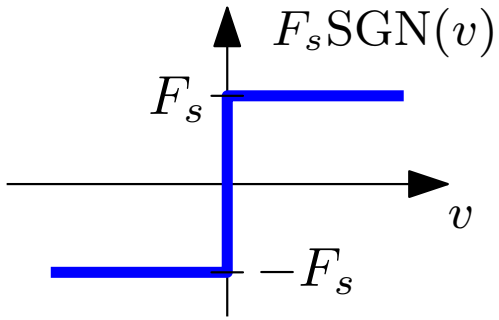
$$\dot{e}_i = s$$
$$\dot{s} = v$$
$$u = -k_i e_i - k_p s - k_v v,$$
$$u - \dot{v} = \begin{cases} +F_s & \text{if } v > 0 \\ \text{sat}_{F_s}(u) & \text{if } v = 0 \\ -F_s & \text{if } v < 0 \end{cases}$$

$$\Downarrow$$

$$\dot{e}_i = s$$
$$\dot{s} = v$$
$$\dot{v} \in \underbrace{-k_i e_i - k_p s - k_v v}_{u=} - F_s \text{SGN}(v)$$

differential inclusions in general





- Physical model: intuitive, but hard to prove existence of solutions and stability properties with a discontinuous right hand side
- Differential inclusion: existence of solutions and *ad hoc* Lyapunov tools

Lemma BASIC (solutions are unique and complete)

For any initial condition $z(0) = (e_i(0), s(0), v(0)) \in \mathbb{R}^3$, the green differential inclusion has a unique solution defined for all $t \geq 0$.

A partial literature overview

The interest in dynamics with friction had its peak in the 1990's.

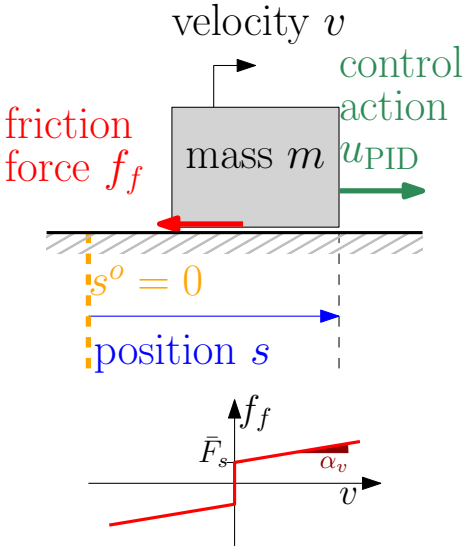
- modeling direction
  - Dahl model:
    P. R. Dahl, A solid friction model. Tech. Rep. of The Aerospace Corporation El Segundo CA, 1968.
  - models by Bliman and Sorine:
    P.-A. Bliman and M. Sorine, *Easy-to-use realistic dry friction models for automatic control*. Proc. of 3rd European Control Conf., 1995.
  - LuGre model:
    C. Canudas-de-Wit, H. Olsson, K. J. Åström, and P. Lischinsky, *A new model for control of systems with friction*. IEEE Trans. Autom. Control, 1995.
    K. J. Åström and C. Canudas-de-Wit, *Revisiting the LuGre friction model*. Control Systems, IEEE, 2008.
    N. Barabanov and R. Ortega, *Necessary and sufficient conditions for passivity of the LuGre friction model*. IEEE Trans. Autom. Control, 2000.
  - Leuven model:
    J. Swevers, F. Al-Bender, C. G. Ganseman, and T. Projogo, *An integrated friction model structure with improved presliding behavior for accurate friction compensation*. IEEE Trans. Autom. Control, 2000.

Set-valued friction and PID control

- use of set-valued mapping for the friction force, and hence differential inclusions
  - uncontrolled multi-degree-of-freedom mechanical systems:
    N. van de Wouw and R. I. Leine, *Attractivity of equilibrium sets of systems with dry friction*. Nonlinear Dynamics, 2004.
  - PD controlled 1 d.o.f. system:
    D. Putra, H. Nijmeijer, and N. van de Wouw, *Analysis of undercompensation and overcompensation of friction in 1 DOF mechanical systems*. Automatica, 2007.
  - combination of set-valued friction laws and Lyapunov tools:
    R. I. Leine and N. van de Wouw, *Stability and convergence of mechanical systems with unilateral constraints*. Springer Science & Business Media, 2007.
  - stability of compact attractors
    V.A. Yakubovich, G.A. Leonov, and A.K. Gelig, *Stability of Stationary Sets in Control Systems with Discontinuous Nonlinearities*, World Scientific, 2004.
- for the same setting (point mass + PID controller), with Coulomb and viscous friction only it was proven that no stick-slip limit cycle (so-called hunting) exist:
  - B. Armstrong-Hélouvry and B. Amin, *PID control in the presence of static friction: exact and describing function analysis*. Amer. Control Conf., 1994.
  - B. Armstrong and B. Amin, *PID control in the presence of static friction: A comparison of algebraic and describing function analysis*. Automatica, 1996.

Coulomb-only friction provides an initially simplified setting

- Coulomb friction experience suggests (slow) convergence and stability



State equations with $z = (e_i, s, v)$ are

$$\begin{bmatrix} \dot{e}_i \\ \dot{s} \\ \dot{v} \end{bmatrix} \in \begin{bmatrix} s \\ v \\ -k_i e_i - k_p s - k_v v - F_s \mathrm{SGN}(v) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k_i & -k_p & -k_v \end{bmatrix} z - \begin{bmatrix} 0 \\ 0 \\ F_s \end{bmatrix} \mathrm{SGN}(v)$$

- Standing assumption about the PID gains is probably necessary for GAS

Assumption LIN

In the absence of friction ($F_s = 0$), the origin is globally asymptotically stable (GAS). Equivalently,

$$k_i > 0,\ k_p > 0,\ k_v k_p > k_i.$$

With Coulomb Friction the largest set of equilibria is GAS

Assumption LIN

In the absence of friction ($F_s = 0$), the origin is globally asymptotically stable (GAS). Equivalently,

$$k_i > 0,\ k_p > 0,\ k_v k_p > k_i.$$

- For $z = (e_i, s, v)$ and

$$\begin{aligned} \dot{e}_i &= s \\ \dot{s} &= v \\ \dot{v} &\in -k_i e_i - k_p s - k_v v - F_s \mathrm{SGN}(v) \end{aligned}$$

  the set of equilibria making $\dot{z} = 0$ are $s = v = 0$ and $|e_i| \leq \frac{F_s}{k_i}$.
- Denote the corresponding set (it depends on k_i!!)

$$\mathcal{A} := \left\{ (e_i, s, v) : \ s = 0,\ v = 0, e_i \in \left[-\frac{F_s}{k_i}, \frac{F_s}{k_i} \right] \right\}.$$

Theorem C-GAS (Coulomb-GAS) Bisoffi et al. [2018]

With Coulomb friction, under Assumption LIN, set $\mathcal{A}$ is 1) globally attractive and 2) Lyapunov stable $\Leftrightarrow \exists \beta \in \mathcal{KL}$ such that $|z(t)|_{\mathcal{A}} \leq \beta(|z(0)|_{\mathcal{A}}, t)$, $\forall t \geq 0$.

Illustration by simulation is informative

▷ $f_c = 1\ \mathrm{m/s^2}$

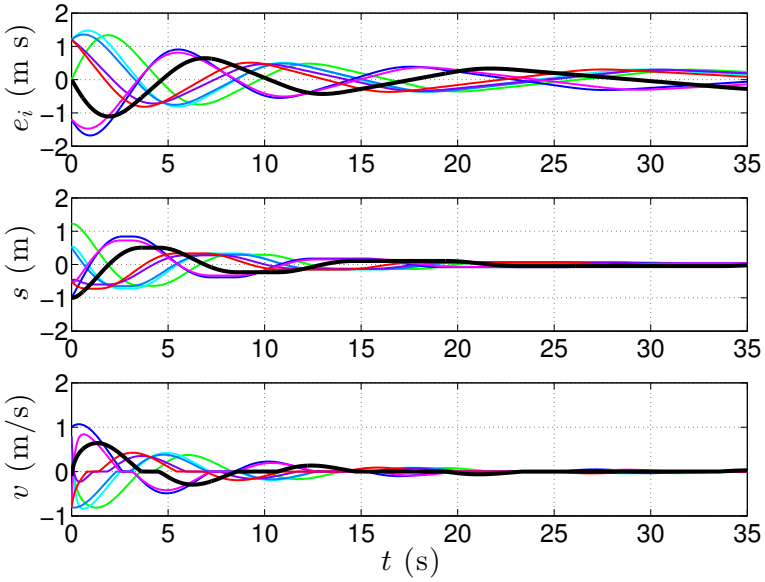
$(k_v, k_p, k_i) = (6.4, 3, 4)$

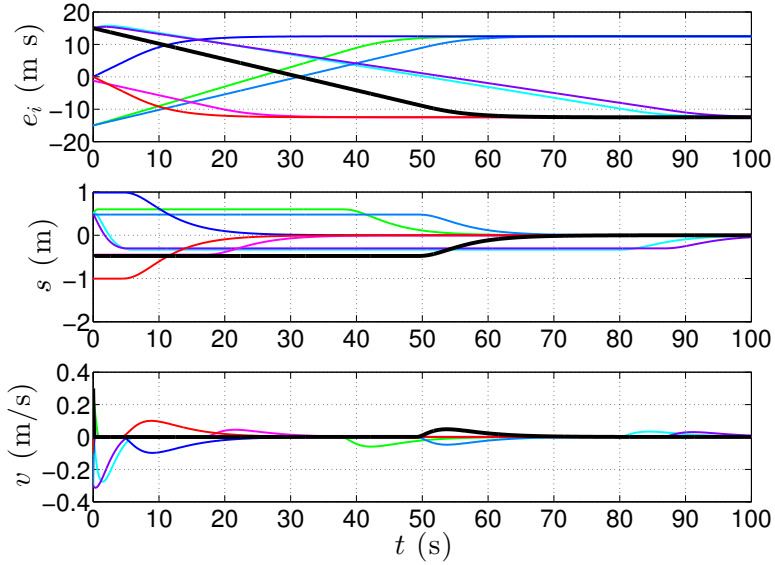
➔ complex conjugate roots

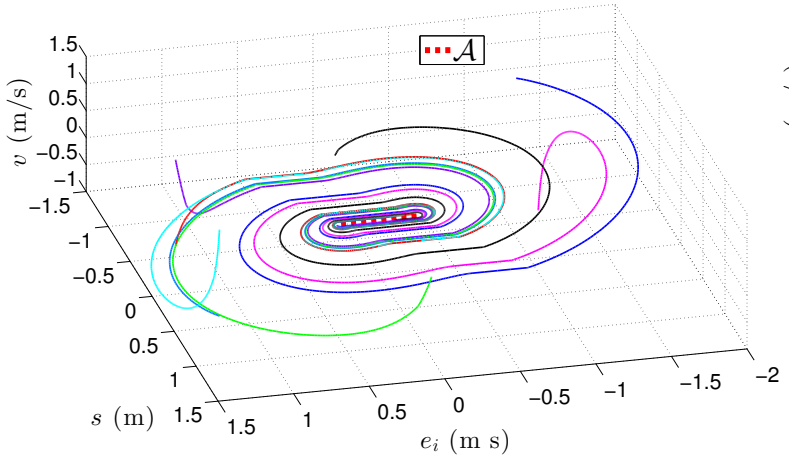
$$\dot{z} \in \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k_i & -k_p & -k_v \end{bmatrix} \begin{bmatrix} e_i \\ s \\ v \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ f_c \end{bmatrix} \mathrm{SGN}(v)$$

