Equivalent Conditions for Synchronization of Identical Linear Systems and Application to Quality-Fair Video Delivery

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Outline

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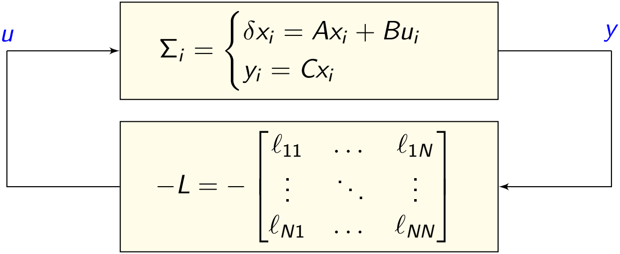
Identical linear systems with networked feedback



- ▷ Network composed by N identical continuos- or discrete- time SISO LTI agents Σ_i, $i = 1, \dots, N$.

$$\Sigma_i = \begin{cases} \delta x_i = Ax_i + Bu_i \\ y_i = Cx_i \end{cases} \qquad \delta x = \dot{x} \backslash x^+, \quad x_i \in \mathbb{R}^n \qquad (\spadesuit)$$

- ▷ Output feedback interconnection:

$$u = -Ly, \qquad (\diamondsuit)$$

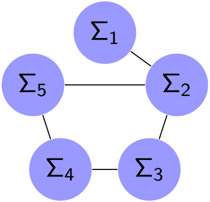
where $u = [u_1 \dots u_N]^\top \in \mathbb{R}^N$, $y = [y_1 \dots y_N]^\top \in \mathbb{R}^N$

Assumptions on the graph and Laplacian matrix

▷ **Laplacian matrix** L in interconnection (♢) characterizes a graph with directed topology satisfying standing assumption.

Assumption (R)

Matrix L has real eigenvalues and the graph has a directed spanning tree



▷ Laplacian matrix defined as $L = L^\top = [\ell_{ij}]\,,\ \ell_{ij} = \begin{cases} -a_{ij}^d & \text{if } i \neq j \\ \sum_{j=1}^N a_{ij}^d & \text{if } i = j \end{cases}$

▷ Properties of Laplacian matrix:

- 0 is an eigenvalue with right eigenvector $\mathbf{1}$ and left eigenvector $v_\mathbf{1}$:

$$L\mathbf{1} = 0 \qquad v_\mathbf{1}^T L = 0$$

▷ There exists an **Orthogonal Transformation** T such that:

$$TLT^\top = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & \dots & \star \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_{N-1} \end{bmatrix}$$

A uniform stability viewpoint on consensus/synchronization

- Consensus/Synchronization set is spanned by (eigen)-vector $\mathbf{1} \otimes I_n$:

$$\mathcal{A} := \left\{ x \in \mathbb{R}^{nN} : x_i - x_j = 0, \forall i, j \in \{1, \dots, N\} \right\}. \qquad (\heartsuit)$$

- May use Lyapunov tools to measure the point-to-set distance from $\mathcal{A}$:

$$|x|_{\mathcal{A}} = \inf_{y \in \mathcal{A}} |x - y|$$

- Our goal: establish necessary and sufficient conditions for UGES of $\mathcal{A}$ for the networked interconnection.

Definition (UGES)

A closed set $\mathcal{A}$ is Uniformly Globally Exponentially stable for the dynamics if $\exists$ positive M and λ such that all solutions ϕ satisfy:

$$|\phi(t)|_{\mathcal{A}} \le M e^{-\lambda t} |\phi(0)|_{\mathcal{A}} \qquad \text{if } t \in \mathbb{R}$$
$$|\phi(t)|_{\mathcal{A}} \le M e^{-\lambda t} |\phi(0)|_{\mathcal{A}} \qquad \text{if } t \in \mathbb{Z}$$

- Note that UGES is nontrivial when $\mathcal{A}$ is **unbounded** (not compact) but linearity helps

Main Result: Equivalent conditions for Synchronization

Theorem (see also [Fax Murray,2004], [Scardovi Sepulchre,2008])

Consider the network of agents (♠) and the interconnection (♢) under Assumption (R). The following statements are equivalent:

1) *Denoting by λ_k are the eigenvalues of L, matrices*

$$A_k := A - \lambda_k BC, \qquad k = 1, \ldots, N-1$$

are Hurwitz [Schur-Cohn].

