Distributed estimation based on multi-hop subspace decomposition

Luca Zaccarian [2,3]

Joint work with A.R. del Nozal[1], P. Millán[1], L. Orihuela[1], A. Seuret[2]

[1]Departamento de Ingeniería, Universidad Loyola Andalucía, Sevilla, Spain
[2]LAAS-CNRS, Université de Toulouse, CNRS, Toulouse, France
[3]Department of Industrial Engineering, University of Trento, Italy

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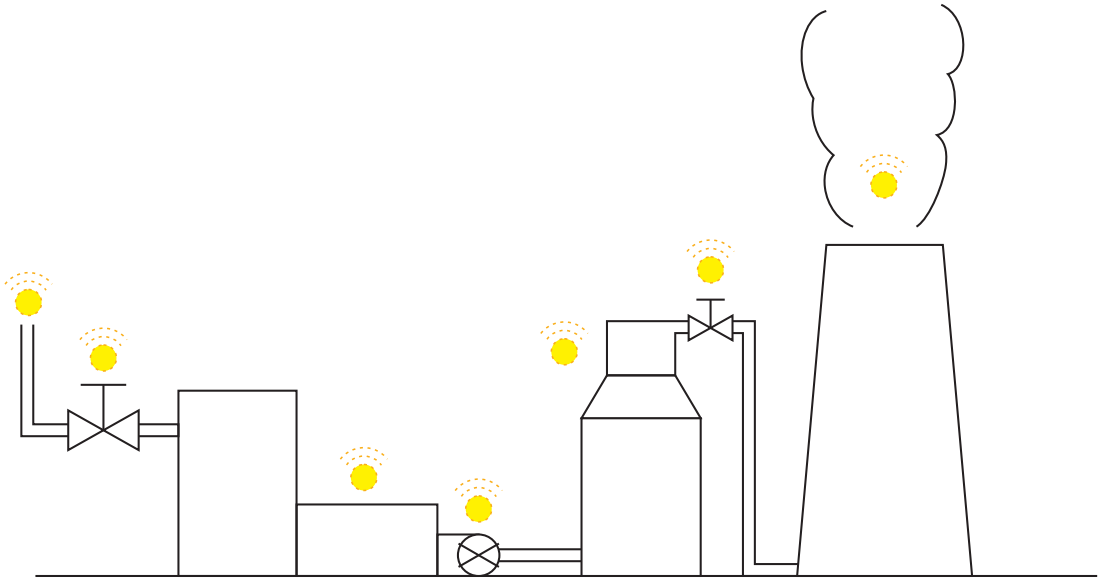
Overview

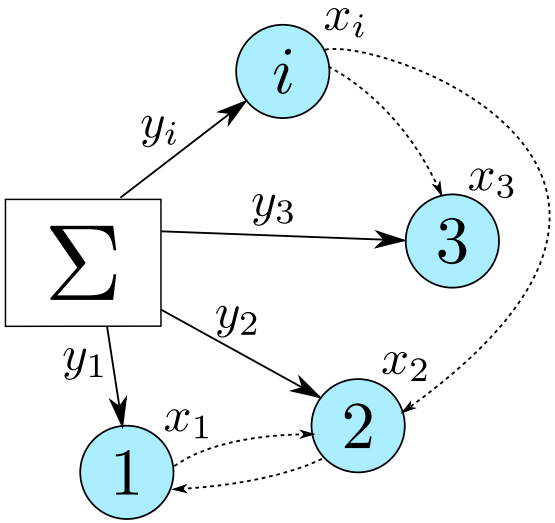
A large plant with scattered sensors

Motivation

- Increasing number of embedded sensors dispersedly integrated at complex plants is fostering the application of distributed estimation and control schemes.
- Distributed strategies offer interesting advantages such as scalability, flexibility, fault tolerance and robustness.