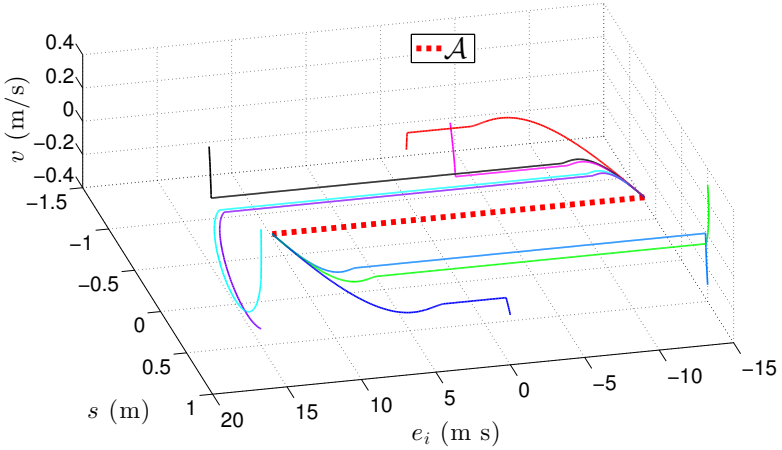
$(k_v, k_p, k_i) = (1.5, 0.66, 0.08)$

➔ three distinct real roots









Change of coordinates simplifies $\mathcal{A}$

- Apply change of coordinates

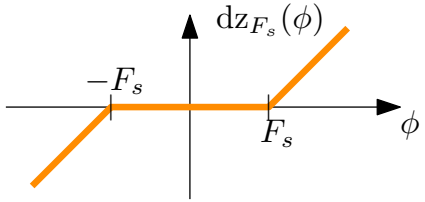
$$\sigma := -k_i s$$
$$\phi := -k_i e_i - k_p s \quad \text{to} \quad \dot{z} := \begin{bmatrix} \dot{e}_i \\ \dot{s} \\ \dot{v} \end{bmatrix} \in \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k_i & -k_p & -k_v \end{bmatrix} z - \begin{bmatrix} 0 \\ 0 \\ F_s \end{bmatrix} \mathrm{SGN}(v)$$
$$v := v$$

- ...and get dynamics

$$\dot{x} := \begin{bmatrix} \dot{\sigma} \\ \dot{\phi} \\ \dot{v} \end{bmatrix} \in \begin{bmatrix} -k_i v \\ \sigma - k_p v \\ \phi - k_v v - F_s \mathrm{SGN}(v) \end{bmatrix} = \begin{bmatrix} 0 & 0 & -k_i \\ 1 & 0 & -k_p \\ 0 & 1 & -k_v \end{bmatrix} \begin{bmatrix} \sigma \\ \phi \\ v \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ F_s \end{bmatrix} \mathrm{SGN}(v)$$
$$= Ax - b\,\mathrm{SGN}(v) =: F(x)$$

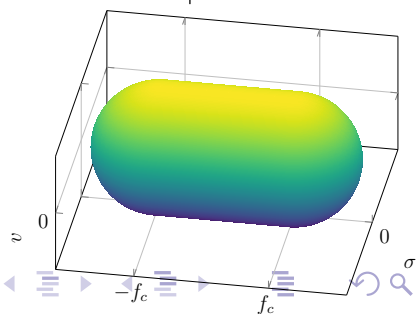
- Attractor (simpler expression independent of k_i)

$$\mathcal{A} = \{(\sigma, \phi, v) \colon |\phi| \leq F_s, \sigma = 0, v = 0\}$$



- Distance to attractor

$$|x|_{\mathcal{A}}{}^2 := \left(\inf_{y \in \mathcal{A}} |x - y|\right)^2 = \sigma^2 + v^2 + \mathrm{dz}_{F_s}(\phi)^2$$

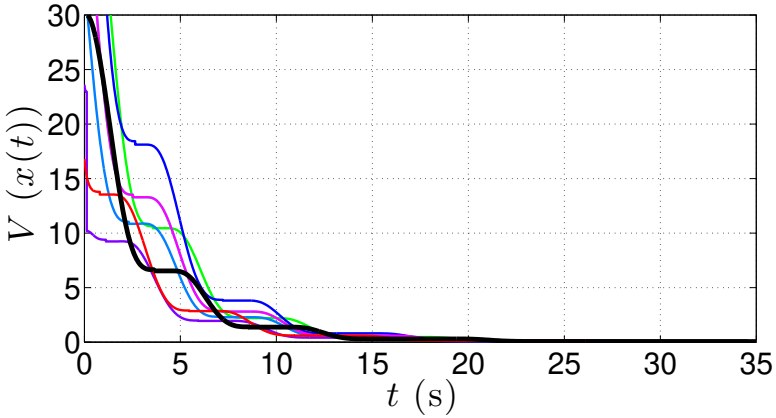


Lyapunov-like function is discontinuous!!

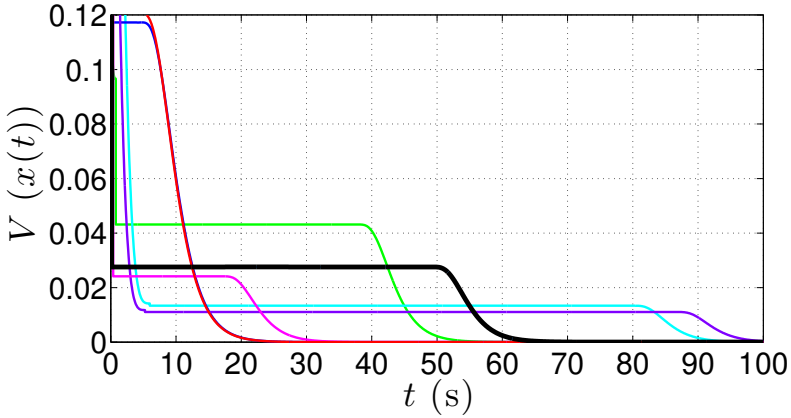
$$V(x) := \begin{bmatrix}\sigma\\ v\end{bmatrix}^T \begin{bmatrix}\frac{k_v}{k_i} & -1\\ -1 & k_p\end{bmatrix}\begin{bmatrix}\sigma\\ v\end{bmatrix} + \min_{f\in F_s\,\mathrm{SGN}(v)} |\phi - f|^2$$

$$= \min_{f\in F_s\,\mathrm{SGN}(v)} \begin{bmatrix}\sigma\\ \phi-f\\ v\end{bmatrix}^T \begin{bmatrix}\frac{k_v}{k_i} & 0 & -1\\ 0 & 1 & 0\\ -1 & 0 & k_p\end{bmatrix}\begin{bmatrix}\sigma\\ \phi-f\\ v\end{bmatrix} = \min_{f\in F_s\,\mathrm{SGN}(v)} \begin{bmatrix}\sigma\\ \phi-f\\ v\end{bmatrix}^T P \begin{bmatrix}\sigma\\ \phi-f\\ v\end{bmatrix}$$

complex conjugate roots



three distinct real roots



- Immediate to check:
  - $V(x) = 0$ if and only if $x \in \mathcal{A}$
  - V is not continuous

for $\{(\sigma_i, \phi_i, v_i)\}_{i=0}^{+\infty} = \{(0, 0, (\frac{1}{2})^i\}_{i=0}^{+\infty}$, V converges to F_s^2 but $V(0) = 0$

Properties of the Lyapunov-like function V

$$V(x) := \begin{bmatrix} \sigma \\ v \end{bmatrix}^T \begin{bmatrix} \frac{k_v}{k_i} & -1 \\ -1 & k_p \end{bmatrix} \begin{bmatrix} \sigma \\ v \end{bmatrix} + \min_{f \in F_s \, \mathrm{SGN}(v)} |\phi - f|^2$$

Properties of V

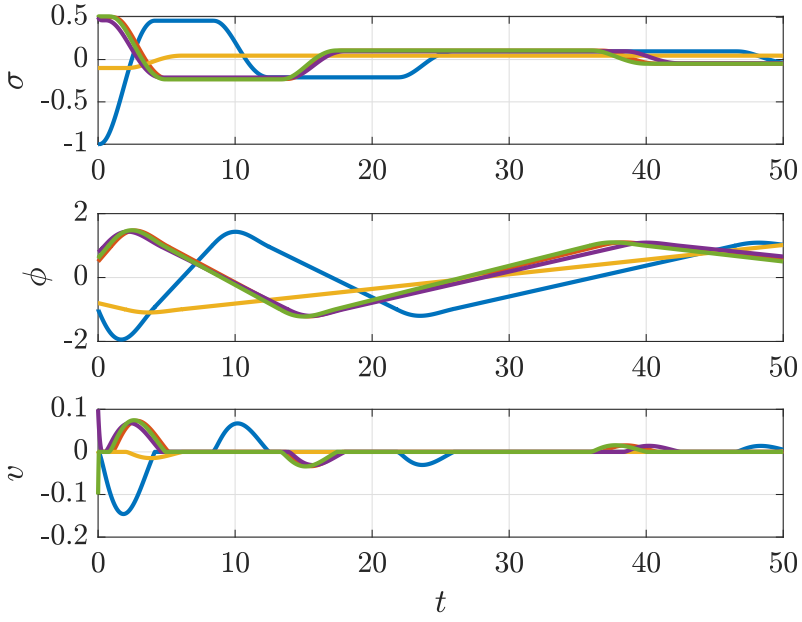
The Lyapunov-like function V is:

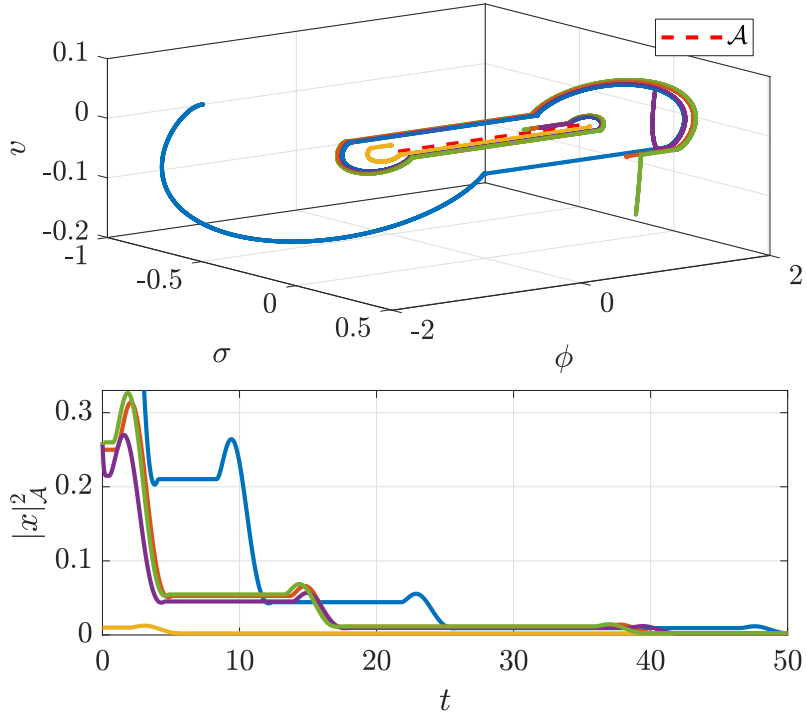
1. **lower semicontinuous** (lsc)
$$V(\bar{x}) \leq \lim_{x \to \bar{x}} V(x), \quad \forall \bar{x} \in \mathbb{R}^3 \qquad \text{(Regularity)}$$
2. **lower bounded**: There exist $c_1, c_2 > 0$ such that
$$c_1 |x|_{\mathcal{A}}{}^2 \leq V(x) \leq c_2 |x|_{\mathcal{A}}{}^2 + 2F_s^2 \quad \forall x \in \mathbb{R}^3 \qquad \text{(Sandwich)}$$
3. **decreasing along trajectories**: $\exists c > 0$: for each solution $x = (\sigma, \phi, v)$,
$$\forall t_2 \geq t_1 \geq 0, \quad V(x(t_2)) - V(x(t_1)) \leq -c \int_{t_1}^{t_2} v(t)^2 dt. \qquad \text{(Flow)}$$