2) *There exists a Lyapunov function $V(x) = x^\top Px$ satisfying:*

$$\bar{c}_1|x|^2_{\mathcal{A}} \leq V(x) \leq \bar{c}_2|x|^2_{\mathcal{A}}, \quad \dot{V}(x)\backslash\Delta V(x) \leq -\bar{c}_3|x|^2_{\mathcal{A}},$$

3) *The closed attractor $\mathcal{A}$ in (♡) is UGES for the system (♠)-(♢).*
4) *The closed loop is such that each sub-state x_i converges exponentially to the unique solution of:*

$$\delta x_\circ = Ax_\circ, \quad x_\circ(0) = \frac{1}{|v_{\mathbf{1}}|_1}\sum_{k=1}^{N} v_{\mathbf{1},k}x_k(0) \quad \left(= \frac{1}{N}\sum_{k=1}^{N} x_k(0)\right)$$

Consensus in Quality-fair Video delivery

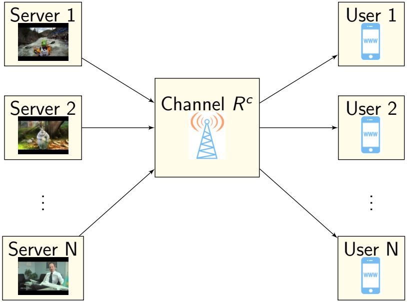
▷ Parallel delivery of N encoded video streams.

▷ Communication channel of limited capacity R^c.

▷ Synchronous control of each video chain.

We want to achieve:

- Fairness among the terminals in terms of some quality video metrics, e.g. the Peak Signal-to-Noise Ratio (PSNR).
- Robustness with respect of the characteristics of the video streams.



Main aspects of the considered application

Video Quality Fairness:

- Encoding Rate of the video streams.
- Transmission Rate through the link.

Control Strategy:

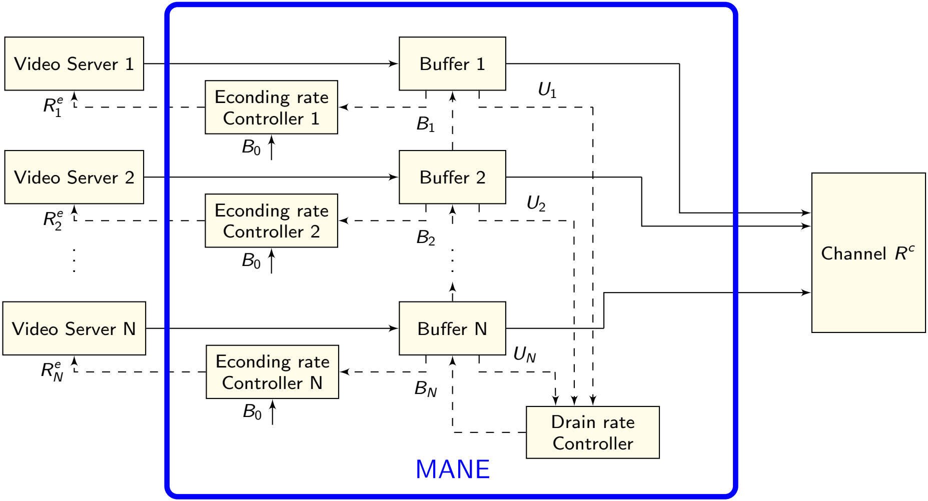
- Feedback Loops based on Proportional Integral (PI) controllers.
- No information is exchanged between the video servers.

Available Measurements:

- Quality informations of the encoded video: utilities $U_1, \ldots, U_N$ inserted in the packet headers.

