

LMI-based algorithms for input-saturated linear time-invariant plants

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XXIII Congresso Brasileiro de Automática
Virtual Congress, November 26, 2020

Outline

- 1 State feedback stabilization with quadratic Lyapunov functions
- 2 Input-Output \mathcal{H}_∞ -type performance with saturation
- 3 Use of Quadratic Functions for Stability and Performance
- 4 Incorporating Robustness in Quadratic Conditions
- 5 Use of nonquadratic functions for analysis and design

(Global) Linear Stability Analysis

- Lyapunov Conditions for GES:

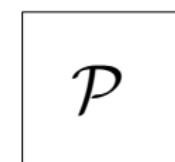
$$V(x) = x^T P x$$

- positive definite
- radially unbounded

$$\dot{V}(x) = \frac{\partial V}{\partial x} A x \\ = 2x^T P A x$$

- negative definite

$$\dot{x} = Ax$$



- LMI-based Conditions for Global Exponential Stability (GES):

$$P > 0$$

$$PA + A^T P < 0$$

$$[\text{He}(PA) < 0]$$

$$\overleftarrow{\quad} \quad Q = P^{-1}$$

$$Q P Q = Q > 0$$

$$Q P A Q + Q A^T P Q = A Q + Q A^T < 0$$

$$[\text{He}(Q P A Q) = \text{He}(A Q) < 0]$$

- LMI formulation allows for efficient polynomial-time stability certification

(Global) LMI-based linear stabilizing state feedback

- Lyapunov Conditions for GES:

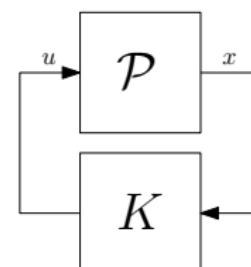
$$\mathbf{V}(\mathbf{x}) = \mathbf{x}^T \mathbf{P} \mathbf{x}$$

- positive definite
- radially unbounded

$$\begin{aligned}\dot{\mathbf{V}}(\mathbf{x}) &= \frac{\partial \mathbf{V}}{\partial \mathbf{x}} (\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{K}\mathbf{x}) \\ &= 2\mathbf{x}^T \mathbf{P}(\mathbf{A} + \mathbf{B}\mathbf{K})\mathbf{x}\end{aligned}$$

- negative definite

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{K}\mathbf{x}$$



- LMI-based Conditions for Global Exponential Stability (GES):

$$\mathbf{P} > 0$$

$$\mathbf{P}(\mathbf{A} + \mathbf{B}\mathbf{K}) + (\star)^T < 0$$

$$[\text{He}(\mathbf{P}\mathbf{A} + \mathbf{P}\mathbf{B}\mathbf{K}) < 0]$$

$$\overleftarrow{\overrightarrow{Q = \mathbf{P}^{-1}}}$$

$$\mathbf{Q}\mathbf{P}\mathbf{Q} = \mathbf{Q} > 0$$

$$\text{He}(\mathbf{A}\mathbf{Q} + \mathbf{B}\mathbf{K}\mathbf{Q}) < 0$$

$$\text{He}(\mathbf{A}\mathbf{Q} + \mathbf{B}\mathbf{X}) < 0$$

Gain selection: $K = XQ^{-1}$

- Left problem (\mathbf{P}, \mathbf{K}) is **nonlinear**. Right problem ($\mathbf{Q}, \mathbf{X} = \mathbf{K}\mathbf{Q}$) is **linear**

Deadzone nonlinearity more convenient than saturation

- **Sector Condition** Khalil [2002]: for all $x \in \mathbb{R}^n$,
 $(Kx - \text{sat}(Kx))^T W \text{sat}(Kx) \geq 0, \forall W > 0$ diagonal

$$\text{dz}(Kx)^T W(Kx - \text{dz}(Kx)) \geq 0, \forall W > 0$$

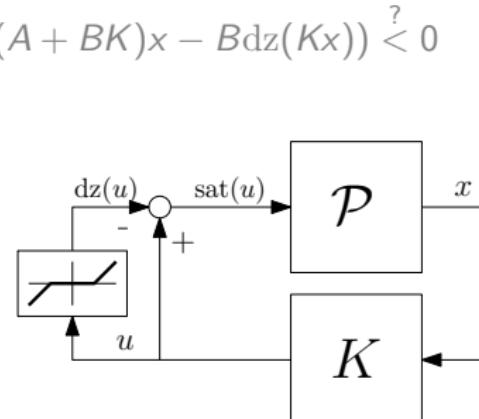
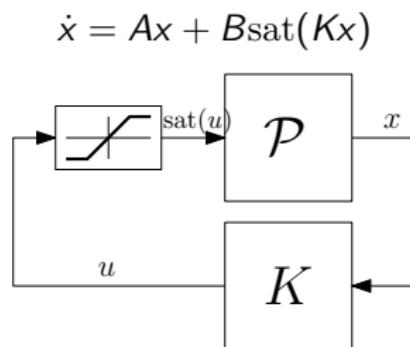
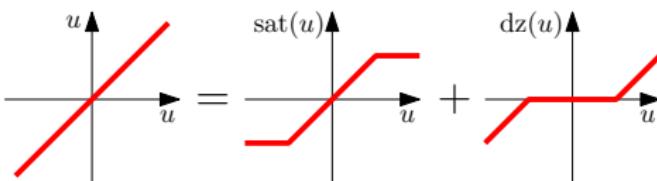
- **Lyapunov-based GES design:**
 $\mathbf{V}(x) = x^T Px > 0$ and radially unbounded

$$\dot{\mathbf{V}}(x) = 2x^T P(Ax + B\text{sat}(Kx)) \stackrel{?}{<} 0$$

$$= 2x^T P(Ax + B(Kx - \text{dz}(Kx))) = 2x^T P((A + BK)x - B\text{dz}(Kx)) \stackrel{?}{<} 0$$

- **Deadzone as the essential nonlinearity:**

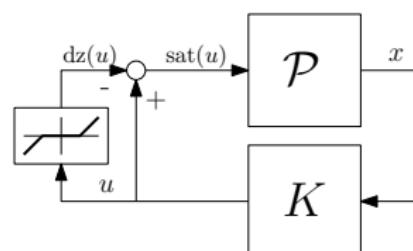
$$u = \text{sat}(u) + \text{dz}(u)$$



Global LMI-based saturated linear state feedback

- **Sector Condition Khalil [2002]:** for all $x \in \mathbb{R}^n$,
 $(Kx - \text{sat}(Kx))^T W \text{sat}(Kx) \geq 0, \forall W > 0$ diagonal
 $\text{dz}(Kx)^T W(Kx - \text{dz}(Kx)) \geq 0, \forall W > 0$ diagonal

$$\dot{x} = (A + BK)x - B\text{dz}(Kx)$$



- **Lyapunov-based GES design:**
 $V(x) = x^T Px > 0$ and radially unbounded

$$\begin{aligned} \dot{V}(x) &= 2x^T P((A + BK)x - B\text{dz}(Kx)) \\ &\leq 2x^T (P(A + BK)x - PB\text{dz}(Kx)) + 2\text{dz}(Kx)^T W(Kx - \text{dz}(Kx)) \\ &= 2 \begin{bmatrix} x \\ \text{dz}(Kx) \end{bmatrix}^T \begin{bmatrix} P(A + BK) & -PB \\ WK & -W \end{bmatrix} \begin{bmatrix} x \\ \text{dz}(Kx) \end{bmatrix} \stackrel{?}{<} 0 \end{aligned}$$

- **LMI-based Globally Exponentially Stabilizing (GES) design:**

$P > 0, W > 0$ diagonal

$$\text{He} \begin{bmatrix} PA + PBK & -PB \\ WK & -W \end{bmatrix} < 0$$

$$\overleftarrow{\overrightarrow{Q = P^{-1},}} \quad \overleftarrow{\overrightarrow{U = W^{-1}}}$$

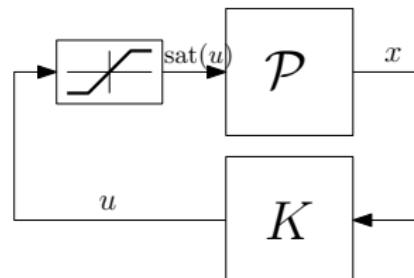
$Q > 0, U > 0$ diagonal

$$\text{He} \begin{bmatrix} A\mathbf{Q} + BK\mathbf{Q} & -B\mathbf{U} \\ K\mathbf{Q} & -U \end{bmatrix} < 0$$

- Left problem (P, K, W) **nonlinear**. Right problem $(Q, X = KQ, U)$ **linear**

Saturation is overkill with global exponential stability

- Saturation: **an abrupt nonlinearity**:
 - Small signals: $\text{sat}(u) = u \Rightarrow$ no effect
 - Large signals: $\text{sat}(u)$ bounded \Rightarrow severe effect



- GES is an **intrinsically defective** property:

$$\begin{bmatrix} I & -B \\ 0 & I \end{bmatrix} \left(\text{He} \begin{bmatrix} A\mathbf{Q} + BK\mathbf{Q} & -B\mathbf{U} \\ K\mathbf{Q} & -\mathbf{U} \end{bmatrix} \right) \begin{bmatrix} I & 0 \\ -B^T & I \end{bmatrix} = \text{He} \begin{bmatrix} \mathbf{A}\mathbf{Q} & \mathbf{B}\mathbf{U} \\ K\mathbf{Q} & -\mathbf{U} \end{bmatrix} < 0$$