- Proof of Theorem C-GAS given in Bisoffi et al. [2018] using:
  - auxiliary function and state partition for **stability**
  - Integral invariance principle of E.P. Ryan (1999) for **attractivity**

A closer look at the slow transients reveals promising ideas

- Solutions show long stick phases in the band $\mathcal{E}_{\text{stick}} := \{x \in \mathbb{R}^3 : \ v = 0, |\phi| \leq F_s\}$





- Lyapunov function suggests reversing the sign of ϕ (reset to $-\phi$) when $\phi v \leq 0$

$$V(x) := \begin{bmatrix} \sigma \\ v \end{bmatrix}^T \begin{bmatrix} \frac{k_v}{k_i} & -1 \\ -1 & k_p \end{bmatrix} \begin{bmatrix} \sigma \\ v \end{bmatrix} + \min_{f \in F_s \, \text{SGN}(v)} |\phi - f|^2$$

- Solutions would then jump across the band $\mathcal{E}_{\text{stick}}$
- Time-regularized solutions (with timer τ) imposes dwell time $t_{k+1} - t_k \geq \delta$

Reset PID Control Design Improves Coulomb Transient

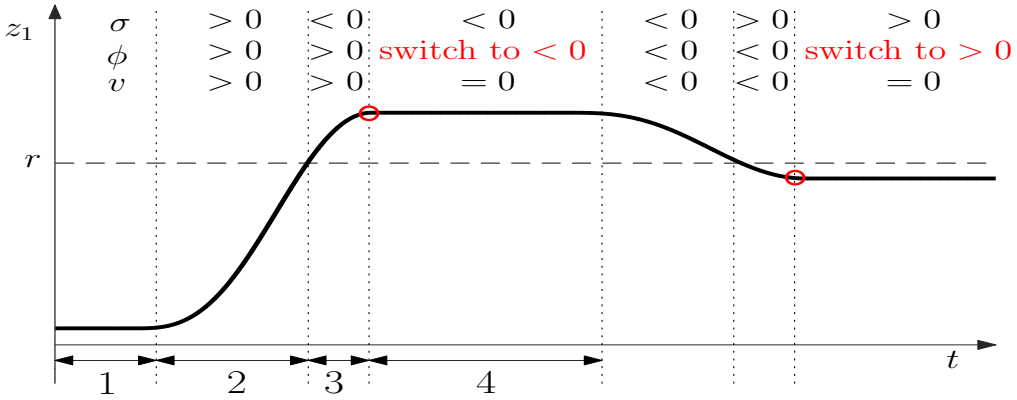
- Overall state involves $x = (\sigma, \phi, v)$ and $\tau \in [0, 2\delta]$, that is $(x, \tau) \in \mathbb{R}^3 \times [0, 2\delta]$
- Hybrid closed loop with reset PID (no knowledge of F_s required)

$$\begin{cases} \dot{x} \in F(x) := \begin{bmatrix} 0 & 0 & -k_i \\ 1 & 0 & -k_p \\ 0 & 1 & -k_v \end{bmatrix} \begin{bmatrix} \sigma \\ \phi \\ v \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ F_s \end{bmatrix} \mathrm{SGN}(v), \\ \dot{\tau} = 1 - \mathrm{dz}(\tau/\delta) \end{cases} \quad (x, \tau) \in \mathcal{C} := \overline{\mathbb{R}^3 \times [0, 2\delta] \setminus \mathcal{D}},$$

$$\begin{cases} x^+ = g(x) := \begin{bmatrix} \sigma & -\alpha\phi & v \end{bmatrix}^\top, \\ \tau^+ = 0 \end{cases} \quad (x, \tau) \in \mathcal{D} := \{(x, \tau) \mid \phi\sigma \leq 0,\ \phi v \leq 0,\ \tau \geq \delta\},$$

F and g are the **flow and jump maps**, $\mathcal{C}$ and $\mathcal{D}$ are the **flow and jump sets**.

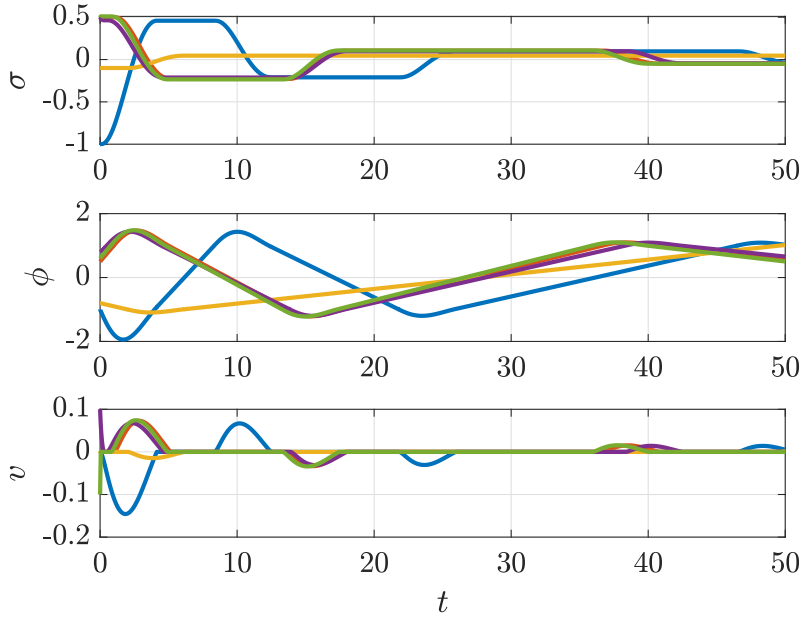
- Explanation of the jump set $\mathcal{D}$:
  - $\phi\sigma \leq 0$ so the solution is overshooting
  - $\phi v \leq 0$ so the Lyapunov function does not increase

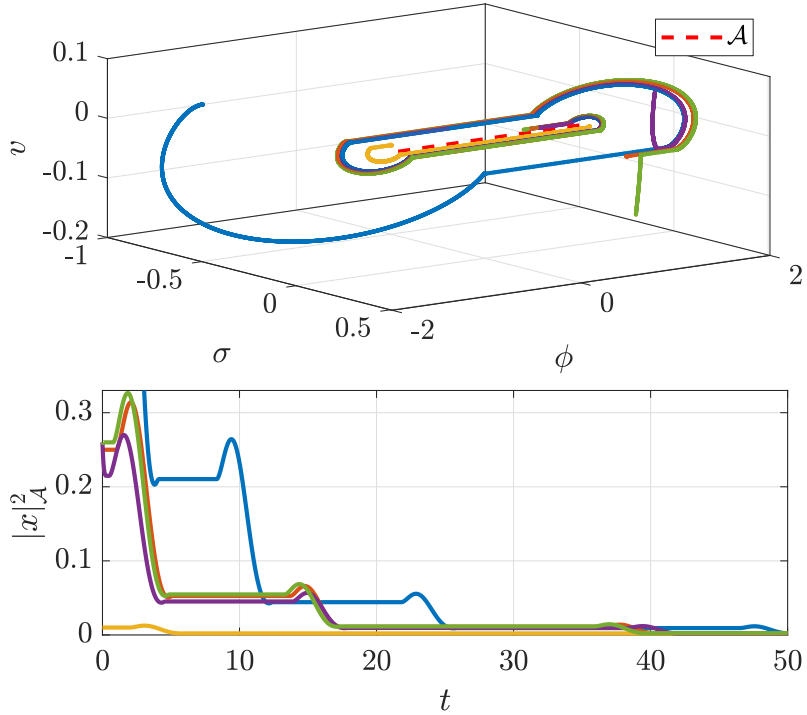


- Parameter $\alpha \in [0, 1]$ tunes **robustness** ($\alpha = 0$) vs **performance** ($\alpha = 1$)

A closer look at the slow transients (recall)

- Solutions show long stick phases in the band $\mathcal{E}_{\text{stick}} := \{x \in \mathbb{R}^3 : \ v = 0, |\phi| \leq F_s\}$



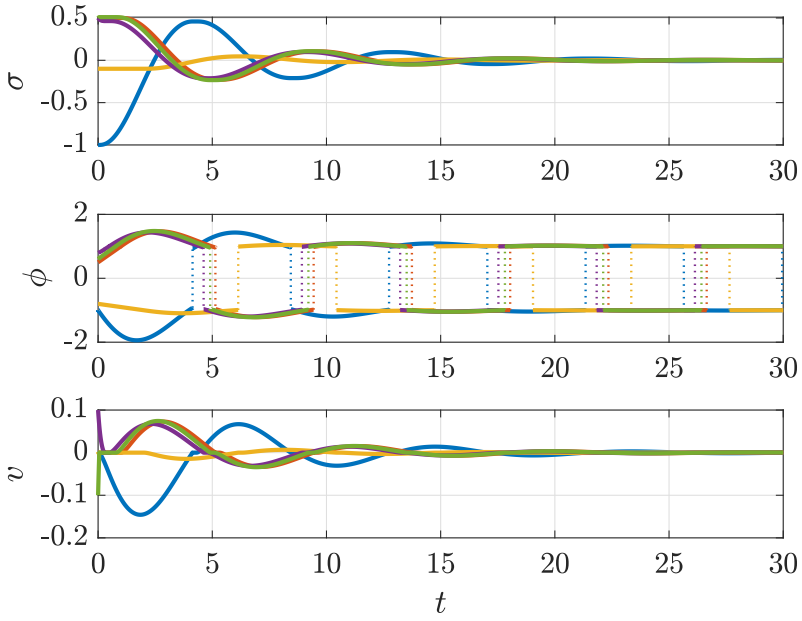


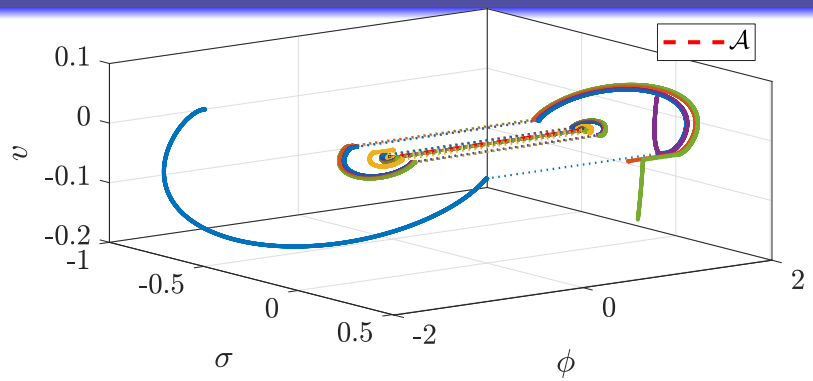
- Lyapunov function suggests reversing the sign of ϕ (reset to $-\alpha\phi$) when $\phi v \leq 0$

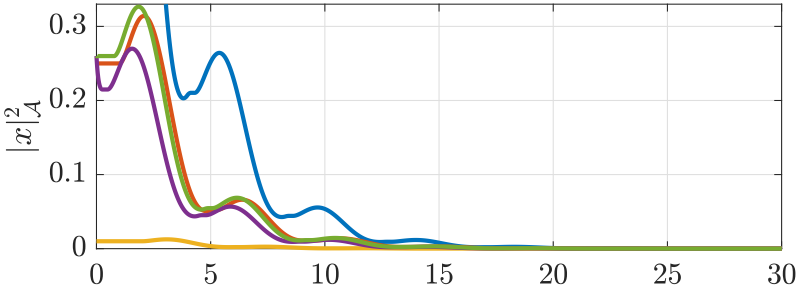
$$V(x) := \begin{bmatrix} \sigma \\ v \end{bmatrix}^T \begin{bmatrix} \frac{k_v}{k_i} & -1 \\ -1 & k_p \end{bmatrix} \begin{bmatrix} \sigma \\ v \end{bmatrix} + \min_{f \in F_s \, \text{SGN}(v)} |\phi - f|^2$$