Abstraction involves multi-agent structure



Plant definition

$$\mathcal{G} = (\mathcal{V}, \mathcal{E}),$$
$$x^+ = Ax,$$
$$y_i = C_i x \quad \forall i \in \mathcal{V}, \qquad (\spadesuit)$$

Goals

- Reconstruct the whole state at every node.
- Design the observer in a distributed way.
- Fix an arbitrary convergence rate for the estimation error.
- Exploit linearity to provide intuitive structure.
- Reduce the exchange of information.

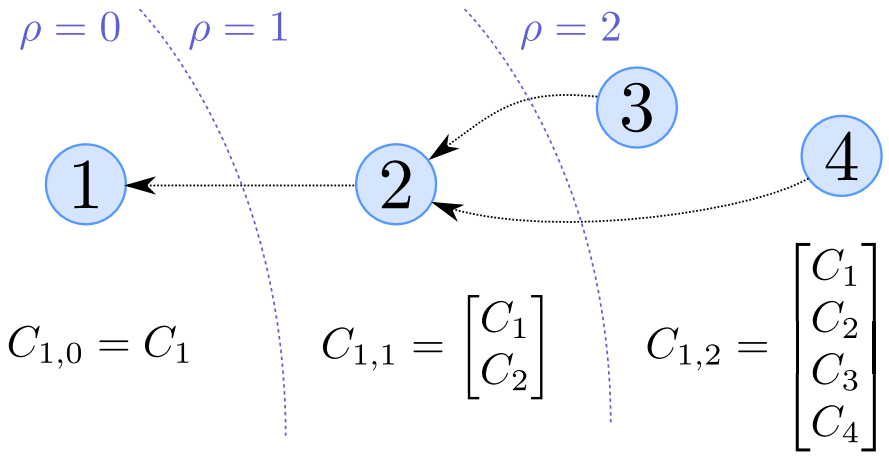
Multi-hop output matrix generalization

The ρ-hop output matrix of agent i

The ρ-hop output matrix of agent i, $C_{i,\rho}$, is composed by its output matrix C_i and the output matrices of all agents j with a direct path to i involving ρ or less edges. That is:

$$C_{i,\rho} \triangleq \begin{bmatrix} C_{i,\rho-1} \\ \mathrm{col}(C_{j,\rho-1})_{j\in\mathcal{N}_i} \end{bmatrix}, \quad \forall \rho > 0,$$

where $C_{i,0} := C_i$.



Multi-hop Unobservable Subspaces

The ρ-hop **unobservable** subspace from agent i, $\bar{\mathcal{O}}_{i,\rho}$

The ρ-hop **unobservable** subspace from agent i, $\bar{\mathcal{O}}_{i,\rho} := Im(\bar{V}_{i,\rho})$, is the unobservable subspace related to pair $(C_{i,\rho}, A)$.

The ρ-hop **observable** subspace from agent i, $\mathcal{O}_{i,\rho}$

The ρ-hop **observable** subspace from agent i, $\mathcal{O}_{i,\rho} := Im(V_{i,\rho})$ is the orthogonal complement of $\bar{\mathcal{O}}_{i,\rho}$.

Property: $\begin{array}{l} \left[\bar{V}_{i,\rho}\ V_{i,\rho}\right] \text{ is nonsingular }, \\ \mathcal{O}_{i,\rho-1} \subseteq \mathcal{O}_{i,\rho}, \end{array} \quad \forall i \in \mathcal{V},\ \rho \in \{1, \ldots, \ell_i\}$

Innovation Matrix $W_{i,\rho}$

Innovation matrix $W_{i,\rho}$ that generates $\mathcal{O}_{i,\rho} \cap (\mathcal{O}_{i,\rho-1})^{\perp}$ satisfies

- $W_{i,\rho}^{\top} W_{i,\rho'} = 0,\ \forall \rho,\ \rho' \in \{1, \ldots, \ell_i\}$ such that $\rho \neq \rho'$.
- $Im(W_{j,\rho-1}) \subseteq Im(V_{i,\rho}),\ \forall j \in \mathcal{N}_i$.