- **Orange** term $\mathbf{A}\mathbf{Q}$ requires \mathbf{A} Hurwitz ($\mathbf{A}\mathbf{Q} + \mathbf{Q}\mathbf{A}^T < 0$):
 - This means that the plant is already GES before feedback (bummer!)
- This limitation is not surprising [Sontag \[1984\]](#), [Lasserre \[1992\]](#)
 - *Global exponential* stability cannot be gained from a bounded input
 - *Global asymptotic* stability possible (if and) only if \mathbf{A} is not exp. unstable

\Rightarrow Need non-global sector conditions [Gomes da Silva Jr and Tarbouriech \[2005\]](#)

$$\text{dz}(\mathbf{H}\mathbf{x}) = 0 \quad \Rightarrow \quad \text{dz}(\mathbf{K}\mathbf{x})^T \mathbf{W}(\mathbf{K}\mathbf{x} - \text{dz}(\mathbf{K}\mathbf{x}) + \mathbf{H}\mathbf{x}) \geq 0$$

Regional LMI-based saturated linear state feedback

- **Regional Sector Condition:**

$$dz(Hx) = 0 \Rightarrow dz(Kx)^T W(Kx - dz(Kx) + Hx) \geq 0,$$

$$\dot{x} = (A + BK)x - Bdz(u)$$

$$u = Kx$$

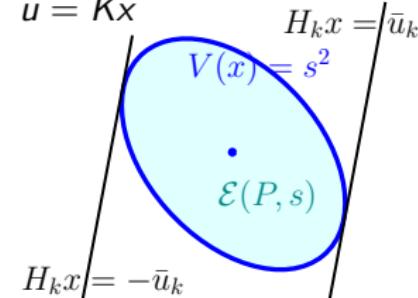
$$H_k x = \bar{u}_k$$

- **Lyapunov-based RES design** $V(x) = x^T Px$

$$V(x) = x^T Px \leq s^2 \implies |H_k x|^2 \leq \bar{u}_k^2$$

$$x^T \frac{H_k^T H_k}{\bar{u}_k^2} x < x^T \frac{P}{s^2} x \Leftrightarrow \begin{bmatrix} P & H_k^T \\ H_k & \frac{\bar{u}_k^2}{s^2} \end{bmatrix} > 0$$

$$\dot{V}(x) = 2x^T P((A + BK)x - Bdz(Kx))$$



$$\leq 2x^T (P(A + BK)x - PBdz(Kx)) + 2dz(Kx)^T W(Kx - dz(Kx) + Hx)$$

$$= 2 \begin{bmatrix} x \\ dz(Kx) \end{bmatrix}^T \begin{bmatrix} P(A + BK) & -PB \\ WK + WH & -W \end{bmatrix} \begin{bmatrix} x \\ dz(Kx) \end{bmatrix} \stackrel{?}{<} 0$$

- **LMI-based Regionally Exponentially Stabilizing (RES) design:**

$$\begin{bmatrix} P & H_k^T \\ H_k & \frac{\bar{u}_k^2}{s^2} \end{bmatrix} > 0, \quad k = 1, \dots, m \quad W > 0 \text{ diagonal}$$

$$\text{He} \begin{bmatrix} PA + PBK & -PB \\ WK + WH & -W \end{bmatrix} < 0$$

$$\overleftarrow{\overrightarrow{Q = P^{-1},}} \quad \overleftarrow{\overrightarrow{U = W^{-1}}}$$

$$\begin{bmatrix} Q & QH_k^T \\ H_k Q & \frac{\bar{u}_k^2}{s^2} \end{bmatrix} > 0, \quad k = 1, \dots, m \quad U > 0 \text{ diagonal}$$

$$\text{He} \begin{bmatrix} AQ + BKQ & -BU \\ KQ + HQ & -U \end{bmatrix} < 0$$

Adding a nonlinear algebraic loop improves performance

Mulder et al. [2001], Zaccarian and Teel [2002], Hu et al. [2006]

- Regional Sector Condition:**

$$\text{dz}(u)^T W \underbrace{(Kx + \text{Ldz}(u))}_{=u} - \text{dz}(u) + \text{Hx} \geq 0,$$

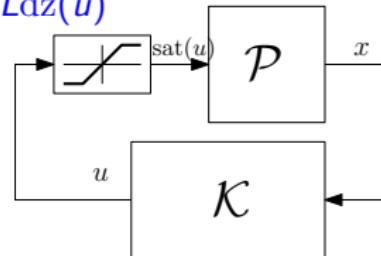
$$\begin{aligned}\dot{x} &= Ax + B(Kx + \text{Ldz}(u)) - B\text{dz}(u) \\ &= (A + BK)x - B\text{dz}(u) - B\text{Ldz}(u) \\ u &= Kx + \text{Ldz}(u)\end{aligned}$$

- Lyapunov-based RES design**

$$\mathbf{V}(x) = x^T Px \leq s^2 \implies |\text{H}_k x|^2 \leq \bar{u}_k^2$$

$$\begin{aligned}\dot{\mathbf{V}}(x) &= 2x^T P((A + BK)x - B\text{dz}(u) + \text{BLdz}(u)) \\ &\leq 2x^T (P(A + BK)x - PB\text{dz}(u) + \text{PBLdz}(u)) \\ &\quad + 2\text{dz}(Kx)^T W(Kx + \text{Ldz}(u) - \text{dz}(Kx) + \text{Hx})\end{aligned}$$

$$= 2 \begin{bmatrix} x \\ \text{dz}(Kx) \end{bmatrix}^T \begin{bmatrix} P(A + BK) & \text{PBL} - PB \\ WK + WH & WL - W \end{bmatrix} \begin{bmatrix} x \\ \text{dz}(Kx) \end{bmatrix} \stackrel{?}{<} 0$$



- LMI-based Regionally Exponentially Stabilizing (RES) design:**

$$\begin{bmatrix} P & H_k^T \\ H_k & \frac{\bar{u}_k^2}{s^2} \end{bmatrix} > 0, \quad k = 1, \dots, m$$

$W > 0$ diagonal

$$\text{He} \begin{bmatrix} PA + PBK & PBL - PB \\ WK + WH & WL - W \end{bmatrix} < 0$$

$$\begin{array}{c} Q = P^{-1}, \\ \text{U} = W^{-1} \end{array}$$

$$\begin{bmatrix} Q & QH_k^T \\ H_k Q & \frac{\bar{u}_k^2}{s^2} \end{bmatrix} > 0, \quad k = 1, \dots, m$$

$U > 0$ diagonal

$$\text{He} \begin{bmatrix} AQ + BKQ & BLU - BU \\ KQ + HQ & LU - U \end{bmatrix} < 0$$

Properties and solution to the nonlinear algebraic loop

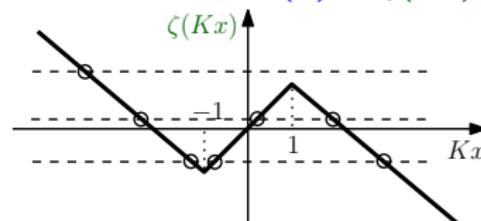
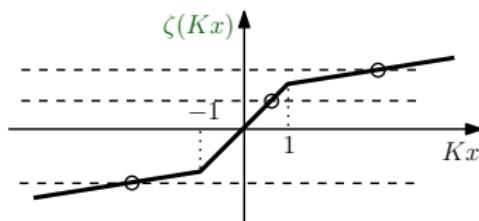
Zaccarian and Teel [2002], Hu et al. [2006]

- Regional Sector Condition:**

$$dz(u)^T W(Kx + Ldz(u) - dz(u) + Hx) \geq 0,$$

$$\dot{x} = Ax + B(Kx + Ldz(u)) - Bdz(u)$$

$$u = Kx + Ldz(u) = \zeta(Kx)$$



- Piecewise affine solution** $\zeta(\cdot)$ invertible & globally Lipschitz [Hu et al. \[2006\]](#)

- if and only if $\det(I - L\Delta) > 0$, $\forall \Delta = \text{diag}\{\delta_1, \dots, \delta_m\}$, with $\delta_i \in \{0, 1\}$
- if $\exists U > 0$ diagonal such that $\text{He}(LU - U) = LU + UL^T - 2U < 0$

- Sufficiency** is guaranteed by all the LMI-based designs (lower right LMI)

- Solution** of the algebraic loop is a QP [Syaichu-Rohman et al. \[2003\]](#) and can be computed in various ways [Riz et al. \[2020\]](#)

- iteratively by **Simulink**
- by **enumeration** of all regions
- by **Dynamic augmentation**:

$$\dot{u} = -dz^*(u - Kx - Ldz(u))$$

$$\begin{bmatrix} Q & QH_k^T \\ H_k Q & \frac{\bar{u}_k^2}{s^2} \end{bmatrix} > 0, \quad k = 1, \dots, m$$

$$\text{He} \begin{bmatrix} AQ + BKQ & BLU - BU \\ KQ + HQ & LU - U \end{bmatrix} < 0$$

Switched and Scheduled Quadratic Constructions

Zaccarian and Teel [2004], Cristofaro et al. [2019], Valmorbida et al. [2017b]