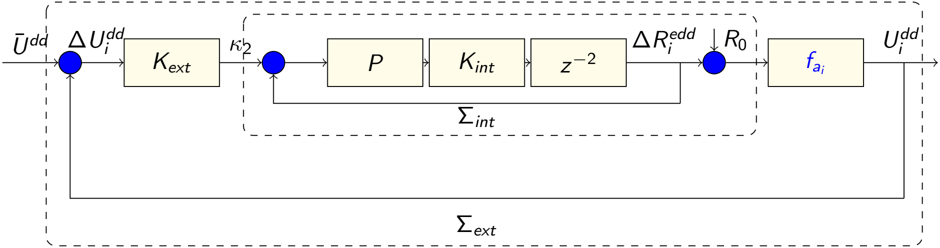
Block diagram of an existing solution (MANE)

▷ MANE = Media Aware Network Element is a centralized controller



Pre-existing model rewritten in Synchronization form

▷ Block Diagram representation of the i-th video stream, $i = 1, \ldots, N$:



▷ Two sets of PI controller gains must be tuned:

- Encoding rate controller K_{int}:

$$K_{int}(z) = \frac{k_I^{int}}{z-1} + k_P^{int}$$

- Transmission rate controller K_{ext}:

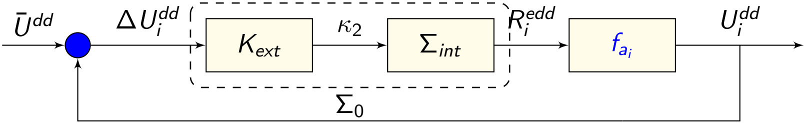
$$K_{ext}(z) = \frac{k_I^{ext}}{\sigma}\frac{1}{z-1} + \frac{k_P^{ext}}{\sigma}$$

▷ σ = normalizing constant allowing for a-dimensional gain tuning

▷ f_{a_i} is the only nonlinearity in the system, and is monotonically increasing

Two PI Control Loops Σ_{ext}, Σ_{int} should ensure consensus

▷ Block Diagram representation of the i-th video stream, $i = 1, \ldots, N$:



▷ State-space representation of the inner blocks containing PI gains:

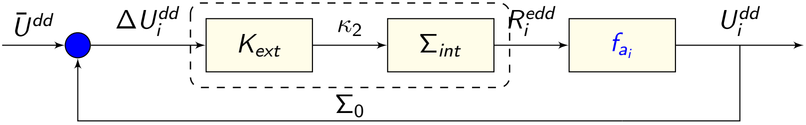
$$K_{ext} : \begin{cases} x_{ext}^{+} = A_{ext} x_{ext} + B_{ext} \Delta U_i^{dd} \\ \kappa_2 = C_{ext} x_{ext} + D_{ext} \Delta U_i^{dd} \end{cases} \qquad \Sigma_{int} : \begin{cases} x_{int}^{+} = A_{int} x_{int} + B_{int} \kappa_2 \\ U_i^{dd} = C_{int} x_{int} \end{cases}$$

▷ State-space representation of the overall linear synchronization feedback:

$$\Sigma_0 : \begin{cases} \begin{bmatrix} x_{ext} \\ x_{int} \end{bmatrix}^{+} = \overbrace{\begin{bmatrix} A_{ext} & 0 \\ B_{int} C_{ext} & A_{int} \end{bmatrix}}^{A_0} \begin{bmatrix} x_{ext} \\ x_{int} \end{bmatrix} + \overbrace{\begin{bmatrix} B_{ext} \\ B_{int} D_{ext} \end{bmatrix}}^{B_0} \Delta U^{dd} \\ R_i^{edd} = \overbrace{\begin{bmatrix} 0 & C_{int} \end{bmatrix}}^{C_0} \begin{bmatrix} x_{ext} \\ x_{int} \end{bmatrix} \end{cases}$$

Two Distinct Actions from the two PI control loops

- ▷ Block Diagram representation of the i-th video stream, $i = 1, \ldots, N$:



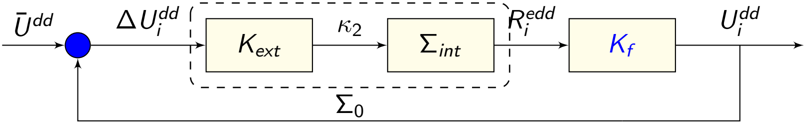
- ▷ Σ_0 is the cascaded interconnection of K_{ext} and Σ_{int}
- ▷ K_{int} stabilizes the streams dynamics rejecting constant bias R_0
- ▷ K_{ext} synchronizes the network of agents rejecting constant bias B_0
  - The coupling among the video streams arises from subtracting the average utility $\bar{U}^{dd}$ from the i-th stream utility U_i^{dd}:

$$U_i^{dd} - \bar{U}^{dd} = U_i^{dd} - \frac{1}{N}\sum_{i=1}^{N} U_i^{dd} = [LU^{dd}]_i,$$

    where L is the **Laplacian** of a fully connected graph.
  - U_i^{dd} is a nonlinear time-varying function of output R_i^{edd}.

Approximated LTI representation from standing assumption

▷ We want to focus on the following linearized dynamics:



▷ To this end we introduce the following standing assumption:

Assumption

There exist scalars h_i, $i = 1, \ldots, N$ and a scalar $K_f > 0$ such that:

$$U_i^{dd} \;=\; f_{a_i}(R_i^{edd}) = h_i + K_f R_i^{edd} \qquad \forall i = 1, \ldots, N$$

▷ The integral action of K_{ext} rejects the constant bias h_i, $\forall i = 1, \ldots, N$.

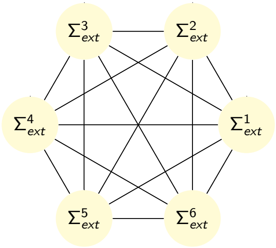
▷ We obtain a linear output feedback interconnection of N identical linear systems, whose dynamics is (after a suitable change of coordinates):

$$\Sigma_0 : \begin{cases} x_i^+ &= A_0 x_i + B_0 \Delta U_i^{dd} \\ U_i^{dd} &= K_f C_0 x_i \end{cases}$$

Closed loop as a synchronization of identical linear systems

▷ Dynamics of each agent with state $x_i \in \mathbb{R}^n$:

$$\Sigma_0 : \begin{cases} x_i^+ &= A_0 x_i + B_0 \Delta U_i^{dd} \\ U_i^{dd} &= K_f C_0 x_i \end{cases}$$



▷ The utility discrepancy corresponds to:

$$\Delta U^{dd} = -\begin{bmatrix} 1-\frac{1}{N} & -\frac{1}{N} & \dots & \dots & -\frac{1}{N} \\ -\frac{1}{N} & 1-\frac{1}{N} & \dots & \dots & -\frac{1}{N} \\ \vdots & \vdots & \ddots & \vdots & \\ -\frac{1}{N} & -\frac{1}{N} & \dots & & 1-\frac{1}{N} \end{bmatrix} U^{dd} = -LU^{dd}$$

where $U^{dd} = \left[\, U_1^{dd} \ \dots \ U_N^{dd} \,\right]^\top$, $\Delta U^{dd} = \left[\, \Delta U_1^{dd} \ \dots \ \Delta U_N^{dd} \,\right]^\top$.

▷ $\mathcal{G}$ is a *fully connected* graph. The eigenvalues of L satisfy $\lambda_o = 0$ and $\lambda_1 = \dots = \lambda_{N-1} = \frac{N}{N-1}$.

▷ The closed-loop dynamics of the N interconnected systems is:

$$\begin{cases} x^+ &= (I_N \otimes A_0)x + (I_N \otimes B_0)(-Ly) \\ y &= U^{dd} = K_f (I_N \otimes C_0)x, \end{cases}$$

Main Result (recall)

Theorem (see also [Fax Murray,2004], [Scardovi Sepulchre,2008])

Consider the network of agents (♠) and the interconnection (♢) where the graph has a directed spanning tree. The following are equivalent:

1) *Denoting by λ_k are the eigenvalues of L, matrices*

$$A_k := A - \lambda_k BC, \qquad k = 1, \dots, N-1$$

   are Hurwitz [Schur-Cohn].