Multi-hop Transformation Matrix

ℓ_i is selected later as a suitable positive integer

Multi-hop transformation matrix T_i

It is orthogonal $T_i^\top = T_i^{-1}$ by construction:

$$T_i := [\underbrace{\bar{V}_{i,\ell_i} \quad W_{i,\ell_i} \quad \cdots \quad W_{i,\rho+1}}_{\bar{V}_{i,\rho}} \quad \underbrace{W_{i,\rho} \quad \cdots \quad W_{i,0}}_{V_{i,\rho}}]$$

Property: $[\bar{V}_{i,\rho}\ V_{i,\rho}]$ is nonsingular , $\mathcal{O}_{i,\rho-1} \subseteq \mathcal{O}_{i,\rho}$, $\quad \forall i \in \mathcal{V},\ \rho \in \{1, \ldots, \ell_i\}$

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- $Im(W_{j,\rho-1}) \subseteq Im(V_{i,\rho}),\ \forall j \in \mathcal{N}_i$.

Multi-hop observable decomposition enjoys nice structure

$$T_i := [\underbrace{\bar{V}_{i,\ell_i} \quad W_{i,\ell_i} \quad \cdots \quad W_{i,\rho+1}}_{\bar{V}_{i,\rho}} \quad \underbrace{W_{i,\rho} \quad \cdots \quad W_{i,0}}_{V_{i,\rho}}]$$

Proposition

Orthogonal transformation T_i provides **Multi-hop Observable Decomposition**:

$$T_i^\top A T_i = \begin{bmatrix} \bar{V}_{i,\ell_i}^\top A \bar{V}_{i,\ell_i} & \bar{V}_{i,\ell_i}^\top A W_{i,\ell_i} & \cdots & \bar{V}_{i,\ell_i}^\top A W_{i,1} & \bar{V}_{i,\ell_i}^\top A W_{i,0} \\ 0 & W_{i,\ell_i}^\top A W_{i,\ell_i} & \cdots & W_{i,\ell_i}^\top A W_{i,1} & W_{i,\ell_i}^\top A W_{i,0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & W_{i,1}^\top A W_{i,1} & W_{i,1}^\top A W_{i,0} \\ 0 & 0 & \cdots & 0 & W_{i,0}^\top A W_{i,0} \end{bmatrix}$$

$$C_{i,0} T_i = [\ 0 \quad 0 \quad \cdots \quad 0 \quad \star\ ]$$

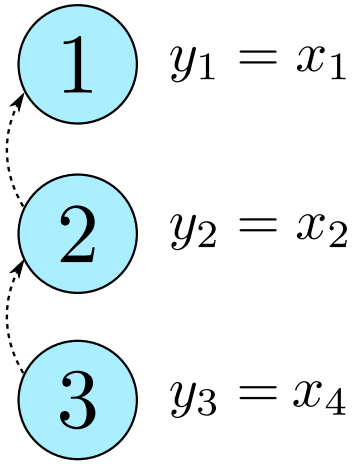
$$\vdots$$

$$C_{i,\ell_i} T_i = [\ 0 \quad \star \quad \cdots \quad \star \quad \star\ ]$$

Example of multi-hop observable decomposition

$$A = \begin{bmatrix} 0.295 & 0 & 0 & 0 \\ 0 & 0.9954 & -0.0876 & 0 \\ 0 & 0.1248 & 0.9945 & 0 \\ 0 & 0 & 0 & 0.6775 \end{bmatrix}, \quad \begin{aligned} C_1 &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \\ C_2 &= \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \\ C_3 &= \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

$$W_{i,0} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \; W_{i,1} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \; W_{i,2} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$



$$T_1^\top A T_1 = \begin{bmatrix} 0.6775 & 0 & 0 & 0 \\ 0 & 0.9954 & -0.0876 & 0 \\ 0 & 0.1248 & 0.9945 & 0 \\ 0 & 0 & 0 & 0.295 \end{bmatrix}$$

$$C_{1,0} T_1 = C_1 T_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C_{1,1} T_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad C_{1,2} T_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Proposed observer structure

$$\hat{x}_i^+ = \underbrace{A\hat{x}_i}_{(A)} + \underbrace{W_{i,0}L_i(y_i - \hat{y}_i)}_{(B)} + \underbrace{\sum_{\rho=1}^{\ell_i}\sum_{j\in\mathcal{N}_i} W_{i,\rho}N_{i,j,\rho}W_{j,\rho-1}^\top(\hat{x}_j - \hat{x}_i)}_{(C)} \quad (\heartsuit)$$