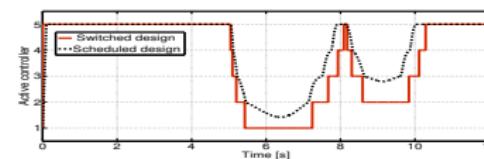
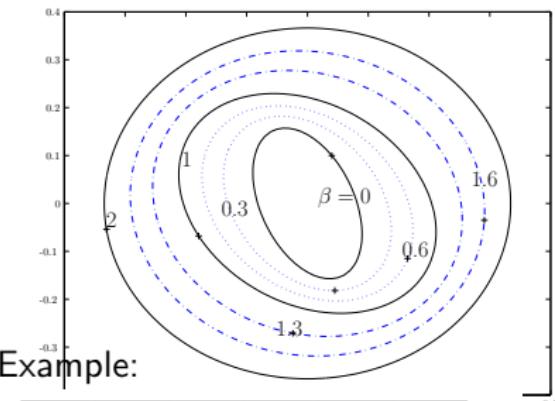
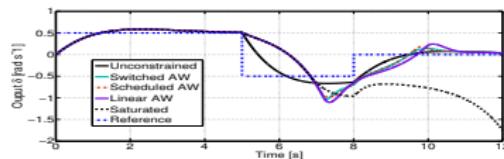
- Building nested ellipsoids with increasing performance levels
(e.g., exponential convergence α) parametrized by $\beta \in \mathbb{R}_{\geq 0}$

$$\text{He} \begin{bmatrix} (A + \alpha(\beta)I)Q(\beta) + BX_1(\beta) & BX_2(\beta) - BU(\beta) \\ X_1(\beta) + Y(\beta) & X_2(\beta) - U(\beta) \end{bmatrix} < 0$$

$$u = K(\beta)x + L(\beta)dz(u), \quad K(\beta) = X_1(\beta)Q(\beta)^{-1}, \quad L(\beta) = X_2(\beta)Q(\beta)^{-1}$$

- Three possible approaches:
 - Switched solution $\beta \in \mathbb{Z}$
 - Scheduled interpolated solution
 - Scheduled polynomial solution
(rational feedback gains)

- Tailless Advanced Fighter Aircraft (TAFA) Example:



\mathcal{H}_∞ (or \mathcal{L}_2) performance and Bounded Real Lemma

- \mathcal{L}_2 norm measures the input w and output z size

$$z \in \mathcal{L}_2 \quad \text{if} \quad \|z\|_2 := \left(\int_0^\infty |z(t)|^2 dt \right)^{\frac{1}{2}} < \infty$$

- **Global** finite \mathcal{L}_2 gain (linear \mathcal{H}_∞ norm) is $\bar{\gamma}_{wz}$

$$\|z\|_2 \leq \bar{\gamma}_{wz} \|w\|_2 \quad \text{for all } w \in \mathcal{L}_2$$

- **Lyapunov-based** characterization of $\bar{\gamma}_{wz}$ using $V(x) = x^T Px > 0$

$$\dot{V} + \bar{\gamma}_{wz}^{-2} z^T z - |w|^2 < 0$$

$$= 2x^T P(Ax + B_w w) + \bar{\gamma}_{wz}^{-2} ([c_z \ D_{zw}] [\begin{smallmatrix} x \\ w \end{smallmatrix}])^T [c_z \ D_{zw}] [\begin{smallmatrix} x \\ w \end{smallmatrix}] - |w|^2 < 0$$

$$\Rightarrow \int_0^t |z(\tau)|^2 d\tau \leq \bar{\gamma}_{wz}^2 V(x(t)) + \int_0^t |z(\tau)|^2 d\tau < \bar{\gamma}_{wz}^2 V(x(0)) + \bar{\gamma}_{wz}^2 \int_0^t |w(\tau)|^2 d\tau$$

- **Bounded real lemma** provides LMI-based expression of $\bar{\gamma}_{wz}$

$$\bar{\gamma}_{wz}^2 = \inf_{\substack{P=P^T > 0, \\ \gamma^2 > 0}} \gamma^2, \quad \text{subject to} \quad \text{He} \begin{bmatrix} PA & PB_w & 0 \\ 0 & -\frac{1}{2}I & 0 \\ C_z & D_{zw} & -\frac{\gamma^2}{2}I \end{bmatrix} < 0$$

\mathcal{H}_∞ (or \mathcal{L}_2) performance and Bounded Real Lemma

- \mathcal{L}_2 norm measures the input w and output z size

$$z \in \mathcal{L}_2 \quad \text{if} \quad \|z\|_2 := \left(\int_0^\infty |z(t)|^2 dt \right)^{\frac{1}{2}} < \infty$$

- **Global** finite \mathcal{L}_2 gain (linear \mathcal{H}_∞ norm) is $\bar{\gamma}_{wz}$

$$\|z\|_2 \leq \bar{\gamma}_{wz} \|w\|_2 \quad \text{for all } w \in \mathcal{L}_2$$

- **Lyapunov-based** characterization of $\bar{\gamma}_{wz}$ using $V(x) = x^T Px > 0$

$$\dot{V} + \bar{\gamma}_{wz}^{-2} z^T z - |w|^2 < 0$$

$$= 2x^T P(Ax + B_w w) + \bar{\gamma}_{wz}^{-2} ([c_z \ D_{zw}] [\begin{smallmatrix} x \\ w \end{smallmatrix}])^T [c_z \ D_{zw}] [\begin{smallmatrix} x \\ w \end{smallmatrix}] - |w|^2 < 0$$

$$\Rightarrow \int_0^t |z(\tau)|^2 d\tau \leq \bar{\gamma}_{wz}^2 V(x(t)) + \int_0^t |z(\tau)|^2 d\tau < \bar{\gamma}_{wz}^2 V(x(0)) + \bar{\gamma}_{wz}^2 \int_0^t |w(\tau)|^2 d\tau$$

- **Bounded real lemma (Dual)** provides LMI-based expression of $\bar{\gamma}_{wz}$

$$\bar{\gamma}_{wz}^2 = \inf_{\substack{Q=Q^T > 0, \\ \gamma^2 > 0}} \gamma^2, \quad \text{subject to} \quad \text{He} \begin{bmatrix} A\mathbf{Q} & B_w & 0 \\ 0 & -\frac{1}{2}I & 0 \\ C_z Q & D_{zw} & -\frac{\gamma^2}{2}I \end{bmatrix} < 0$$

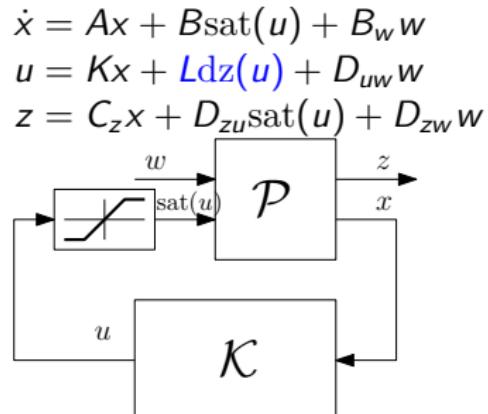
Performance with saturation depends on size of disturbance

- Saturation: **an abrupt nonlinearity**:
 - Small signals: $\text{sat}(u) = u \Rightarrow$ no effect
 - Large signals: $\text{sat}(u)$ bounded \Rightarrow severe effect
- Need **nonglobal** \mathcal{L}_2 performance metric
- Closed-loop performance metrics:
 - **Global** Finite \mathcal{L}_2 gain (linear \mathcal{H}_∞ norm):

$$\|z\|_2 \leq \bar{\gamma}_{wz} \|w\|_2 \quad \text{for all } w \in \mathcal{L}_2$$
 - **Nonlinear \mathcal{L}_2 gain**: Megretski [1996] a function $s \mapsto \gamma_{wz}(s)$:

$$\|z\|_2 \leq \gamma_{wz}(s) \|w\|_2 \quad \text{for all } w \text{ satisfying } \|w\|_2 \leq s$$
- **LMI-based estimation** of $\gamma_{wz}^2(s) = \inf_{Q=Q^T > 0, \dots} \gamma^2$ subject to

$$\text{He} \begin{bmatrix} A\mathbf{Q} + B\mathbf{K}\mathbf{Q} & BLU - BU \\ K\mathbf{Q} + HQ & LU - U \\ \hline 0 & 0 \\ C_z\mathbf{Q} + C_z\mathbf{K}\mathbf{Q} & D_{zu}LU - D_{zu}U \end{bmatrix} \begin{bmatrix} B_w & 0 \\ D_{uw} & 0 \\ \hline -\frac{1}{2}I & 0 \\ D_{zw} & -\frac{\gamma^2}{2}I \end{bmatrix} < 0, \quad \begin{bmatrix} \mathbf{Q} & QH_k^T \\ H_k\mathbf{Q} & \frac{\bar{u}_k^2}{s^2} \\ \hline k = 1, \dots, m \end{bmatrix} > 0, \quad \mathbf{U} > 0 \text{ diagonal}$$