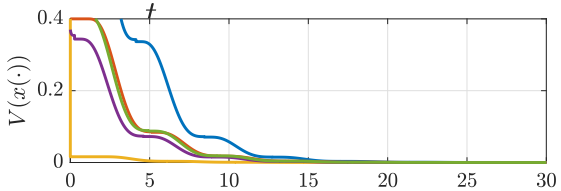
- Solutions would then jump across the band $\mathcal{E}_{\text{stick}}$
- Time-regularized solutions (with timer τ) imposes dwell time $t_{k+1} - t_k \geq \delta$

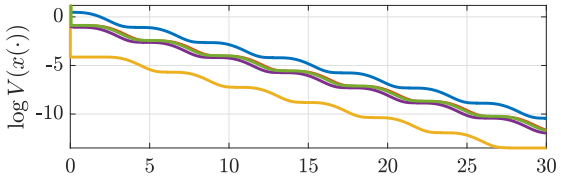
Reset PID (with $\alpha = 1$) successfully jumps across $\mathcal{E}_{\text{stick}}$

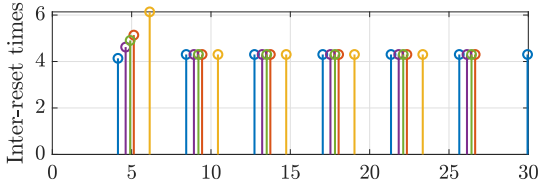












- Lyapunov decrease, and decrease of $|x|_{\mathcal{A}}$ suggests exponential convergence
- Bad solutions sequence from $x_k(0) = (\epsilon_k, 0, 0)$ satisfy:

$$|x_k(t)|_{\mathcal{A}} = |x_k(0)|_{\mathcal{A}} = \epsilon_k \text{ for all } t \leq T_k,$$

$$\text{with } \lim_{k\to\infty} \epsilon_k = 0 \text{ and } \lim_{k\to\infty} T_k = +\infty.$$

thus **disproving exponential convergence**

- However exponential convergence seems to often occur

The same Lyapunov function helps in the reset context

Recall the Lyapunov-like function:

$$V(x) := \begin{bmatrix} \sigma \\ v \end{bmatrix}^T \begin{bmatrix} \frac{k_v}{k_i} & -1 \\ -1 & k_p \end{bmatrix} \begin{bmatrix} \sigma \\ v \end{bmatrix} + \min_{f \in F_s \,\mathrm{SGN}(v)} |\phi - f|^2$$

Properties of V carry over from non-hybrid case

Function V is **lower semicontinuous** and there exist $c_1, c_2, c > 0$ such that:

$$c_1 |x|_{\mathcal{A}}{}^2 \leq V(x) \leq c_2 |x|_{\mathcal{A}}{}^2 + 2F_s^2 \qquad \forall x \in \mathbb{R}^3 \qquad \text{(Sandwich)}$$

$$V(x(t_2,j)) - V(x(t_1,j)) \leq -c \int_{t_1}^{t_2} v(t,j)^2 dt, \qquad \forall \text{ solution } (x,\tau) \quad \forall (t_2,j) \geq (t_1,j) \in \operatorname{dom} x \qquad \text{(Flow)}$$

$$V(g(x)) - V(x) \leq 0, \qquad \forall (x,\tau) \in \mathcal{D} \qquad \text{(Jump)}$$

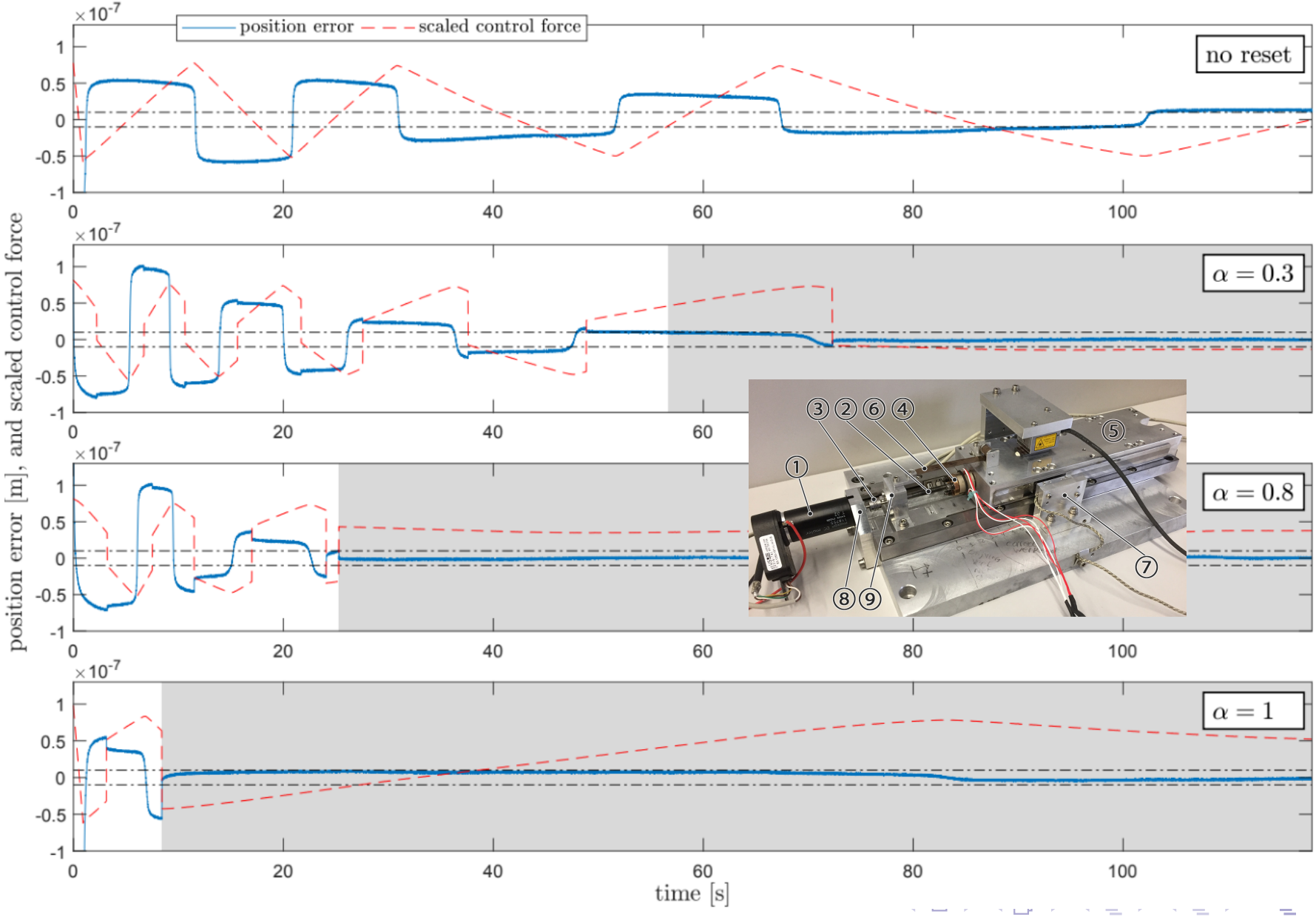
Theorem RC-GAS (Reset-Coulomb-GAS) Beerens et al. [2019]

With Coulomb friction, under Assumption LIN, set $\mathcal{A}$ is $\mathcal{KL}$-GAS.

- **Stability proof**: same as before using extra (Jump) condition
- **Global attractivity proof**: uses meagre-limsup hybrid invariance principle

Experimental response confirms transient improvement

Beerens et al. [2019]

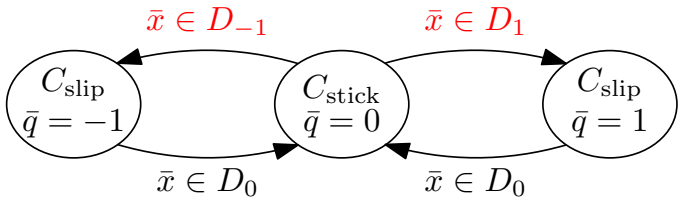


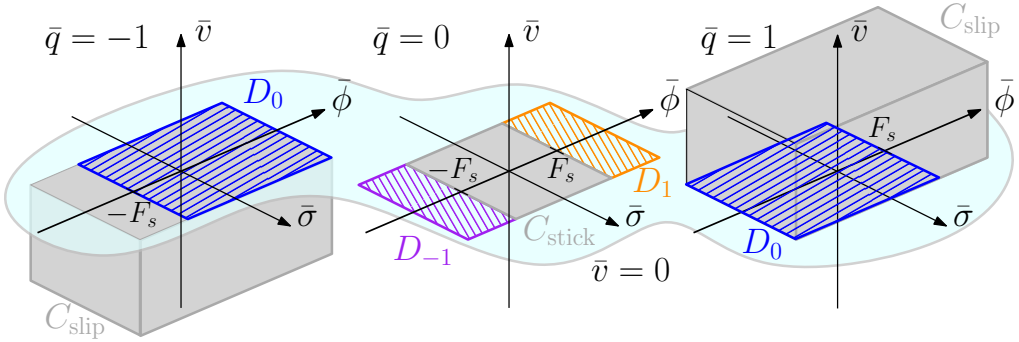
Without resets, alternative hybrid automaton model

Bisoffi et al. [2019]

- Extended state $\bar{x}$ includes timer $\bar{\tau}$ and logic variable $\bar{q}$ such that $\bar{q}\bar{v} \geq 0$

$$\bar{x} := (\bar{\sigma}, \bar{\phi}, \bar{v}, \bar{q}, \bar{\tau}) \in \bar{\Xi} := \{\bar{x} \in \mathbb{R}^3 \times \{-1, 0, 1\} \times [0, 2\delta] \mid \bar{q}\bar{v} \geq 0\},$$





- Hybrid automaton $\mathcal{H}_\delta$ (Coulomb, no resets) – semiglobally correct

$$\mathcal{H}_\delta : \begin{cases} \dot{\bar{x}} = \bar{f}(\bar{x}), & \bar{x} \in \bar{C} \\ \bar{x}^+ = \bar{g}(\bar{x}), & \bar{x} \in \bar{D} \end{cases}$$

$$\bar{C} := C_{\text{slip}} \cup C_{\text{stick}}$$

$$\bar{D} := D_{-1} \cup D_0 \cup D_1$$

$$C_{\text{slip}} := \{\bar{x} \in \bar{\Xi} : |\bar{q}| = 1\}$$

$$C_{\text{stick}} := \{\bar{x} \in \bar{\Xi} : \bar{q} = 0, \bar{v} = 0, |\bar{\phi}| \leq F_s\}$$

$$D_1 := \{\bar{x} \in \bar{\Xi} : \bar{q} = 0, \bar{v} = 0, \bar{\phi} \geq F_s, \bar{\tau} \in [\delta, 2\delta]\}$$

$$D_{-1} := \{\bar{x} \in \bar{\Xi} : \bar{q} = 0, \bar{v} = 0, \bar{\phi} \leq -F_s, \bar{\tau} \in [\delta, 2\delta]\}$$

$$D_0 := \{\bar{x} \in \bar{\Xi} : |\bar{q}| = 1, \bar{v} = 0, \bar{q}\bar{\phi} \leq F_s\}$$