(A) Model-based **open-loop term**.

(B) **Luenberger-term** for the *observable modes*: The difference between the locally measured y_i and predicted $\hat{y}_i = C_i\hat{x}_i$ outputs is multiplied by gain L_i whose action is limited to the (local) subspace $W_{i,0}$.

Recall that $y_i - \hat{y}_i = C_i x - C_i\hat{x}_i = C_i\varepsilon_i$

(C) **Consensus-term** for the *unobservable modes*: The estimation difference with the neighbors $\hat{x}_i - \hat{x}_j$ is multiplied by the gain $N_{i,j,\rho}$, whose action is limited to the (non-local) subspaces $W_{i,\rho}$.

Recall that $\hat{x}_j - \hat{x}_i = \hat{x}_j - x + x - \hat{x}_i = \varepsilon_i - \varepsilon_j$

Design goal and basic (necessary) assumption

Problem α

Given $\alpha \in (0,1)$, plant (♠), and the interconnection graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, design the gains L_i and $N_{i,j,\rho}$ in (♡) such that all estimates $\hat{x}_i$ converge to x exponentially fast with exponential rate α.

Collective α-detectability

Plant (♠) is collectively α-detectable if for each agent $i \in \mathcal{V}$, there exist a finite number of hops $\ell_i \in \mathbb{Z} > 0$ such that pair (C_{i,ℓ_i}, A) is α-detectable.

Collective α-detectability is necessary

Collective α-detectability is a *necessary and sufficient* condition for solving Problem α.

Assumption α

Given $\alpha \in (0,1)$, plant (♠) is collectively α-detectable.

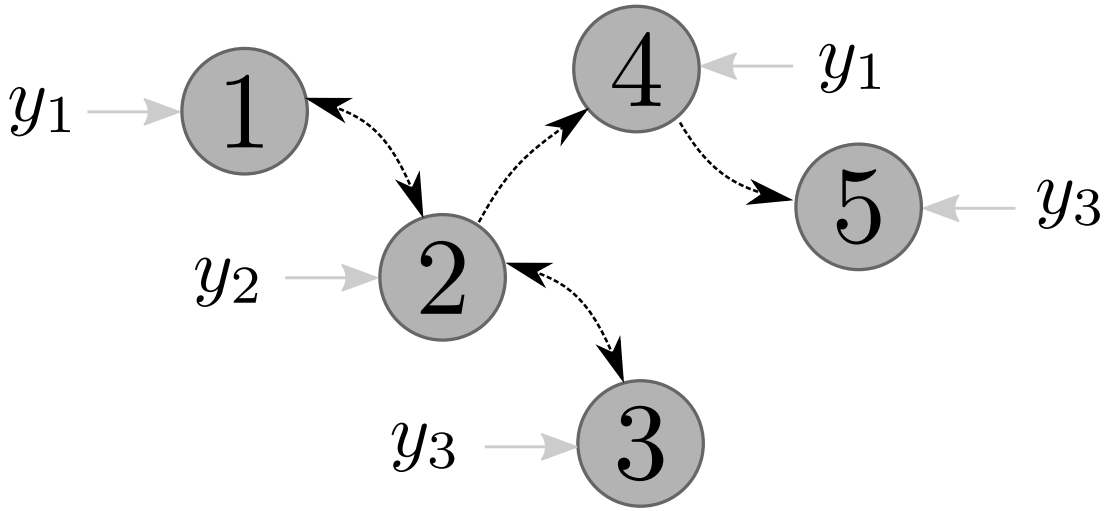


Figure 1: Assume that pair $(\tilde{C}, A)$ with $\tilde{C} = [C_1^\top, C_2^\top, C_3^\top]^\top$ is α-detectable. Although strong connectivity does not hold, Assumption is met.