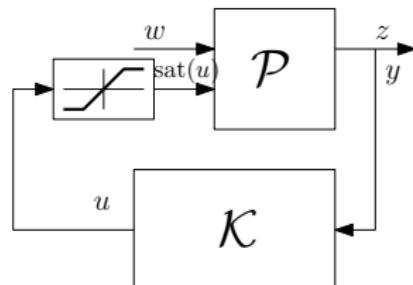


Example shows possible nonlinear gains trends

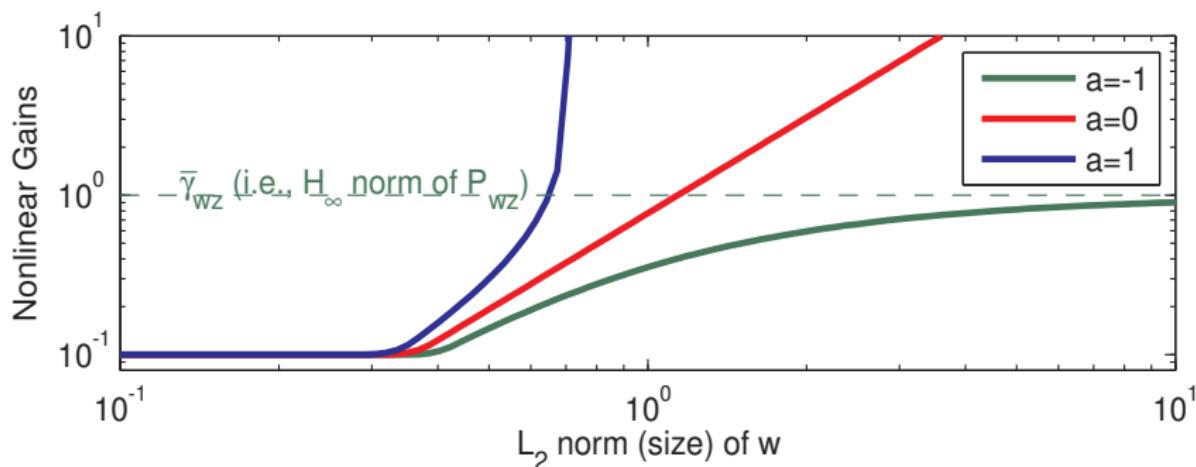
Controller \mathcal{K} cancels the plant dynamics
and **stabilizes** (before saturation)

$$\mathcal{P} : \dot{z} = az + \text{sat}(u) + w$$

$$\mathcal{K} : u = -az - 10z$$

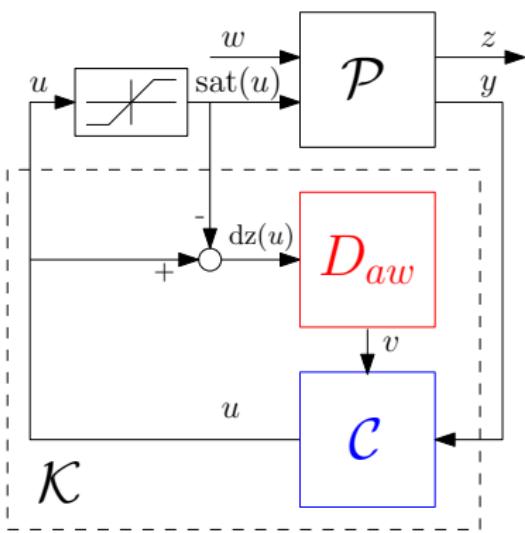


Three representative cases Sontag [1984], Lasserre [1992]



\mathcal{H}_∞ design generalizes to linear input-saturated plants

Dai et al. [2009a]



- Given \mathcal{P} linear, **design** \mathcal{K} , namely
 - \mathcal{C} linear plant-order
 - \mathcal{AW} static: linear gain
 - Performance objective:**
given s^* , minimize $\gamma_{wz}(s^*)$
 - Linear controller \mathcal{K} equations**

$$\dot{x}_c = Ax_c + By + D_{aw,1}dz(u)$$

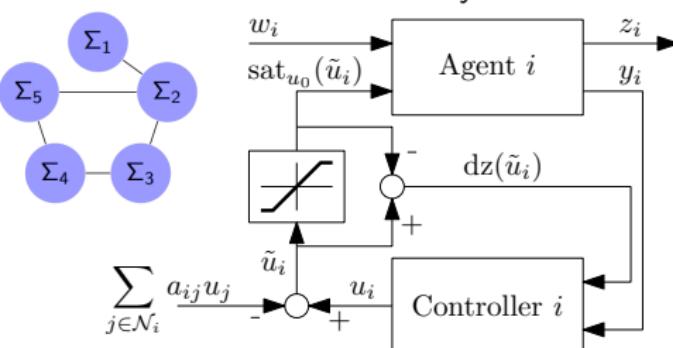
$$u = Cx_c + Dy + D_{aw,2}dz(u)$$
- (recall the **nonlinear algebraic loop!**)

- Synthesis is a convex problem (generalizes LMI- \mathcal{H}_∞ Gahinet and Apkarian [1994], Iwasaki and Skelton [1994])
- Synthesis without \mathcal{AW} is nonconvex
- Generalized hybrid approaches for sampled-data design Dai et al. [2010]

\mathcal{H}_∞ paradigm applies to bounded synchronization

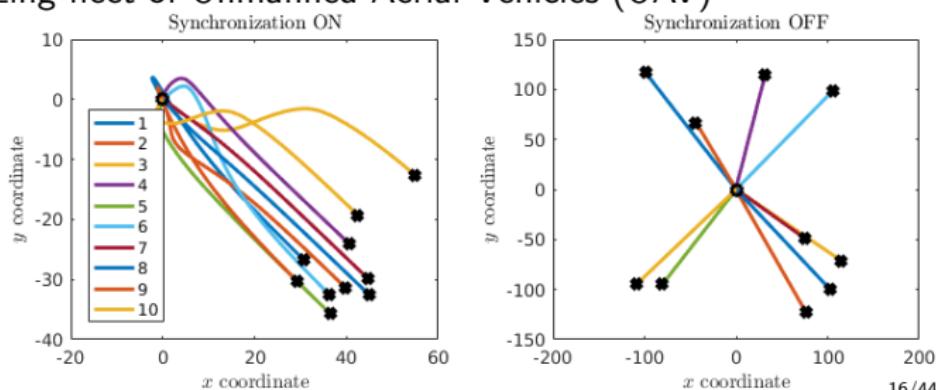
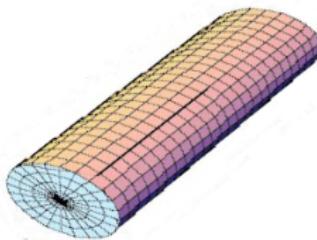
Dal Col et al. [2019]

- Decentralized bounded synchronization of identical linear agents



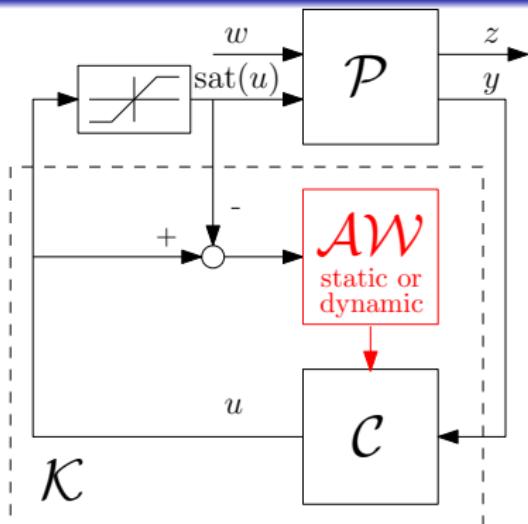
- Performance objective:** given s^* , minimize $\tilde{\gamma}_{wz}(s^*)$
 - Distributed controller equations**
- $$\dot{x}_{ci} = Ax_{ci} + By_i + D_{aw,1}dz(\tilde{u}_i)$$
- $$u_i = Cx_{ci} + Dy_i + D_{aw,2}dz(\tilde{u}_i)$$

- Example of synchronizing fleet of Unmanned Arial Vehicles (UAV)



Direct Linear Anti-Windup (DLAW) design is also convex

Grimm et al. [2003a], Gomes da Silva Jr and Tarbouriech [2005], Hu et al. [2008]

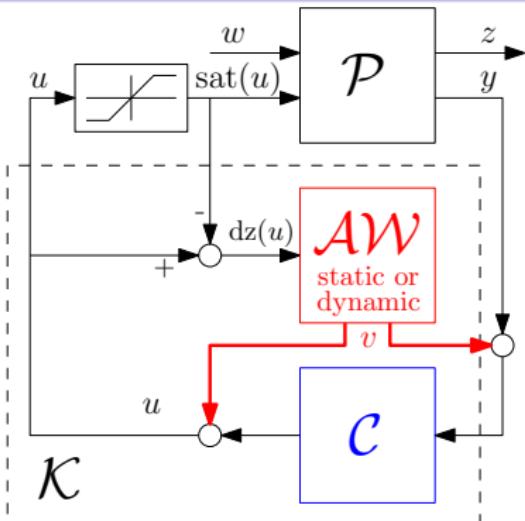


- Given \mathcal{P} linear, \mathcal{C} linear, **only** design
 - AW linear static or plant-order
- Performance objective:**
given s^* , minimize $\gamma_{wz}(s^*)$
- Necessary conditions:**
 - linear feedback $(\mathcal{P}, \mathcal{C})$ exp stable
 - $(\exists \mathcal{V}([\begin{smallmatrix} x_p \\ x_c \end{smallmatrix}]) = [\begin{smallmatrix} x_p \\ x_c \end{smallmatrix}]^T [\begin{smallmatrix} Q_p & Q_{pc} \\ Q_{pc}^T & Q_c \end{smallmatrix}]^{-1} [\begin{smallmatrix} x_p \\ x_c \end{smallmatrix}])$
 - $\exists \mathcal{F}$ s.t. $A + BF$ exp stable $(\mathcal{V}_F(x_p) = x_p^T \overline{Q}_p^{-1} x_p, |\mathcal{F}| \xrightarrow{s^* \rightarrow \infty} 0)$

- Static anti-windup construction (convex, LMIs)
 - feasible if $Q_p = \overline{Q}_p$: quasi-common quadratic Lyapunov function
- Plant-order anti-windup construction (convex, LMIs)
 - always feasible as long as \mathcal{V}_F and \mathcal{V} above exist
- Reduced order design (BMIs)** Galeani et al. [2006] and design for **rate saturation** (LMIs) Galeani et al. [2008]

External Anti-Windup design: same feasibility conditions

Grimm et al. [2004a]