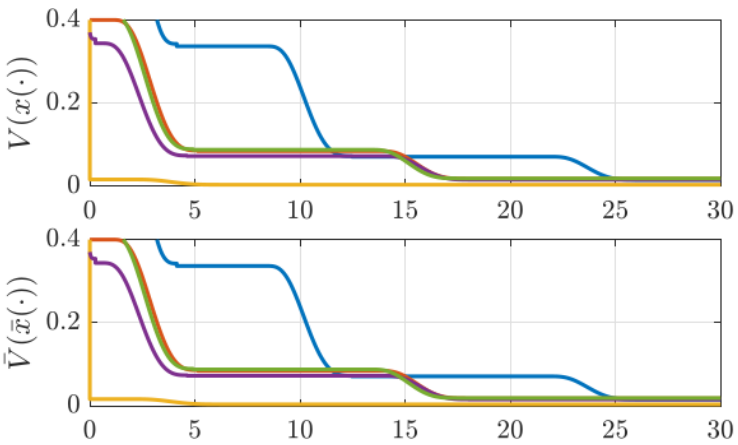
- **Smooth** Lyapunov function certifies GAS of $\bar{\mathcal{A}} := \{\bar{x} : \bar{\sigma} = \bar{v} = 0, \bar{\phi} \in F_s \mathrm{SGN}(\bar{q})\}$

$$\bar{V}(\bar{x}) := \begin{bmatrix} \bar{\sigma} \\ \bar{v} \end{bmatrix}^\top \begin{bmatrix} \frac{k_v}{k_i} & -1 \\ -1 & k_p \end{bmatrix} \begin{bmatrix} \bar{\sigma} \\ \bar{v} \end{bmatrix} + |\bar{q}|(\bar{\phi} - \bar{q}F_s)^2 + (1 - |\bar{q}|)\mathrm{dz}_{F_s}^2(\bar{\phi}).$$

Alternative proof of Theorem C-GAS uses function $\bar{V}$

Bisoffi et al. [2019]

- Same exact evolution for V (along original sol'ns x) and $\bar{V}$ (along sol'ns $\bar{x}$ to $\mathcal{H}_\delta$)



Properties of smooth $\bar{V}$ are convenient

Function $\bar{V}$ is **smooth** and there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $c > 0$ such that:

$$\alpha_1(|\bar{x}|_{\bar{\mathcal{A}}}) \leq \bar{V}(\bar{x}) \leq \alpha_2(|\bar{x}|_{\bar{\mathcal{A}}}) \quad \forall \bar{x} \in \bar{\Xi} \qquad \text{(Sandwich)}$$

$$\langle \nabla \bar{V}(\bar{x}), \bar{f}(\bar{x}) \rangle = -c\bar{v}^2, \quad \forall \bar{x} \in C_{\text{slip}} \cup C_{\text{stick}} \qquad \text{(Flow)}$$

$$\bar{V}(\bar{g}_i(\bar{x})) - \bar{V}(\bar{x}) \leq 0, \quad \forall \bar{x} \in D_i, i \in \{1, -1, 0\} \qquad \text{(Jump)}$$

- Alternative proof of Theorem C-GAS given in Bisoffi et al. [2019] using:
  - auxiliary function and state partition for **stability**
  - proof of **attractivity** using the following arguments
    - Original solutions x are uniformly bounded
    - Solutions $\bar{x}$ of hybrid automaton $\mathcal{H}_\delta$ semiglobally reproduces any original solution x evolving in a compact set $\mathcal{K}(\delta)$ such that $\lim_{\delta \to 0} \mathcal{K}(\delta) = \mathbb{R}^3$
    - Smooth Lyapunov function $\bar{V}$ certifies global attractivity for $\mathcal{H}_\delta$
    - Attract. for $\mathcal{H}_\delta \Rightarrow$ semiglobal Attract. $\Rightarrow$ Attract. of the original system
- Interesting connections with (bi)simulation concepts found in computer science

With resets, extended automaton includes extra variable

- Extended state $\bar{x} := (\bar{\sigma}, \bar{\phi}, \bar{v}, \bar{q}, \bar{\tau}, \bar{a}) \in \bar{\Xi}$ with logic variable $\bar{a}$ such that $\bar{a}\bar{v} \geq 0$

$$\bar{\Xi} := \left\{\bar{x} \in \mathbb{R}^3 \times \{-1, 0, 1\} \times [0, 2\delta] \times \{-1, 1\} \mid \ \bar{q}\bar{v} \geq 0,\ \bar{a}\bar{\phi} \geq 0,\ \bar{a}\bar{v} \geq 0\right\},$$

- Hybrid automaton for overshooting solutions, wherein $\bar{\phi}\bar{v} \geq 0$, then $\bar{a} = \text{sign}(\phi)$

$$\mathcal{H}_\delta : \begin{cases} \dot{\bar{x}} = \bar{f}(\bar{x}), & \bar{x} \in \bar{C} \\ \bar{x}^+ = \bar{g}(\bar{x}), & \bar{x} \in \bar{D} \end{cases}$$

$$\bar{C} := C_{\text{slip}} \cup C_{\text{stick}}$$

$$\bar{D} := D_{-1} \cup D_0 \cup D_1$$

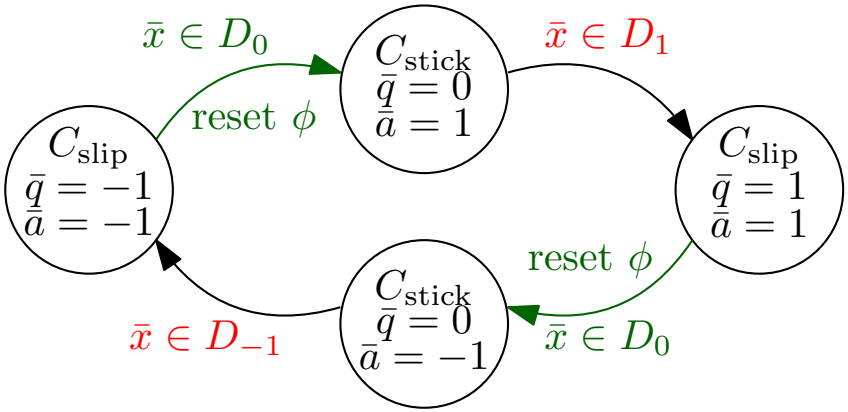
$$C_{\text{slip}} := \{\bar{x} \in \bar{\Xi} \colon |\bar{q}| = 1\}$$

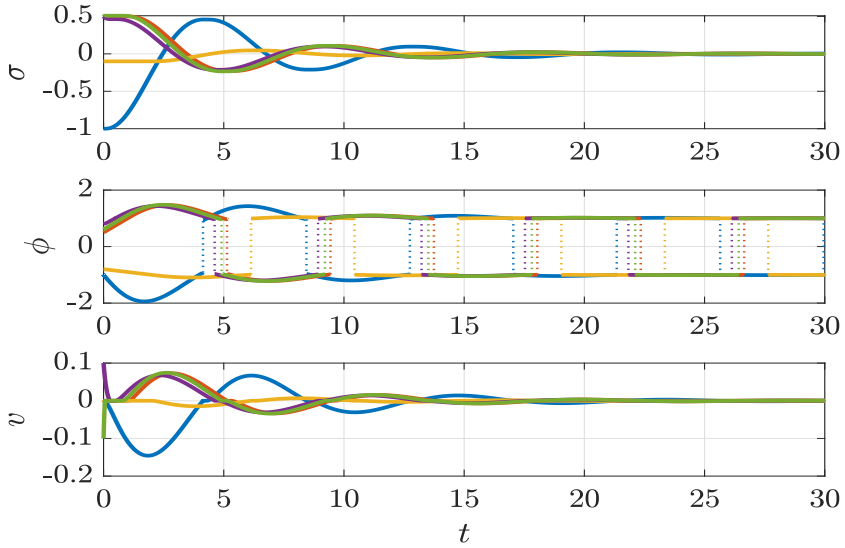
$$C_{\text{stick}} := \{\bar{x} \in \bar{\Xi} \colon \bar{q} = 0, \bar{v} = 0, \bar{a}\bar{\phi} \leq F_s\}$$

$$D_1 := \{\bar{x} \in \bar{\Xi} \colon \bar{q} = 0, \bar{v} = 0, \bar{a}\bar{\phi} \geq F_s, \bar{\tau} \in [\delta, 2\delta]\}$$

$$D_{-1} := \{\bar{x} \in \bar{\Xi} \colon \bar{q} = 0, \bar{v} = 0, \bar{a}\bar{\phi} \geq F_s, \bar{\tau} \in [\delta, 2\delta]\}$$

$$D_0 := \{\bar{x} \in \bar{\Xi} \colon |\bar{q}| = 1, \bar{v} = 0, \bar{a}\bar{\phi} \leq F_s\},$$





Homogeneous automaton explains exponential convergence

- State transformation provides homogeneous hybrid dynamics

$$\bar{x} := (\bar{\sigma}, \bar{\phi}, \bar{v}, \bar{q}, \bar{\tau}, \bar{a}) \quad \mapsto \quad \hat{x} := (\hat{\sigma}, \hat{\phi}, \hat{v}, \hat{q}, \hat{\tau}, \hat{a}) = (\bar{\sigma}, \bar{\phi} - \bar{a}F_s, \bar{v}, \bar{q}, \bar{\tau}, \bar{a}).$$

- With $\alpha = 1$, denoting $\hat{x}_0 = (\hat{\sigma}, \hat{\phi}, \hat{v})$, we get partial homogeneity in $\hat{x}_0$

$$\begin{cases} \dot{\hat{x}}_0 = A_F(\hat{q}, \hat{a})\hat{x}_0, & \hat{x} \in \hat{C} \\ \hat{x}_0^+ = A_J(\hat{q}, \hat{a})\hat{x}_0, & \hat{x} \in \hat{D} \end{cases}$$

$$\hat{C}_{\text{slip}} := \{\hat{x} \colon |\hat{q}| = 1\}$$

$$\hat{C}_{\text{stick}} := \{\hat{x} \colon \hat{q} = 0, \hat{v} = 0, \hat{a}\hat{\phi} \leq 0\}$$

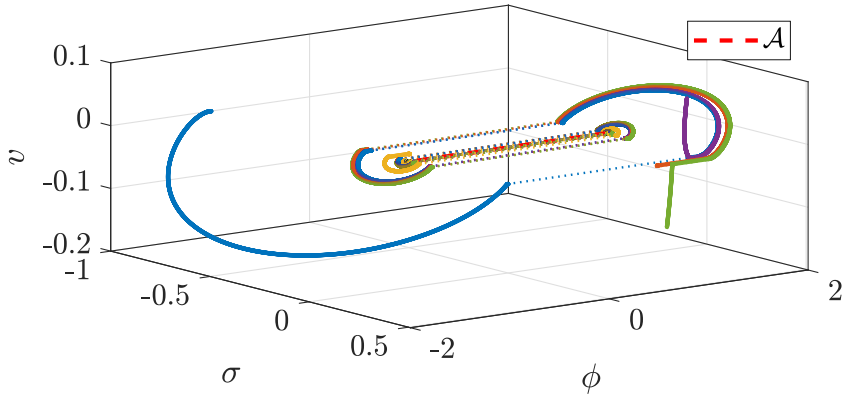
$$\hat{D}_1 := \{\hat{x} \colon \hat{q} = 0, \hat{v} = 0, \hat{a}\hat{\phi} \geq 0, \bar{\tau} \in [\delta, 2\delta]\}$$