Collective α-detectability is necessary

Collective α-detectability is a *necessary and sufficient* condition for solving Problem α.

Main Result: design guidelines

Theorem α (Design of the distributed observer)

Consider plant (♠) and observer structure (♡). If matrices

$$\underbrace{W_{i,0}^\top A W_{i,0} - L_i C_i W_{i,0}}_{=D_{i,(0,0)}}, \quad \underbrace{W_{i,\rho}^\top A W_{i,\rho} - \sum_{j\in\mathcal{N}_i} N_{i,j,\rho} W_{j,\rho-1}^\top W_{i,\rho}}_{=D_{i,(\rho,\rho)}},$$

for all $\rho \in \{1,\dots,\ell_i\}$, have spectral radius smaller than α, then Problem α is solved:

$$|\hat{x}_i(k) - x(k)| \leq M\alpha^k \left\| \begin{bmatrix} \hat{x}_1(0) - x(0) \\ \vdots \\ \hat{x}_N(0) - x(0) \end{bmatrix} \right\|, \ \forall i \in \mathcal{V}.$$

Coarse bound from the Theorem

Interesting additional bounds emerge from multi-hop decomposition (next slide).

Sketch of the proof

Transformed estimation error dynamics of agent i at hop ρ ($\varepsilon_{i,\rho} = W_{i,\rho}^\top (x - \hat{x}_i)$):

$$\varepsilon_{i,0}^+ = [W_{i,0}^\top A W_{i,0} - L_i C_i W_{i,0}]\varepsilon_{i,0}, \qquad \varepsilon_0 := \mathrm{col}(\varepsilon_{i,0})$$

$$\varepsilon_{i,\rho}^+ = \sum_{r=0}^{\rho} D_{i,(\rho,r)}\varepsilon_{i,r} + \sum_{j\in\mathcal{N}_i} N_{i,j,\rho}\varepsilon_{j,\rho-1}, \quad \varepsilon_\rho := \mathrm{col}(\varepsilon_{i,\rho})_{i\in\mathcal{V}:\ell_i+1\geq\rho}$$

Stacking the estimation error at each hop for every agent

$$\begin{bmatrix} \varepsilon_{\bar{\ell}} \\ \vdots \\ \varepsilon_1 \\ \varepsilon_0 \end{bmatrix}^+ = \begin{bmatrix} \triangle_{\bar{\ell}} & \dots & \star & \star \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \triangle_1 & \star \\ 0 & \dots & 0 & \triangle_0 \end{bmatrix} \begin{bmatrix} \varepsilon_{\bar{\ell}} \\ \vdots \\ \varepsilon_1 \\ \varepsilon_0 \end{bmatrix} \quad \text{with } \triangle_\rho = \begin{bmatrix} D_{1,(\rho,\rho)} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & D_{p,(\rho,\rho)} \end{bmatrix}$$

$$T_i := [\underbrace{\bar{V}_{i,\ell_i} \quad W_{i,\ell_i} \quad \cdots \quad W_{i,\rho+1}}_{\bar{V}_{i,\rho}} \quad \underbrace{W_{i,\rho} \quad \cdots \quad W_{i,0}}_{V_{i,\rho}}]$$

Design feasibility

It is always possible, under Assumption α, to find a set of matrices L_i and $N_{i,j,\rho}$ that satisfy the conditions of Theorem α

Sketch of the proof

Existence of L_i from observability of pair $(C_i W_{i,0}, W_{i,0} A W_{i,0})$.