- Given \mathcal{P} linear, \mathcal{C} linear, **only** design
 - $\textcolor{red}{AW}$ linear static or plant-order
- Performance objective:**
given s^* , minimize $\gamma_{wz}(s^*)$
- Necessary conditions:**
 - linear feedback $(\mathcal{P}, \mathcal{C})$ exp stable
 $(\exists \mathcal{V}([\begin{smallmatrix} x_p \\ x_c \end{smallmatrix}]) = [\begin{smallmatrix} x_p \\ x_c \end{smallmatrix}]^T \begin{bmatrix} Q_p & Q_{pc} \\ Q_{pc}^T & Q_c \end{bmatrix}^{-1} [\begin{smallmatrix} x_p \\ x_c \end{smallmatrix}])$
 - $\exists F$ s.t. $A + BF$ exp stable
 $(V_F(x_p) = x_p^T \overline{Q}_p^{-1} x_p, |F| \xrightarrow{s^* \rightarrow \infty} 0)$

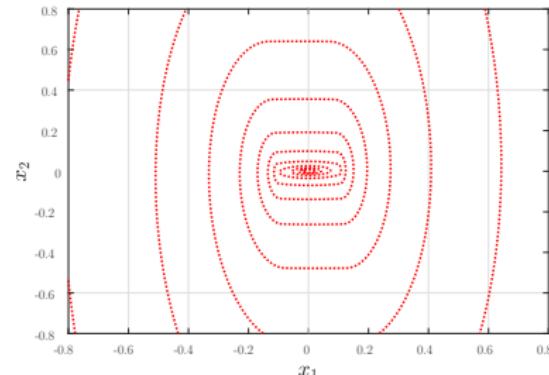
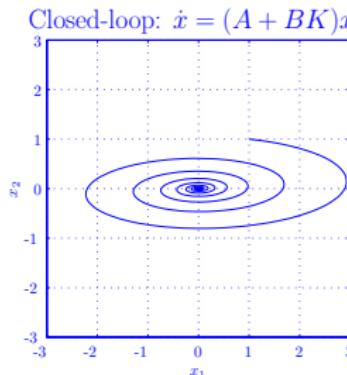
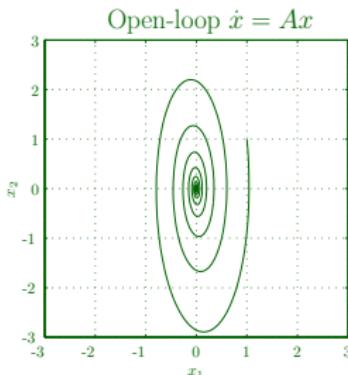
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 - feasible if $Q_p = \overline{Q}_p$: quasi-common quadratic Lyapunov function
- Plant-order anti-windup construction (convex, LMIs)
 - always feasible as long as V_F and \mathcal{V} above exist

Static quadratic feasibility conditions are quite restrictive

Zaccarian and Teel [2011]

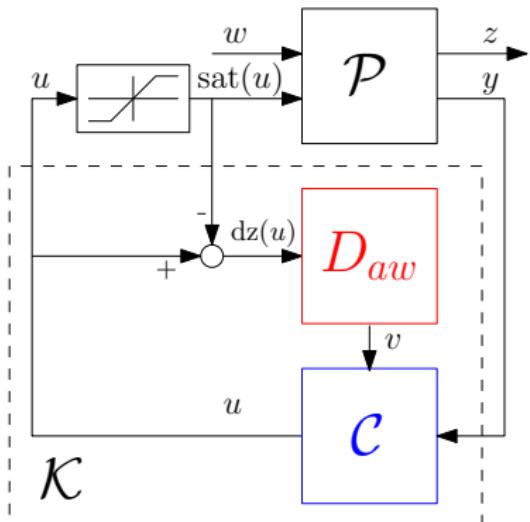
- Planar example: **static anti-windup** with quadratic certificates is **infeasible**
- Planar example: (Quadratic) **Plant-order anti-windup** is **feasible**
- Dynamics shows twofold behavior between open- and closed-loop:

$$\dot{x} = \underbrace{\begin{bmatrix} -0.05 & 1 \\ -10 & -0.5 \end{bmatrix}}_A x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_B \text{sat}\left(\underbrace{\begin{bmatrix} 9.9 & 0.495 \end{bmatrix}}_K x\right) = \underbrace{\begin{bmatrix} -0.05 & 1 \\ -0.1 & -0.005 \end{bmatrix}}_{A+BK} x - B d z(u)$$



Direct Linear static anti-windup design (LMI)

Mulder et al. [2001], Grimm et al. [2003a], Gomes da Silva Jr and Tarbouriech [2005], Hu et al. [2008]



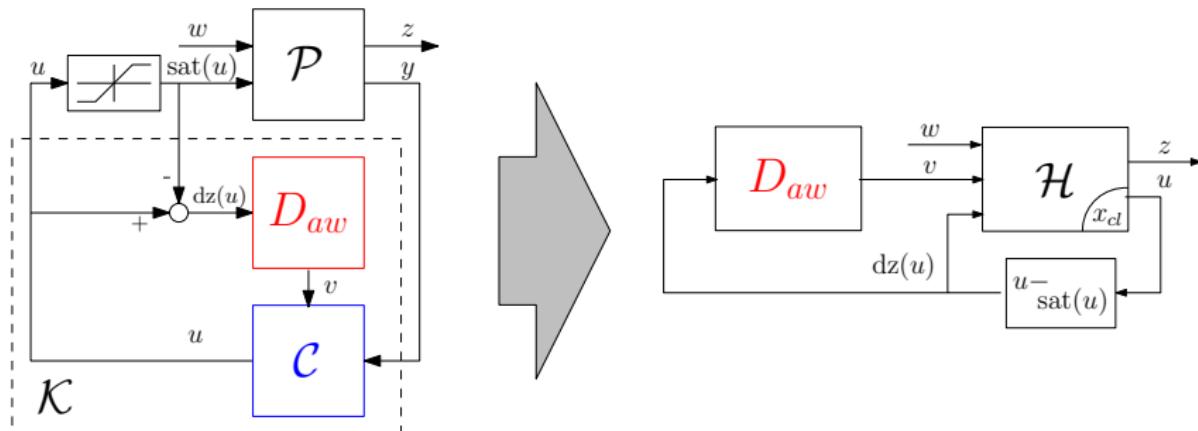
- Given \mathcal{P} linear, \mathcal{C} linear, **design** only
 - linear anti-windup gain $D_{aw} = \begin{bmatrix} D_{aw,1} \\ D_{aw,2} \end{bmatrix}$
- Performance objective:**
given s^* , minimize $\gamma_{wz}(s^*)$
- Linear **controller** \mathcal{K} equations

$$\dot{x}_c = Ax_c + By + D_{aw,1}(u - \text{sat}(u))$$

$$y_c = Cx_c + Dy + D_{aw,2}(u - \text{sat}(u))$$
- LMI-based design Mulder et al. [2001], Grimm et al. [2003a], Gomes da Silva Jr and Tarbouriech [2005], Hu et al. [2008]

- Preserve** of *small signal response* (D_{aw} multiplies $dz(u) = u - \text{sat}(u)$)
Asymptotically recover *large signal response* (global not always possible)
- Results generalize nontrivially to the plant-order case

Compact representation of the closed-loop system



$$\mathcal{H} : \left\{ \begin{array}{lcl} \dot{x}_{cl} & = & A_{cl}x_{cl} + B_{cl,d}(u - \text{sat}(u)) + B_{cl,v}v + B_{cl,w}w \\ u & = & C_{cl,u}x_p + D_{cl,ud}(u - \text{sat}(u)) + D_{cl,uv}v + D_{cl,uw}w \\ z & = & C_{cl,z}x_p + D_{cl,zd} \underbrace{(u - \text{sat}(u))}_{\text{dz}(u)} + D_{cl,zv}v + D_{cl,zw}w, \end{array} \right.$$

Quadratic analysis conditions are convex

Mulder et al. [2001], Gomes da Silva Jr and Tarbouriech [2005], Hu et al. [2008]

Proposition: Given the NOMINAL system and $s > 0$, if the LMI problem

$$\gamma_{wz}^2(s) = \min_{\{\gamma^2, Q, Y, U\}} \gamma^2 \text{ subject to } Q = Q^T > 0, \quad U > 0 \text{ diagonal,}$$

$$\text{He} \begin{bmatrix} A_{cl}Q & B_{cl,d}U + B_{cl,v}D_{aw}U + Y^T & B_{cl,w} & 0 \\ C_{cl,u}Q & D_{cl,ud}U + D_{cl,uv}D_{aw}U - U & D_{cl,uw} & 0 \\ 0 & 0 & -I/2 & 0 \\ C_{cl,z}Q & D_{cl,zd}U + D_{cl,zv}D_{aw}U & D_{cl,zw} & -\frac{\gamma^2}{2}I \end{bmatrix} < 0, \quad \begin{bmatrix} Q & Y_{[k]}^T \\ Y_{[k]} & \bar{u}_k^2/s^2 \end{bmatrix} > 0, \quad k = 1, \dots, n_u$$

is feasible, then the following holds for the saturated closed-loop:

- ① **[Stab]** the origin is **locally exponentially stable** with region of attraction containing the set $\mathcal{E}(Q, s) := \{x : x^T Q^{-1} x \leq s^2\}$;
- ② **[Reach]** the **reachable set** from $x(0) = 0$ with $\|w\|_2 \leq s$ is contained in $\mathcal{E}(Q, s)$;
- ③ **[\mathcal{L}_2 Perf]** for each w such that $\|w\|_2 \leq s$, the zero state solution satisfies the \mathcal{L}_2 gain bound:

$$\|z\|_2 \leq \gamma_{wz}(s) \|w\|_2$$

Quadratic analysis conditions easily lead to synthesis

Mulder et al. [2001], Gomes da Silva Jr and Tarbouriech [2005], Hu et al. [2008]