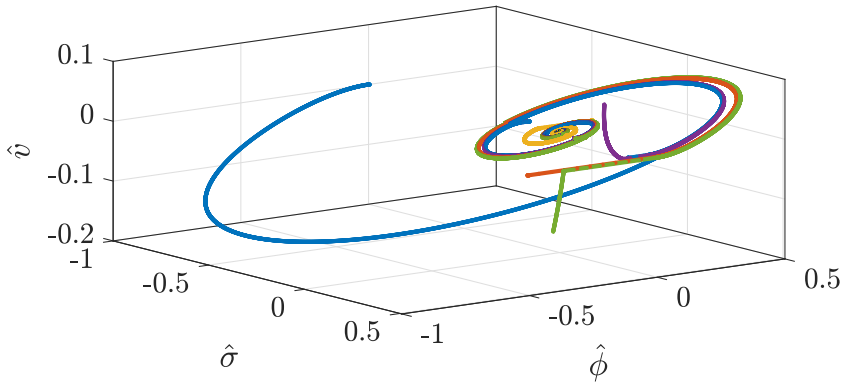
$$\hat{D}_{-1} := \{\hat{x} \colon \hat{q} = 0, \hat{v} = 0, \hat{a}\hat{\phi} \geq 0, \bar{\tau} \in [\delta, 2\delta]\}$$

$$\hat{D}_0 := \{\hat{x} \colon |\hat{q}| = 1, \hat{v} = 0, \hat{a}\hat{\phi} \leq 0\}.$$

$$\hat{C} := \hat{C}_{\text{slip}} \cup \hat{C}_{\text{stick}}$$

$$\hat{D} := \hat{D}_{-1} \cup \hat{D}_0 \cup \hat{D}_1$$



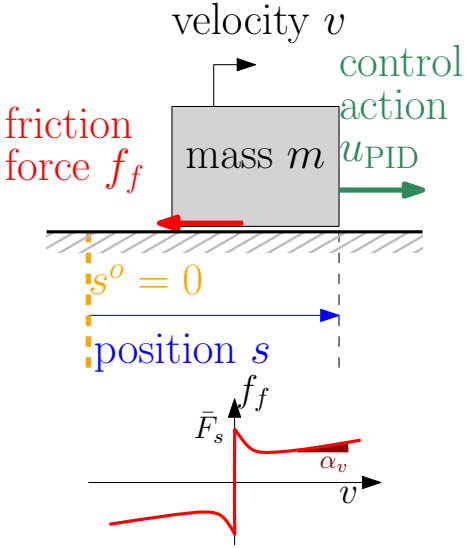


- Exploiting $\bar{a} = \text{sign}(\bar{\phi})$, with $\alpha = 1$ we can prove $\exists M > 0, \mu > 0$ satisfying

$$|(\sigma, \phi - \text{sign}(\phi)F_s, v)| \leq M e^{-\mu t}|\sigma_0|$$

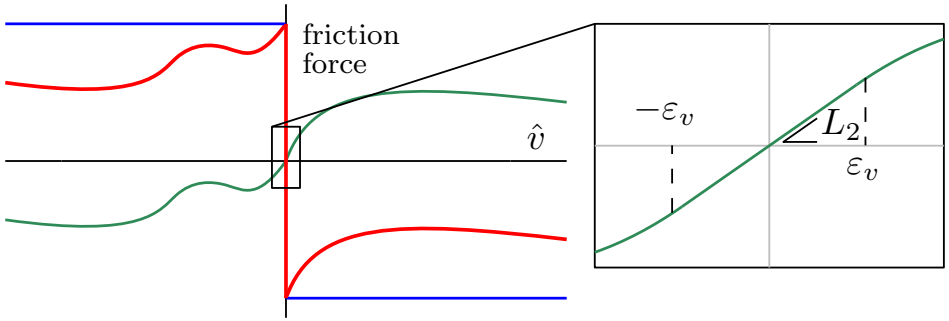
for all solutions starting at stick-to-slip transition $\hat{x}_0 = (\sigma_0, 0, 0)$.

Stribeck model includes extra nonlinearity



- **New velocity weakening** function ψ (previously zero):

$$\begin{bmatrix}\dot{\sigma}\\ \dot{\phi}\\ \dot{v}\end{bmatrix}\begin{bmatrix}0 & 0 & -k_i\\ 1 & 0 & -k_p\\ 0 & 1 & -k_v\end{bmatrix}\begin{bmatrix}\sigma\\ \phi\\ v\end{bmatrix}-\begin{bmatrix}0\\ 0\\ F_s\end{bmatrix}(\text{SGN}(v)+\psi(v)),$$

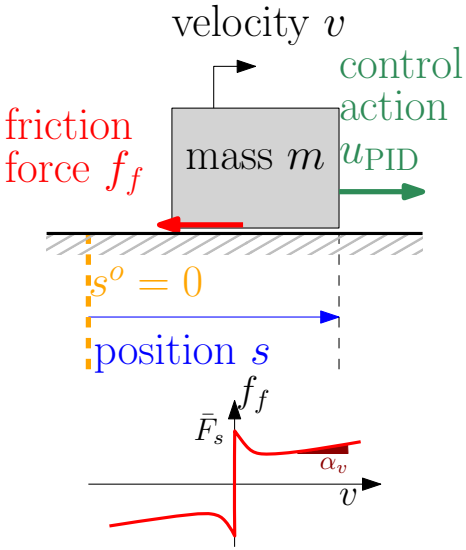


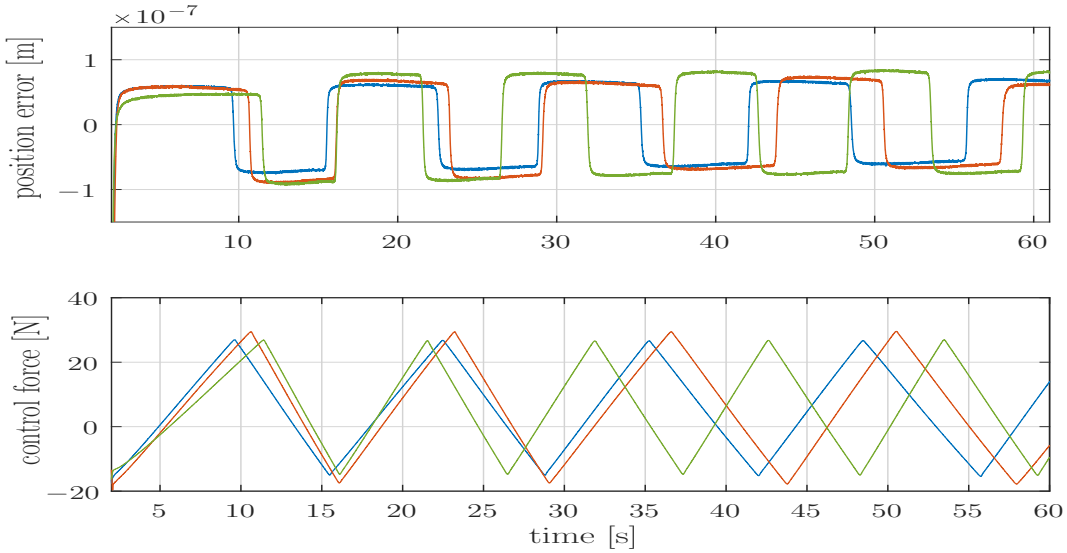
Assumption STRIB

Assumption LIN holds ($k_i > 0$, $k_p > 0$, $k_v k_p > k_i$). Moreover, the velocity weakening function ψ is globally Lipschitz and satisfies

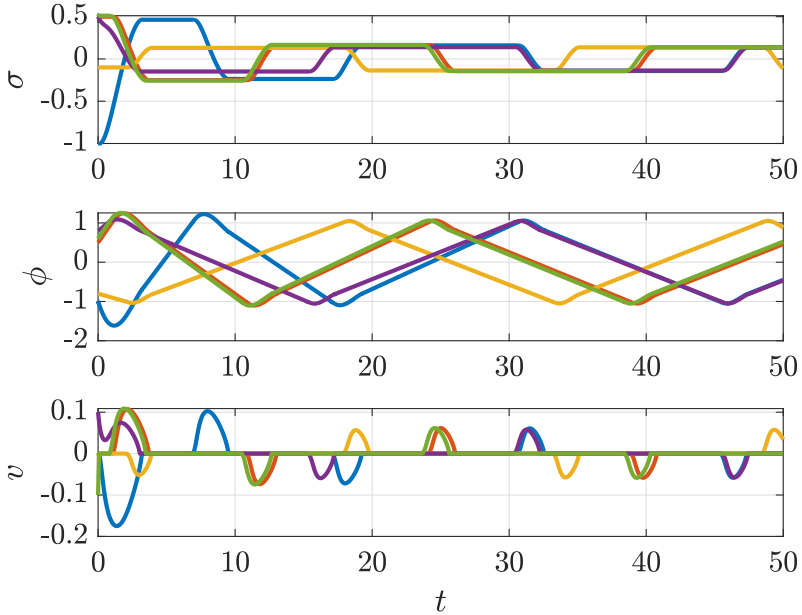
- $|\psi(v)| \leq F_s$
- $v\psi(v) \geq 0$ for all v
- it is linear in a small enough interval around zero
  (namely, for some ε_v, $|v| \leq \varepsilon_v \;\Rightarrow\; \psi(v) = L_2 v$).

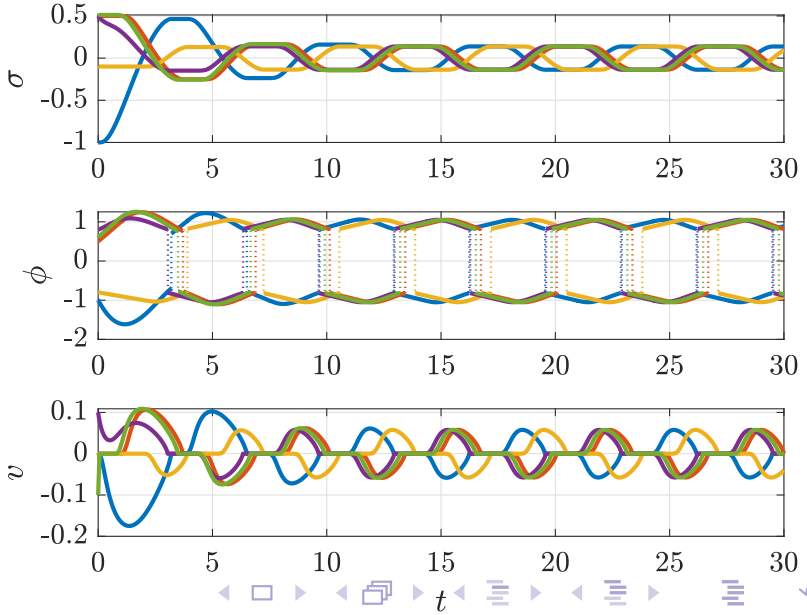
Stribeck "hunting instability" needs a different solution





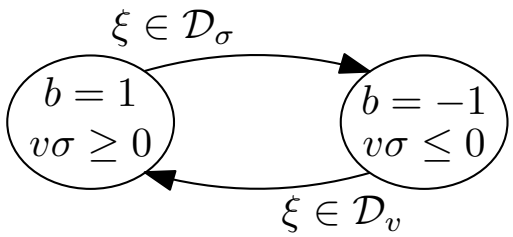
- Reset PID solution solving Coulomb is not successful for Stribeck hunting effect





A two-stage reset PID proposed in Beerens et al. [2020]

- Add boolean state $b \in \{-1, 1\}$ such that $bv\sigma \geq 0$:
  1) $b = 1$ in the overshooting phase $v\sigma \geq 0$
  2) $b = -1$ in the approaching phase $v\sigma \leq 0$



- Ensure that the integral action e_i points in the direction of position error s
  This corresponds to imposing $\phi\sigma \geq \frac{k_p}{k_i}\sigma^2$
- Overall state $\xi := (x, b) := (\sigma, \phi, v, b)$ evolves in Ξ, where

$$\Xi := \{(x, b) \in \mathbb{R}^3 \times \{-1, 1\} \colon bv\sigma \geq 0,\ \sigma\phi \geq \tfrac{k_p}{k_i}\sigma^2,\ bv\phi \geq 0\}.$$

- Jumps at zero-crossing of σ and v, wherein state b alternates between -1 and 1

$$\begin{bmatrix} \sigma^+ \\ \phi^+ \\ v^+ \\ b^+ \end{bmatrix} = g_\sigma(\xi) := \begin{bmatrix} \sigma \\ -\phi \\ v \\ -b \end{bmatrix}, \qquad \xi \in \mathcal{D}_\sigma := \{\xi \in \Xi : \sigma = 0, b = 1\}$$

$$\begin{bmatrix} \sigma^+ \\ \phi^+ \\ v^+ \\ b^+ \end{bmatrix} = g_v(\xi) := \begin{bmatrix} \sigma \\ \frac{k_p}{k_i}\sigma \\ v \\ -b \end{bmatrix}, \qquad \xi \in \mathcal{D}_v := \{\xi \in \Xi : v = 0, b = -1\}.$$