For $N_{i,j,\rho}$ rewrite expression

$$D_{i,(\rho,\rho)} = W_{i,\rho}^\top A W_{i,\rho} - \sum_{j \in \mathcal{N}_i} N_{i,j,\rho} W_{j,\rho-1}^\top W_{i,\rho},$$

as

$$W_{i,\rho}^\top A W_{i,\rho} - \bar{N}_{i,\rho} \Lambda_{i,\rho} \text{ with } \begin{array}{l} \bar{N}_{i,\rho} = \mathrm{col}(N_{i,j,\rho}^\top)_{j \in \mathcal{N}_i}^\top, \\ \Lambda_{i,\rho} = \mathrm{col}(W_{j,\rho-1}^\top)_{j \in \mathcal{N}_i}^\top W_{i,\rho}, \end{array}$$

The the pair $(\Lambda_{i,\rho}, W_{i,\rho}^\top A W_{i,\rho})$ is observable from the Popov-Belevitch-Hautus test.

Distributed observer setup

For every agent/node i do:

a. Compute $\mathcal{O}_{i,0}$ and construct matrix $W_{i,0}$. Set $\rho = 0$.

b. Perform the two steps:
   - Exchange $W_{i,\rho}$ with the neighbors.
   - Construct $\mathcal{O}_{i,\rho+1}$ and construct matrix $W_{i,\rho+1}$.

c. If the ρ-hop unobservable modes have speed α, then stop. Otherwise increment ρ and go to (b).

Gain selection phase

Each agent designs gains $(L_i, N_{i,j,\rho})$.

Running phase

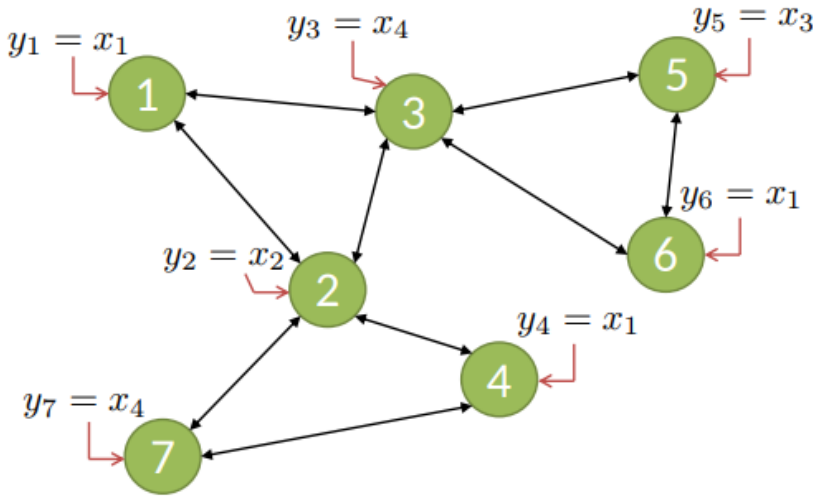
Each agent will exchange with its neighbors a portion of the state defined by $W_{j,\rho-1}\hat{x}_i$, for all $\rho = 1, \ldots, \ell_i$.