Proposition: Given the NOMINAL system and $s > 0$. If the LMI problem

$$\gamma_{wz}^2(s) = \min_{\{\gamma^2, Q, Y, U\}} \gamma^2 \text{ subject to } Q = Q^T > 0, \quad U > 0 \text{ diagonal,}$$

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Quadratic synthesis conditions are convex

Mulder et al. [2001], Gomes da Silva Jr and Tarbouriech [2005], Hu et al. [2008]

Proposition: Given the NOMINAL system and $s > 0$. If the LMI problem

$$\gamma_{wz}^2(s) = \min_{\{\gamma^2, Q, Y, U, X\}} \gamma^2 \text{ subject to } Q = Q^T > 0, \quad U > 0 \text{ diagonal,}$$

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is feasible, then, selecting the static AW gain as

$$D_{aw} = XU^{-1}$$

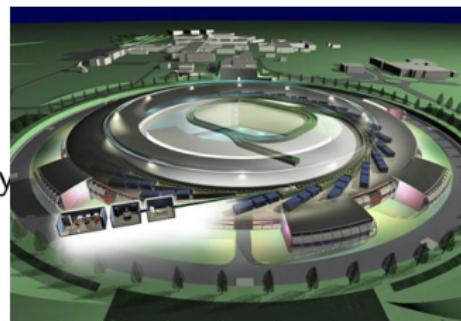
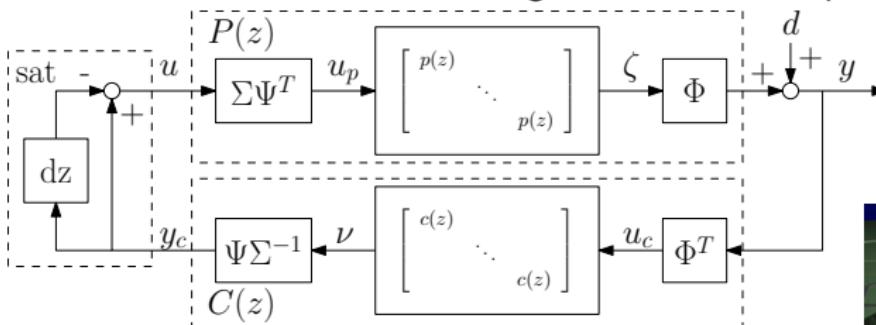
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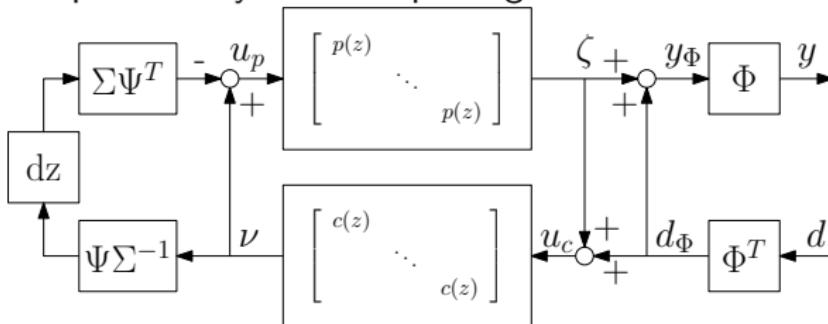
Cross-directional dynamics application: Synchrotron

Queinnec et al. [2015]

- Model is based on a suitable Singular Value Decomposition (SVD)



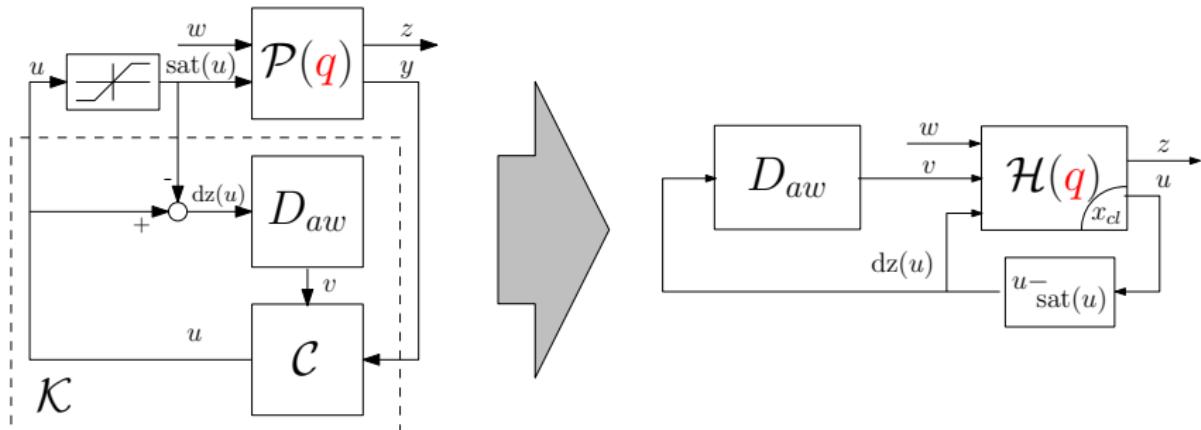
- Equivalent dynamics requires generalized nonlinearity



- Collaboration with Diamond Light Source synchrotron (Oxfordshire, UK)

Closed loop now depends on uncertain parameter $q \in \mathbb{Q}$

Turner et al. [2007], Grimm et al. [2004b], Formentin et al. [2017]



$$\mathcal{H}(q) : \left\{ \begin{array}{lcl} \dot{x}_{cl} & = & A_{cl}(q)x_{cl} + B_{cl,d}(q)(u - \text{sat}(u)) + B_{cl,v}(q)v + B_{cl,w}(q)w \\ u & = & C_{cl,u}(q)x_p + D_{cl,ud}(q)(u - \text{sat}(u)) + D_{cl,uv}(q)v + D_{cl,uw}(q)w \\ z & = & C_{cl,z}(q)x_p + D_{cl,zd}(q) \underbrace{(u - \text{sat}(u))}_{dz(u)} + D_{cl,zv}(q)v + D_{cl,zw}(q)w, \end{array} \right.$$

- Robust designs may follow a deterministic worst case paradigm, **imposing heavy convexity** conditions Turner et al. [2007], Grimm et al. [2004b]

Static AW synthesis based on scenario with certificates

Formentin et al. [2017]

Theorem (Robust static AW using scenario with certificates)

Fix a positive value $s \geq \|w\|_2$, $\varepsilon \in (0, 1)$, $\beta \in (0, 1)$, and select N satisfying $B(N, \varepsilon, n_\theta) \leq \beta$, with $n_\theta = 1 + n_u + n_u(n_u + n_c)$

Extract N samples of the uncertain matrices according to the probability distribution

Solve

$$\gamma_{wz}^2(s) = \min_{\{\gamma^2, U, X\}, \{Q_i, Y_i\}} \gamma^2, \quad \text{subject to } Q_i = Q_i^T > 0, \quad U > 0 \text{ diagonal,}$$

$$\text{He} \begin{bmatrix} A_{cl}^{(i)} Q_i & B_{cl,d}^{(i)} U + B_{cl,v}^{(i)} X + Y_i^T & B_{cl,w}^{(i)} & 0 \\ C_{cl,u}^{(i)} Q_i & D_{cl,ud}^{(i)} U + D_{cl,uv}^{(i)} X - U & D_{cl,uw}^{(i)} & 0 \\ 0 & 0 & -I/2 & 0 \\ C_{cl,z}^{(i)} Q_i & D_{cl,zd}^{(i)} U + D_{cl,zv}^{(i)} X & D_{cl,zw}^{(i)} & -\frac{\gamma^2}{2} I \end{bmatrix} < 0, \quad \begin{bmatrix} Q_i & Y_{i[k]}^T \\ Y_{i[k]} & \bar{u}_k^2/s^2 \end{bmatrix} > 0, \quad \forall k = 1, \dots, n_u \quad \forall i = 1, \dots, N$$

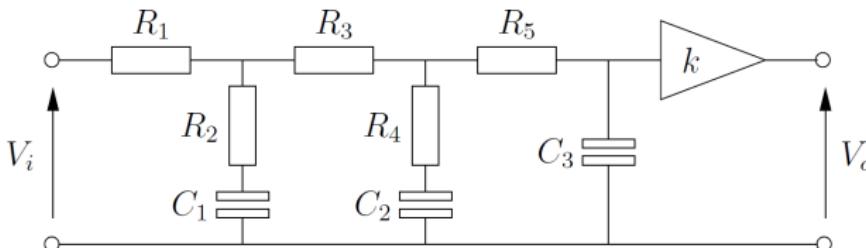
If the above LMIs are feasible, select the static anti-windup gain $D_{aw} = XU^{-1}$

Then, for each $\|w\|_2 < s$, the zero initial state solution of the closed loop satisfies $\Pr(\|z\|_2 > \gamma_{sc}(s) \|w\|_2) \leq \varepsilon$, with probability no smaller than $1 - \beta$.