2) *There exists a Lyapunov function $V(x) = x^\top Px$ satisfying:*

$$\bar{c}_1 |x|^2_{\mathcal{A}} \le V(x) \le \bar{c}_2 |x|^2_{\mathcal{A}}, \quad \dot{V}(x) \backslash \Delta V(x) \le -\bar{c}_3 |x|^2_{\mathcal{A}},$$

3) *The closed attractor $\mathcal{A}$ in (♡) is UGES for the system (♠)-(♢).*

4) *The closed loop is such that each sub-state x_i converges exponentially to the unique solution of:*

$$\delta x_\circ = A x_\circ, \quad x_\circ(0) = \frac{1}{|v_{\mathbf{1}}|_1} \sum_{k=1}^{N} v_{\mathbf{1},k} x_k(0) \quad \left(= \frac{1}{N} \sum_{k=1}^{N} x_k(0) \right)$$

Main result straightforwardly applies to our case

▷ Necessary and sufficient conditions for consensus from previous theorem

Theorem

The following statements are equivalent:

1) *Given any solution, there exists* $\bar{U} \in \mathbb{R}$ *such that* $\lim_{t\to+\infty} y_i(t) = \bar{U},\ \forall i = 1, \ldots, N$.
2) *The consensus set* $\mathcal{A} := \left\{x : x_i - x_j = 0, \quad \forall i, j \in \{1, \ldots, N\}\right\}$ *is uniformly globally exponentially stable for the closed loop and matrix* A_{int} *is Schur-Cohn.*
3) *Matrix* A_{int} *and matrix* $A_f = A_0 - K_f \left(\frac{N-1}{N}\right) B_0 C_0$ *are both Schur-Cohn.*

▷ Item 2) requires synchronization to an open-loop dynamics having one single eigenvalue in zero (the integral action in K_{ext})

▷ Item 3) exploits the fact that all nonzero eigenvalues of L coincide

▷ Item 3) will be used for PI gains tuning (two approaches)

Design of K_{int} using Jury criterion

- From item 3) we must ensure A_{int} to be Schur-Cohn
- Using Jury's criterion, we can derive the stability region for A_{int} as function of k_P^{int} and k_I^{int}
- The suboptimal parameters selection maximize the convergence rate of the internal system

Lemma

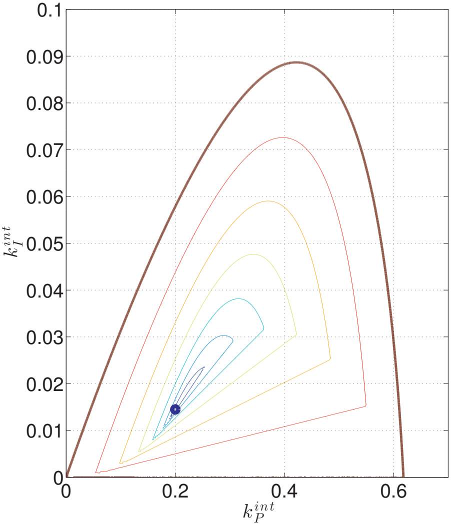
Matrix A_{int} is Schur-Cohn if and only if the following conditions hold:

$$k_I^{int} > 0$$

$$k_P^{int} + \frac{1-\sqrt{5}}{2} \leq k_I^{int} < k_P^{int}$$

$$(k_I^{int} - k_P^{int} - 1)^2 (k_I^{int} - k_P^{int}) - (k_P^{int} + 2)(2k_I^{int} - k_P^{int}) > 0.$$