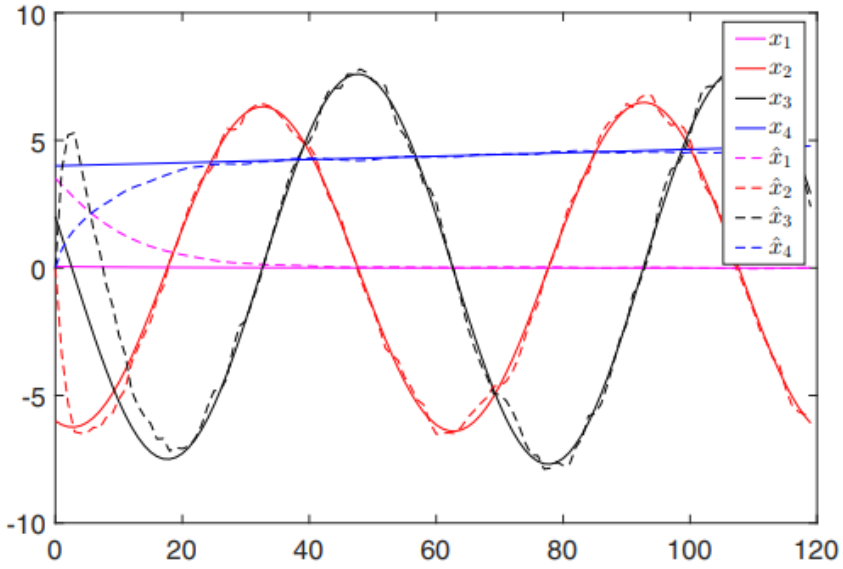
Simulation example with 7 nodes

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}^+ = \begin{bmatrix} 0.95 & 0 & 0 & 0 \\ 0 & 0.995 & -0.0876 & 0 \\ 0 & 0.125 & 0.994 & 0 \\ 0 & 0 & 0 & 1.002 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

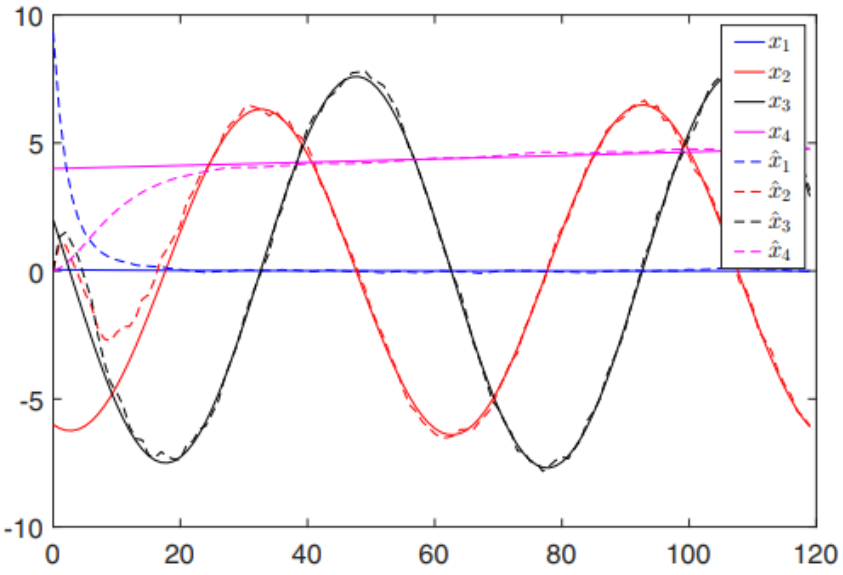
$y_1 = x_1, \quad y_2 = x_2, \quad y_3 = x_4, \quad y_4 = x_1,$