Analysis conditions can also be easily formulated

Illustrative example: A passive network

Grimm et al. [2003b], Formentin et al. [2017]



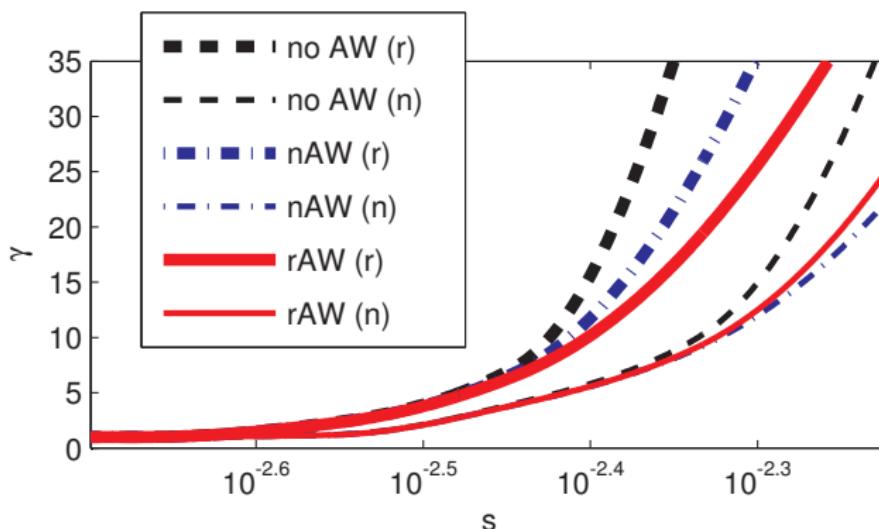
- Uncertain parameters with (known) Gaussian distribution

parameter	mean	std dev	parameter	mean	std dev
R_1	313Ω	± 10	R_5	$10 F$	± 10
R_2	20Ω	± 10	C_1	$0.01 F$	± 10
R_3	315Ω	± 10	C_2	$0.01 F$	± 10
R_4	17Ω	± 10	c_3	$0.01 F$	± 10

- Input generator voltage constrained:
 $u(t) = V_i(t) \in [-\bar{u}, \bar{u}] = [-1 \text{ Volt}, 1 \text{ Volt}]$
- Design parameters are $\varepsilon = 0.01$, $\beta = 10^{-6}$, $s = 0.003$, $n_\theta = 35$
 $\Rightarrow N = 2270$ (not 7565) for design based on sequential algorithm

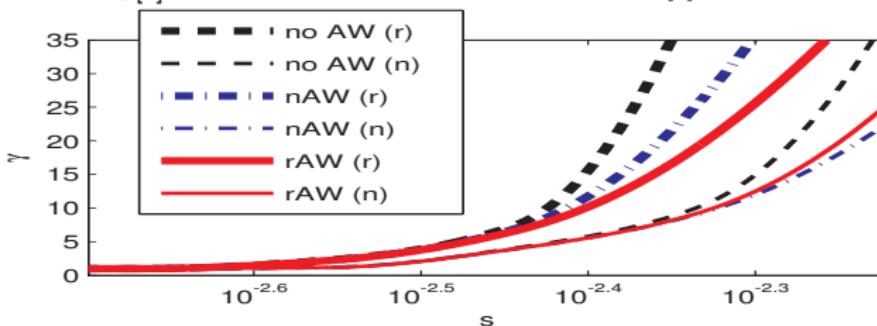
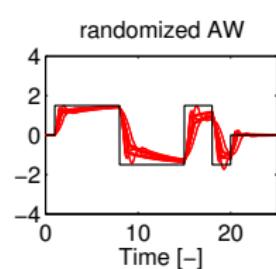
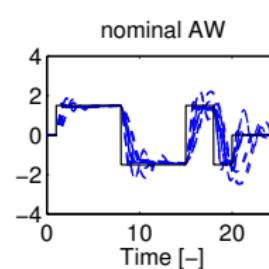
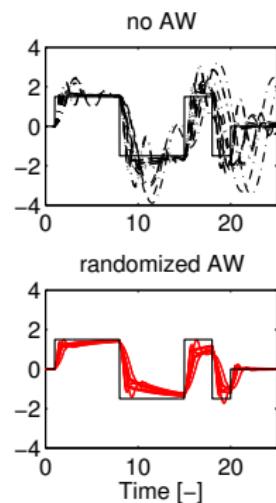
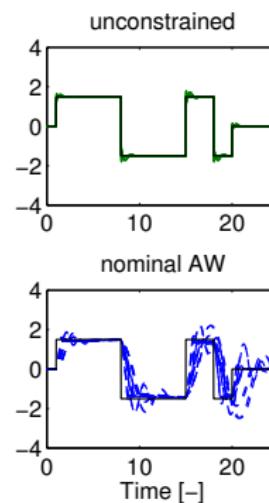
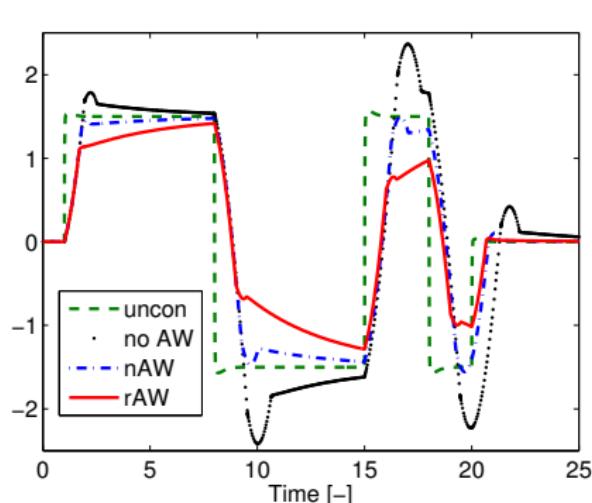
Deterministic and Randomized nonlinear \mathcal{L}_2 gains

- Robust compensator shows better robust performance (red curves)
- The nominal behavior slightly deteriorated (thin curves)



Without anti-windup (black dashed), with nominal anti-windup (blue dashed-dotted) and with robust anti-windup (red solid)

Time responses confirm nonlinear \mathcal{L}_2 gain trends



Nonlinear \mathcal{L}_2 gains are estimated using Lyapunov functions

Hu et al. [2006], Dai et al. [2009b], Garulli et al. [2013]

- (Scheduled) **Quadratic** functions

$$V_1(x) = x^T P x$$

- Max of quadratics (BMIs)

$$V_2(x) = \max_{j \in \{1, \dots, J\}} x^T P_j x$$

- Convex Hull of quadratics (BMIs)

$$V_3(x) = \min_{\gamma_j \geq 0: \sum_j \gamma_j = 1} x^T \left(\sum_j \gamma_j Q_j \right)^{-1} x$$

- Piecewise quadratic (LMI-BMI)

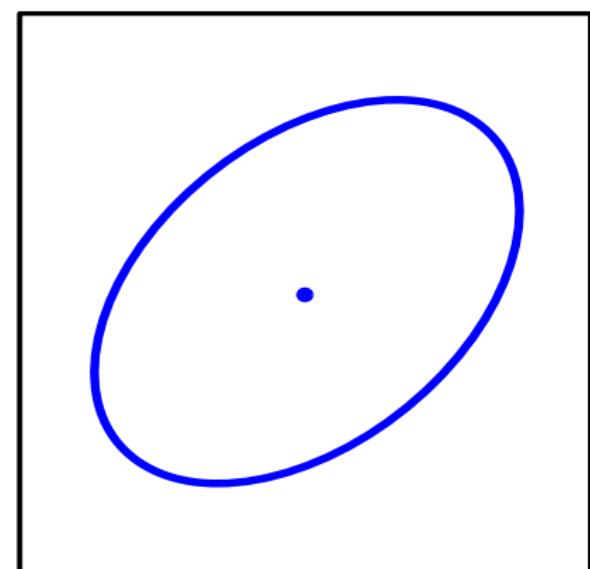
$$V_4(x) = \begin{bmatrix} x \\ dz(u(x)) \end{bmatrix}^T \bar{P} \begin{bmatrix} x \\ dz(u(x)) \end{bmatrix}$$

- Piecewise Polynomial (LMI-BMI)

$$V_5(x) = \begin{bmatrix} x \\ dz(u(x)) \end{bmatrix}^{\{m\}T} \hat{P} \begin{bmatrix} x \\ dz(u(x)) \end{bmatrix}^{\{m\}}$$

Natural tool for estimating \mathcal{L}_{2m} gains

$$\dot{V} + \frac{1}{\gamma_{wz}(s)^2} |z|^2 - |w|^2 < 0$$



A possible level set

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- **Piecewise quadratic (LMI-BMI)**