- Can prove that ϕ is never zero along sol'ns, so $\mathcal{D}_\sigma$ and $\mathcal{D}_v$ robustly implemented as

$$\mathcal{D}_\sigma^{\mathrm{r}} := \{\xi : \sigma\phi \leq 0, b = 1\}, \quad \mathcal{D}_v^{\mathrm{r}} := \{\xi : v\phi \geq 0, b = -1\},$$

The reset closed-loop eliminates the hunting instability

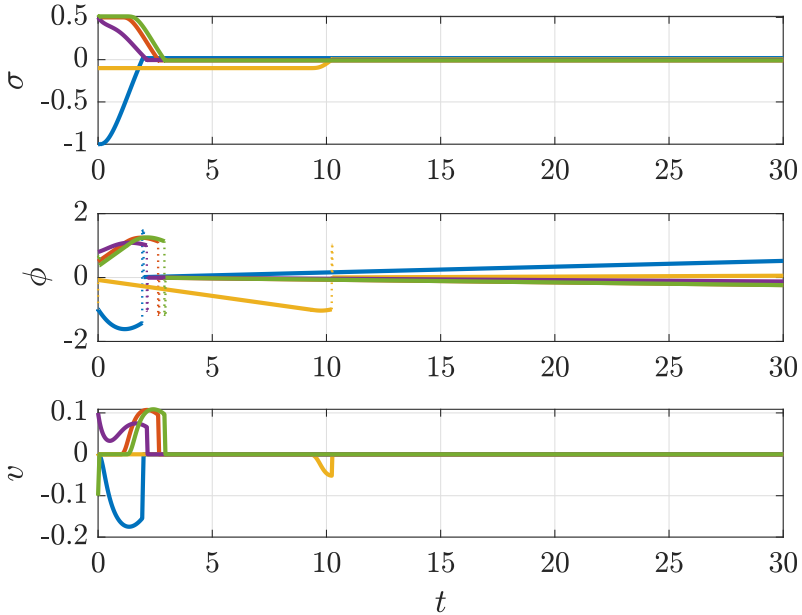
Theorem STRI-GAS (Reset-Stribeck-GAS) Beerens et al. [2020]

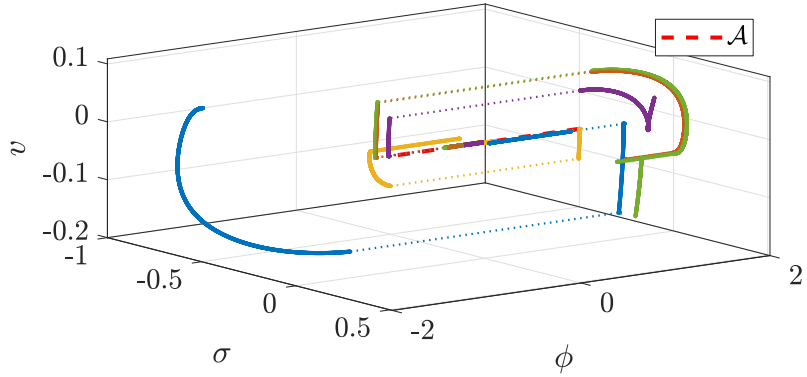
Under Assumption STRIB, the compact set

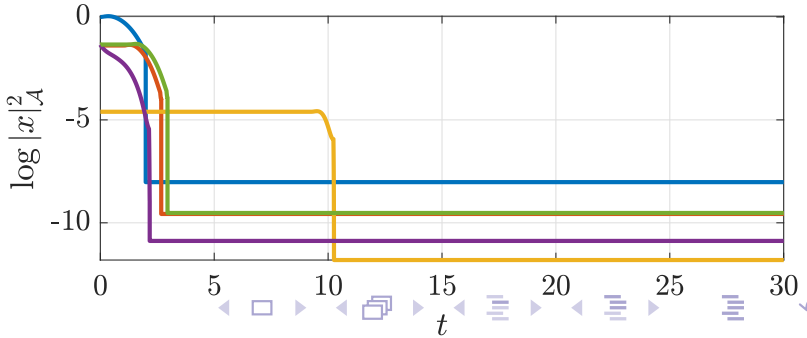
$$\mathcal{A}_e := \mathcal{A} \times \{-1, 1\} = \{\xi \in \Xi \colon \sigma = v = 0, |\phi| \leq F_s\}.$$

is $\mathcal{KL}$-globally asymptotically stable.

- The proof of Theorem STRI-GAS requires using the hybrid automaton trick (a new automaton, a new "smooth" Lyapunov function).





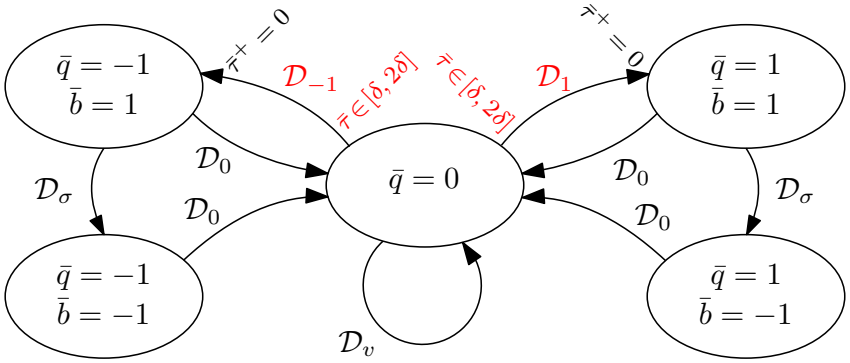


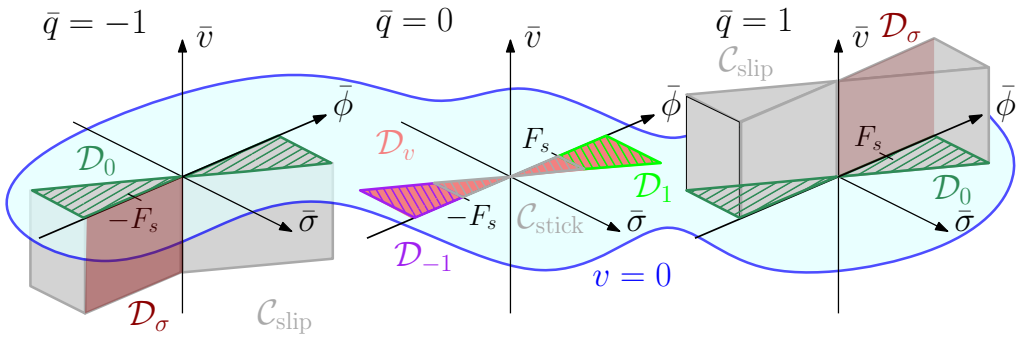
Stribeck hybrid automaton is more sophisticated

Beerens et al. [2020]

- Extended state $\bar{\xi}$ includes timer $\bar{\tau}$ and logic variable $\bar{q}$ such that $\bar{q}\bar{v} \geq 0$

$$\bar{\xi} := (\bar{\sigma}, \bar{\phi}, \bar{v}, \bar{b}, \bar{q}, \bar{\tau}) \in \bar{\Xi} := \left\{\bar{\xi} \mid \bar{q}\bar{v} \geq 0, \bar{b}\bar{q}\bar{\sigma} \geq 0, \bar{\sigma}\bar{\phi} \geq \frac{k_p}{k_i}\bar{\sigma}^2, \bar{b}\bar{q}\bar{\phi} \geq 0\right\}.$$





- Hybrid automaton $\mathcal{H}_\delta$ (Stribeck, resets) – semiglobally correct

$$\mathcal{H}_\delta : \begin{cases} \dot{\bar{\xi}} = \bar{f}(\bar{\xi}), & \bar{\xi} \in \bar{\mathcal{C}} \\ \bar{\xi}^+ = \bar{g}(\bar{\xi}), & \bar{\xi} \in \bar{\mathcal{D}} \end{cases} \qquad \begin{aligned} \bar{\mathcal{C}} &:= \mathcal{C}_{\text{slip}} \cup \mathcal{C}_{\text{stick}} \\ \bar{\mathcal{D}} &:= \mathcal{D}_{-1} \cup \mathcal{D}_0 \cup \mathcal{D}_1 \cup \mathcal{D}_\sigma \cup \mathcal{D}_v \end{aligned}$$

- **Lipschitz** Lyapunov function shows GAS of $\bar{\mathcal{A}}_e := \{\bar{\xi} \mid \bar{\sigma} = \bar{v} = 0, \bar{\phi} \in F_s \mathrm{SGN}(\bar{b}\bar{q})\}$

$$\bar{V}_e(\bar{\xi}) := \begin{bmatrix} \bar{\sigma} \\ \bar{v} \end{bmatrix}^\top \begin{bmatrix} \frac{k_v}{k_i} & -1 \\ -1 & k_p \end{bmatrix} \begin{bmatrix} \bar{\sigma} \\ \bar{v} \end{bmatrix} + |\bar{q}|(\bar{\phi} - \bar{b}\bar{q}F_s)^2 + (1 - |\bar{q}|)\mathrm{dz}_{F_s}^2(\bar{\phi})$$
$$+2\frac{k_p}{k_i}F_s\left(\bar{b}\bar{q}\bar{\sigma} + (1 - |\bar{q}|)|\bar{\sigma}|\right)$$

Proof of Theorem STRI-GAS uses function $\bar{V}_e$

- Function $\bar{V}_e$ is not smooth but Lipschitz $\Rightarrow$ can use Clarke nonsmooth tools

Properties of Lipschitz $\bar{V}_e$ are convenient

Function $\bar{V}_e$ is **Lipscthiz** and there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $c > 0$ such that:

$$\alpha_1(|\bar{\xi}|_{\bar{\mathcal{A}}_e}) \leq \bar{V}_e(\bar{\xi}) \leq \alpha_2(|\bar{\xi}|_{\bar{\mathcal{A}}_e}) \quad \forall \bar{\xi} \in \bar{\Xi} \tag{Sandwich}$$

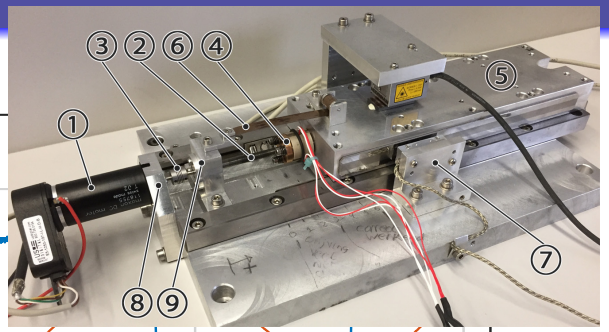
$$\bar{V}_e^\circ(\bar{\xi}) := \max_{\nu \in \partial \bar{V}_e(\bar{\xi})} \langle \nu, \bar{f}(\bar{\xi}) \rangle \leq -c\bar{v}^2, \quad \forall \bar{x} \in \mathcal{C}_{\text{slip}} \cup \mathcal{C}_{\text{stick}} \tag{Flow}$$