$y_5 = x_3, \quad y_6 = x_1, \quad y_7 = x_4$





Estimates of Agent 2

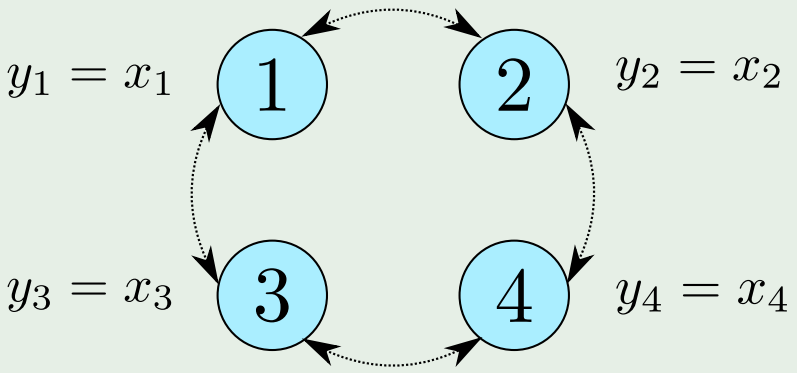


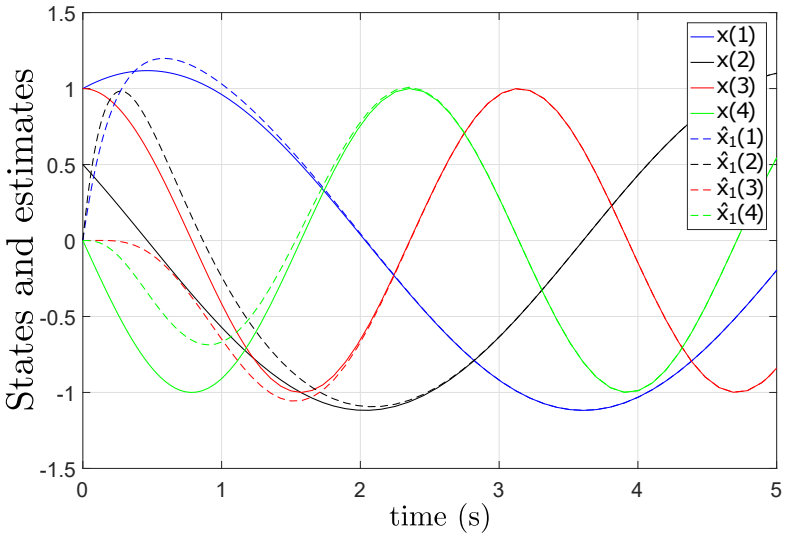
Estimates of Agent 5

Results directly extend to continuous-time

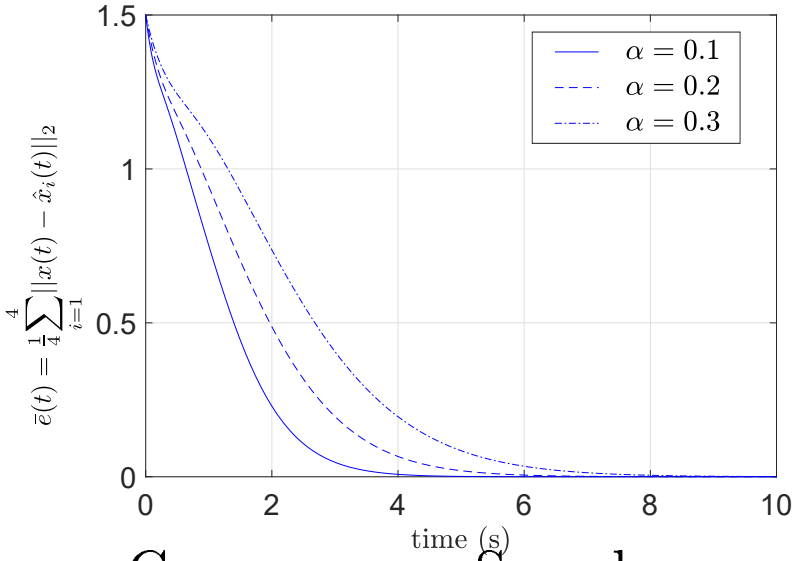
Continuous-time Simulation Example

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{bmatrix} x,$$





Estimates for Agent 1



Convergence Speed α

Conclusions

- Distributed observer with distributed design
- Arbitrary convergence rate α of the estimation error
- Intuitive method based on observable decomposition
  - Continuous- and Discrete-time results
  - Necessary and sufficient conditions

Future work

- Optimal design of gains L_i and $N_{i,j,\rho}$ (LMI-based)
- Time-varying graphs, delays or packet losses
- Designs reducing online information exchange

References

- P. Millan Gata, A. Rodriguez del Nozal, L. Zaccarian, L. Orihuela Espina, and A. Seuret. *Distributed implementation and design for state estimation.* In Proc. of the IFAC World Congress, pages 6667–6672, Toulouse, France, July 2017.
- A. Rodriguez del Nozal, P. Millan Gata, L. Orihuela Espina, and A. Seuret, L. Zaccarian. *Distributed Estimation based on multi-hop subspace decomposition.* Submitted to Automatica.