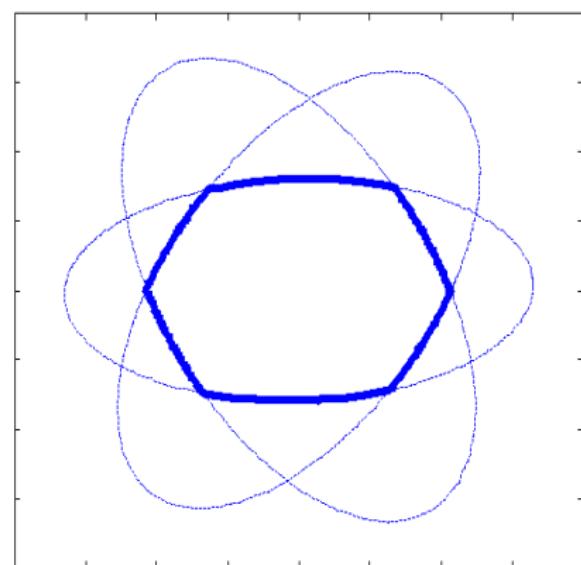
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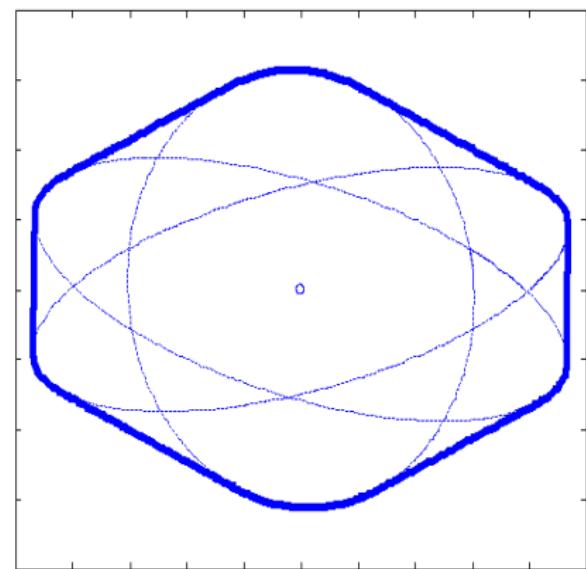
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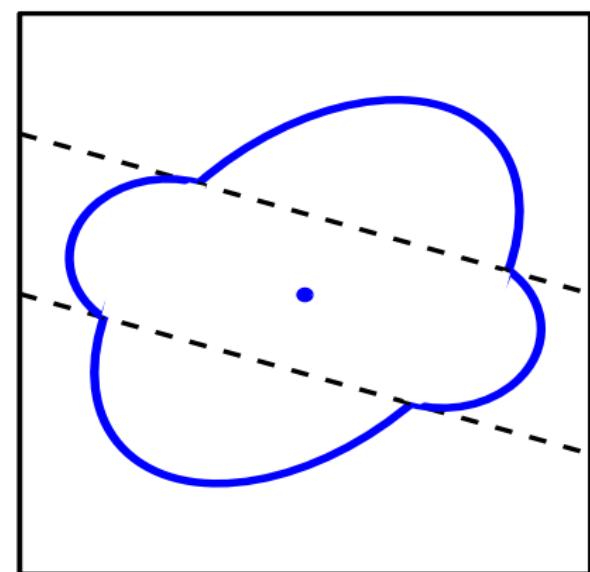
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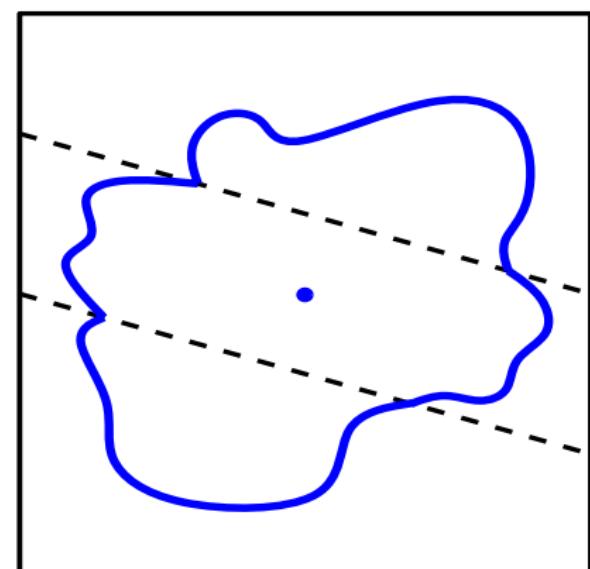
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A possible level set

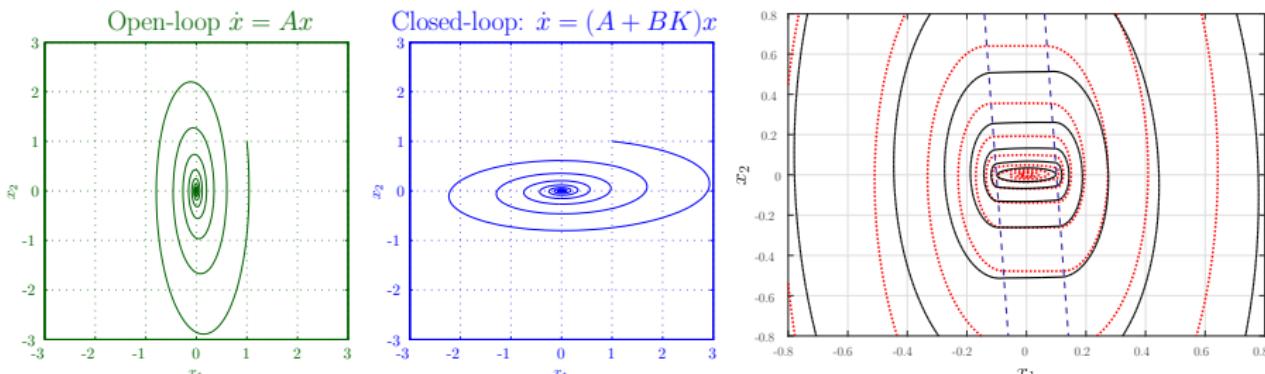
Piecewise quadratic Lyapunov analysis

Dai et al. [2009b]

- Quadratic form with $P = \begin{bmatrix} P_1 & P_{12} \\ P_{12}^T & P_2 \end{bmatrix} > 0$
- $V_4(x) = \begin{bmatrix} x \\ dz(u) \end{bmatrix}^T \begin{bmatrix} P_1 & P_{12} \\ P_{12}^T & P_2 \end{bmatrix} \begin{bmatrix} x \\ dz(u) \end{bmatrix}$
- Exploit identity $2 \int_0^{u_k} dz(s) ds = dz(u_k)^2$
- Global analysis is an LMI
- Regional analysis is a BMI
- LMIs/BMIs stem from sector conditions involving $\text{sat}(u)$, $dz(u)$ and $\frac{d}{dt}dz(u)$
- A global solution can be found for the quadratically infeasible example

to capture Lure-Postnikov function

$$V_{\text{dz}}(x) = x^T Q x + 2\lambda_i \sum_{k=1}^m \int_0^{u_k} dz(s) ds$$



Sign-definite piecewise quadratic synthesis

Queinnec et al. [2020]

- Quadratic form with $P = \begin{bmatrix} P_1 & P_{12} \\ P_{12}^T & P_2 \end{bmatrix} \geqslant 0$

$$V_4(x) = \begin{bmatrix} x \\ dz(u) \end{bmatrix}^T \begin{bmatrix} P_1 & P_{12} \\ P_{12}^T & P_2 \end{bmatrix} \begin{bmatrix} x \\ dz(u) \end{bmatrix}$$

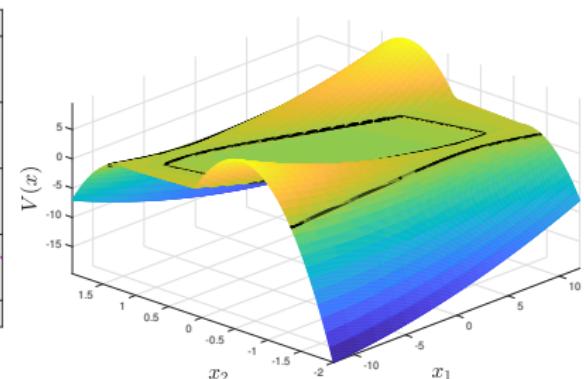
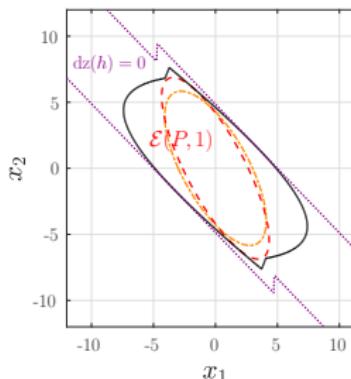
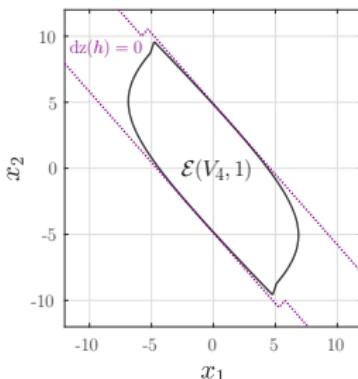
- Global analysis/synthesis is an LMI
- Regional analysis/synthesis is a BMI
- Purple signal $h = Hx$ generalized to $h = H_1x + H_2dz(u)$
- Sign indefinite quadratic forms can become negative outside the set $dz(h) = 0$

- Exploit identity

$$2 \int_0^{u_k} \text{sat}(s) ds = u_k^2 - dz(u_k)^2$$

to capture Lure-Postnikov function

$$V_{\text{sat}}(x) = x^T Q x + 2\lambda_i \sum_{k=1}^m \int_0^{u_k} \text{sat}(s) ds$$



Summary of the presented works with main references

▷ Topics covered in this talk:

- State-feedback quadratic stabilization with saturation Zaccarian and Teel [2004], Cristofaro et al. [2019], Valmorbida et al. [2017b]
- Role of algebraic loops Mulder et al. [2001], Zaccarian and Teel [2002], Hu et al. [2006], Syaichu-Rohman et al. [2003]
- \mathcal{H}_∞ design and anti-windup design with saturation Dai et al. [2009a], Dal Col et al. [2019]
- LMI-based Direct Linear Anti-Windup Mulder et al. [2001], Grimm et al. [2003a], Hu et al. [2008], Zaccarian and Teel [2011]
- Robust anti-windup designs Grimm et al. [2004b], Turner et al. [2007], Formentin et al. [2017]
- Use of nonquadratic Lyapunov certificates Dai et al. [2009b], Garulli et al. [2013], Hu et al. [2006], Valmorbida et al. [2017a], Queinnec et al. [2020]

▷ Thanks to my coauthors and collaborators for the enriching exchanges



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