Level sets of the spectral radius for K_{int}



- The figure shows different level sets of the spectral radius:

$$\rho(A_{int}) := \max_i |\lambda_i(A_{int})|$$

- The external line represents the stability limit, i.e., $\rho(A_{int}) = 1$.
- Inspecting the level sets we obtain the minimum $\rho_{\min}(A_{int}) = 0.7964$.
- The (sub)optimal parameter values are:

$$\hat{k}_I^{int} = 0.0145 \quad \hat{k}_P^{int} = 0.2$$

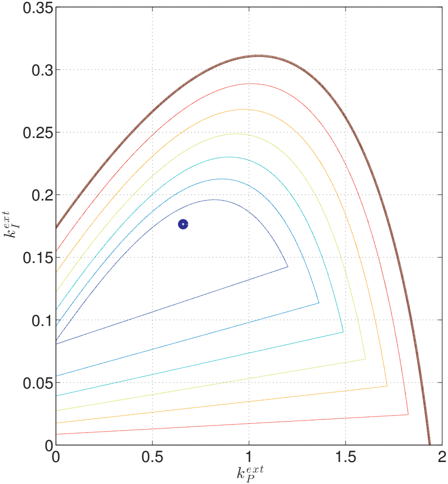
Jury's Criterion: design of K_{ext}

- ▷ From item 3) we must ensure that A_f be Schur-Cohn
- ▷ We fix the optimized values of the internal PI loop $\hat{k}_I^{int}$ and $\hat{k}_P^{int}$
- ▷ Let now consider matrix $A_f = A_0 - K_f \frac{N-1}{N} B_0 C_0$. Conveniently choosing $\sigma := K_f \frac{N-1}{N}$ we obtain:

$$A_f = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ -k_I^{ext} & 1 & 0 & 0 & k_P^{ext}+1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & -\hat{k}_P^{int} & -\hat{k}_I^{int} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

- ▷ We can apply the same numerical procedure adopted for the controller K_{int}, in order to chose the (sub)optimal parameters k_I^{ext}, k_P^{ext} minimizing the spectral radius of A_f.

Level sets of the spectral radius for K_{ext}



- The figure shows different level sets of the spectral radius:

$$\rho(A_f) := \max_i |\lambda_i(A_f)|$$

- The external line represents the stability limit, i.e., $\rho(A_f) = 1$.
- Inspecting the level sets we obtain the minimum $\rho_{\min}(A_f) = 0.9399$.
- The (sub)optimal parameter values are:

$$\hat{k}_I^{ext} = 0.1765 \quad \hat{k}_P^{ext} = 0.6590$$

Gain selection based on Convex Optimization (LMIs)

▷ The problem can be cast as a general static output feedback design:

$$A_{int} = A_1 - B_1 K_{int} \underbrace{\begin{bmatrix} I & 0 \end{bmatrix}}_{:=C_1}, \quad A_f = A_2 - B_2 K_{ext} \underbrace{\begin{bmatrix} I & 0 \end{bmatrix}}_{:=C_2}$$

▷ Maximizing the convergence rate to consensus is a specific discrete-time Lyapunov equation

Problem (Convergence Rate Maximization)

Given (A, B, C), we want to solve:

$$\max_{\alpha, P = P^\top > 0, K} \quad \alpha \text{ subject to:}$$

$$(A - BKC)^\top P (A - BKC) - P \leq -\alpha P$$

▷ This Problem is Non-convex (the optimization variables appear bilinearly)

▷ Algorithm 1 computes iteratively the controller gains: it alternates between two main steps, each of them requiring the solution of a quasiconvex optimization problem, based on LMIs and bisection.

Iterative algorithm for PI gain tuning

Definition

A pair (α_L, α_U) is *admissible* for a LMI, if the LMI is feasible with $\alpha = \alpha_L$ and infeasible with $\alpha = \alpha_U$.

Algorithm 1 Rate α and controllers K.