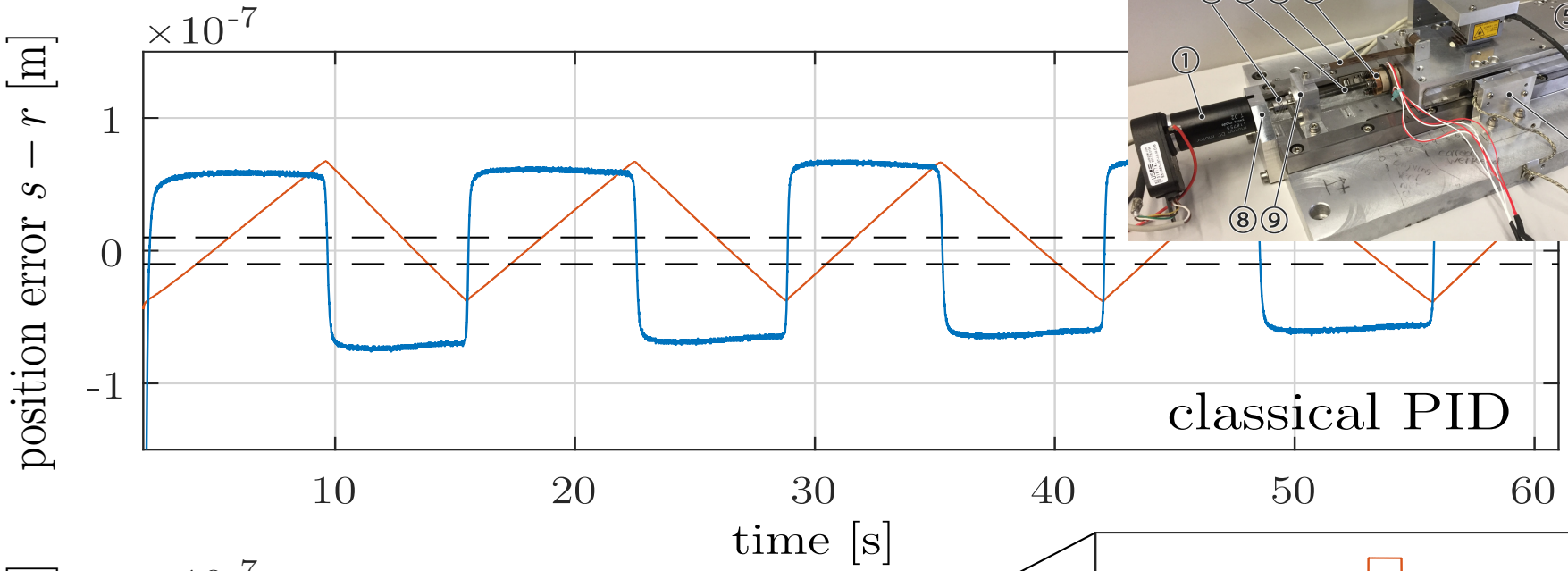
$$\bar{V}_e(\bar{g}_i(\bar{\xi})) - \bar{V}_e(\bar{\xi}) \leq 0, \quad \forall \bar{\xi} \in \mathcal{D}_i, i \in \{1, -1, 0, \sigma, v\} \tag{Jump}$$

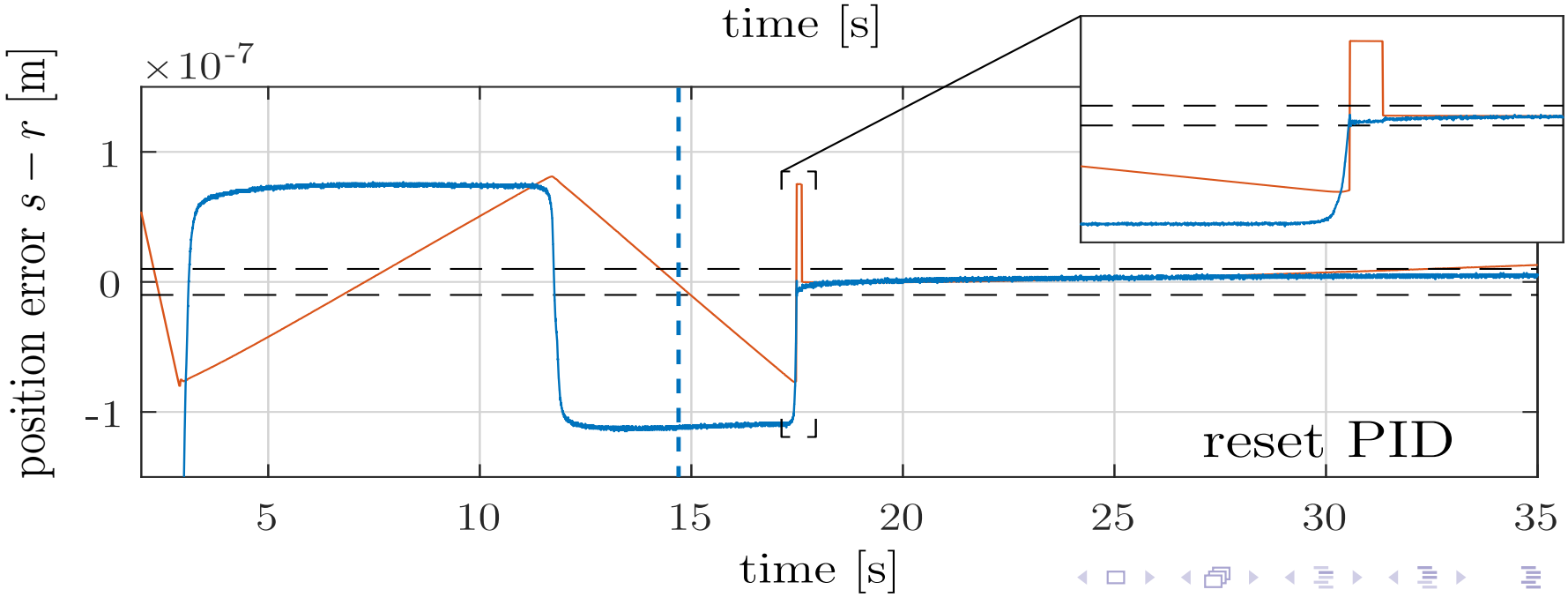
- Proof of Theorem STRI-GAS given in Beerens et al. [2020] using:
  - proof of **uniform global attractivity (UGA)** using the following arguments
    - Original solutions x are uniformly bounded (not as trivial as with Coulomb)
    - Solutions $\bar{\xi}$ of hybrid automaton $\mathcal{H}_\delta$ semiglobally reproduces any original solution ξ evolving in a compact set $\mathcal{K}(\delta)$ such that $\lim_{\delta \to 0} \mathcal{K}(\delta) = \mathbb{R}^3$
    - Lipschitz Lyapunov function $\bar{V}_e$ certifies UGA for $\mathcal{H}_\delta$
    - UGA for $\mathcal{H}_\delta \Rightarrow$ semiglobal UA $\Rightarrow$ UGA of the original system
  - UGA and strong forward invariance of $\bar{\mathcal{A}}_e$ implies **stability**.
- Interesting connections with (bi)simulation concepts found in computer science

Experimental response confirms GAS recovery

Beerens et al. [2020]



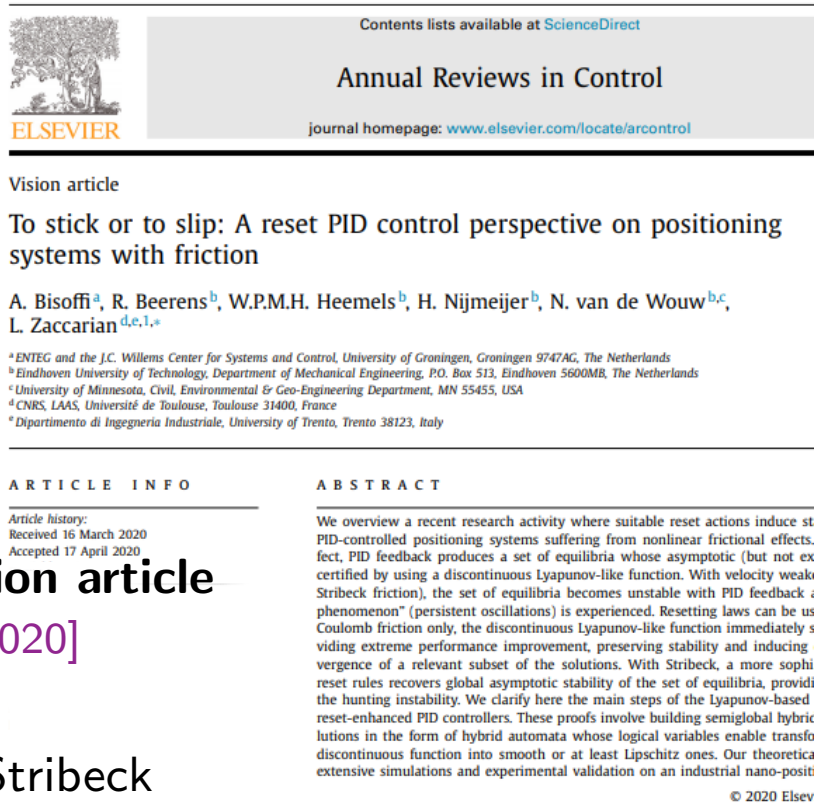




Wrap up and acknowledgements

Conclusions:

- Differential inclusion model for PID controlling sliding mass with Coulomb/Stribeck friction effects
- Coulomb: Lyapunov-based proof of Global Asymptotic (not exponential) Stability
- Coulomb: Reset PID improves transient response (exponential convergence)
- Stribeck: Reset PID resolves "hunting" instability
- The presented results in a **recently published vision article** (IFAC Annual Reviews in Control) Bisoffi et al. [2020]



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Vision article

To stick or to slip: A reset PID control perspective on positioning systems with friction

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ABSTRACT

We overview a recent research activity where suitable reset actions induce st
PID-controlled positioning systems suffering from nonlinear frictional effects.
fect, PID feedback produces a set of equilibria whose asymptotic (but not ex
certified by using a discontinuous Lyapunov-like function. With velocity weak
Stribeck friction), the set of equilibria becomes unstable with PID feedback
phenomenon" (persistent oscillations) is experienced. Resetting laws can be us
Coulomb friction only, the discontinuous Lyapunov-like function immediately s
viding extreme performance improvement, preserving stability and inducing
vergence of a relevant subset of the solutions. With Stribeck, a more sophi
reset rules recovers global asymptotic stability of the set of equilibria, providi
the hunting instability. We clarify here the main steps of the Lyapunov-based
reset-enhanced PID controllers. These proofs involve building semiglobal hybric
lutions in the form of hybrid automata whose logical variables enable transfo
discontinuous function into smooth or at least Lipschitz ones. Our theoretica
extensive simulations and experimental validation on an industrial nano-positi

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Future Work:

- Combine resets for exponential convergence and Stribeck
- Address the case of asymmetric friction laws

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Bibliography I

R. Beerens, A. Bisoffi, L. Zaccarian, M. Heemels, H. Nijmeijer, and N. van de Wouw. Reset integral control for improved settling of PID-based motion systems with friction. *Automatica*, 107:483–492, 2019.

R. Beerens, A. Bisoffi, L. Zaccarian, H. Nijmeijer, M. Heemels, and N. van de Wouw. Reset PID design for motion systems with Stribeck friction. *Submitted to IEEE Transactions on Control Systems Technology*, 2020.

A. Bisoffi, M. Da Lio, A.R. Teel, and L. Zaccarian. Global asymptotic stability of a PID control system with Coulomb friction. *IEEE Transactions on Automatic Control*, 63(8):2654–2661, 2018.

A. Bisoffi, R. Beerens, L. Zaccarian, M. Heemels, H. Nijmeijer, and N. van de Wouw. Hybrid model formulation and stability of a PID-controlled motion system with Coulomb friction. In *Symposium on Nonlinear Control Systems (NOLCOS)*, pages 116–121, Vienna, Austria, September 2019.

A. Bisoffi, R. Beerens, M. Heemels, H. Nijmeijer, N. van de Wouw, and L. Zaccarian. To stick or to slip: A reset PID control perspective on positioning systems with friction. *IFAC Annual Reviews in Control, to appear*, 2020.