Input: Matrices A, B, C and tolerance $\delta > 0$.
Initialization: Set $M = 0$, and initialize $(\alpha_L, \alpha_U) = \left(1 - \bar{\sigma}^2(A), 1.1\right)$.
Iteration:

Step 1: Given M and (α_L, α_U), solve using bisection with $\delta > 0$, the GEVP:

$$\max_{W=W^\top>0, G_{11}, G_{21}, G_{22}, X_1, \alpha} \alpha \qquad (1)$$

$$\text{s.t.} \begin{bmatrix} -W + \alpha W & AG(M) - BX(M) \\ \star & -G(M) - G^\top(M) + W \end{bmatrix} \leq 0$$

Determine an admissible pair (α_L, α_U) for (1) and set $\bar{K} = \bar{G}_{11}^{-1}\bar{X}_1$ for the next step.

Step 2: Given $\bar{K}$ and (α_L, α_U) solve using bisection with tolerance $\delta > 0$, the GEVP:

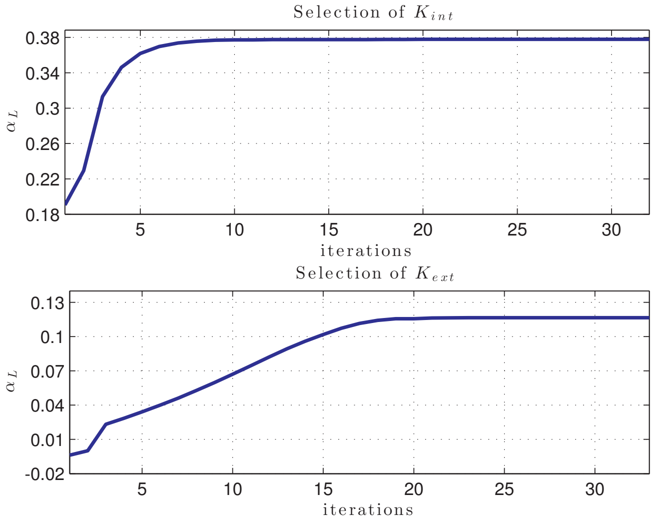
$$\max_{\alpha, W=W^\top>0} \alpha \qquad (2)$$

$$\text{s.t.} \quad A_{cl} W A_{cl}^\top - W \leq -\alpha W$$

Determine an admissible pair (α_L, α_U) for (2) and set $M = \bar{W}_{11}^{-1}\bar{W}_{12}$ for next step.

until: α_L does not increase more than δ over three consecutive steps.
Output: $K_{out} = \bar{K}$ and $\alpha_{out} = \alpha_L$.

Evolution of optimization parameter wrt the iterations



Proposition: Algorithm *initialization* always feasible. Solution carries between *subsequent steps*. *Terminal solution* is good if $\alpha_L > 0$

Comparison between the two proposed design techniques

▷ The two design techniques lead to the same gains

Method	k_P^{int}	k_I^{int}	α_L
Jury Criterion	0.2	0.0145	0.365747
Algorithm 1	0.19256	0.012915	0.37789

Method	k_P^{ext}	k_I^{ext}	α_L
Jury Criterion	0.6590	0.1765	0.1166
Algorithm 1	0.65801	0.17645	0.1165

▷ This confirms that the suboptimal iterative construction works well

Simulation Results compare three techniques

▷ Description of the simulation parameters:

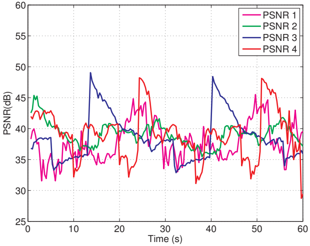
- 6 video streams of different types have been encoded during 60 s with H.264 format at various bit rates, delivered to $N = 4$ clients
- The considered utility U_i is the Peak Signal-to-Noise Ratio (PSNR)
- The linearization constant K_f evaluated based on 4 streams (Progs 1–4)

▷ Description of the simulation results:

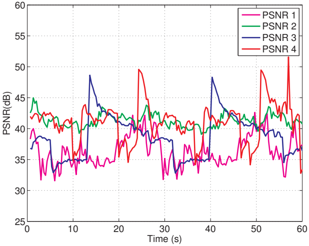
- Robustness of the controller evaluated with other streams (Progs 3–6)
- Five control schemes are comparatively considered:
- With metric $\overline{\Delta U} = \frac{1}{MN} \sum_{j=1}^{M} \sum_{k=1}^{N} \left| U_k(j) - \overline{U}(j) \right|$ we obtain:

Method	K_{int}	K_{ext}	$\overline{\Delta U}$ (1-4)	$\overline{\Delta U}$ (3-6)
Algorithm 1	$\begin{bmatrix}0.192 & 0.013\end{bmatrix}$	$\begin{bmatrix}0.658 & 0.176\end{bmatrix}$	2.28	3.22
[Changuel, 2014]	$\begin{bmatrix}0.152 & 0.002\end{bmatrix}$	$\begin{bmatrix}2.67 & 0.0013\end{bmatrix}$	2.37	–
TRF	$\begin{bmatrix}0.152 & 0.002\end{bmatrix}$	$\begin{bmatrix}0 & 0\end{bmatrix}$	4.12	3.66
UMMF	–	–	0.88	1.45
CMUM	–	–	1.53	1.19

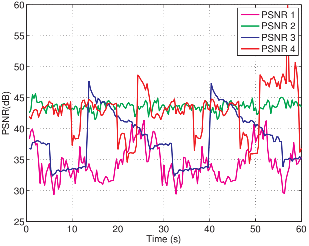
Evolutions of PSNR reveal advantages of proposed solution



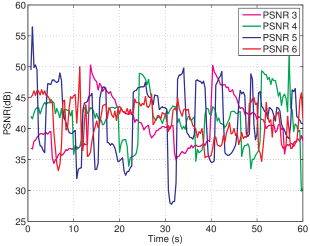
Algorithm 1



[Changuel, 2014]



TRF



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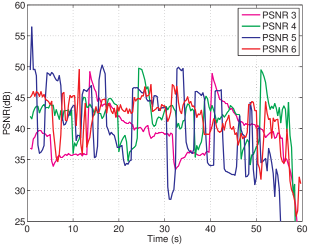
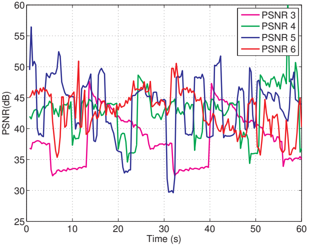
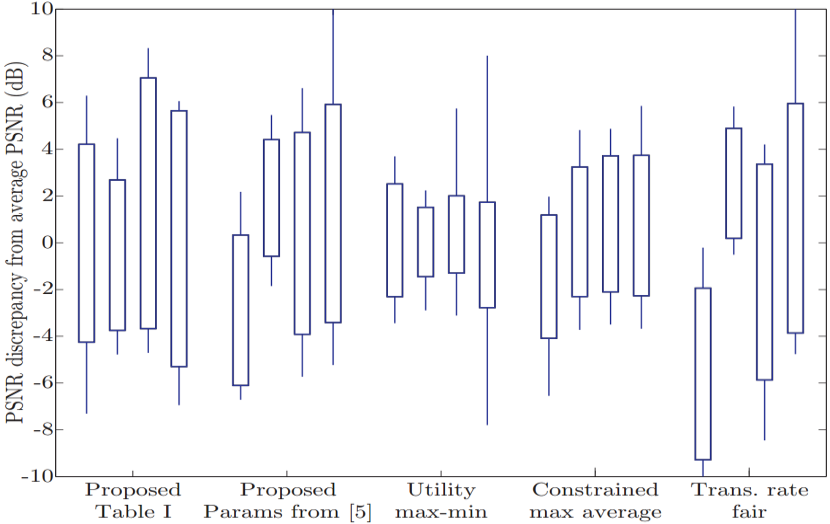


[Changuel, 2014]



TRF

Range of PSNR variations for Progs 1–4 with 5 solutions



Algorithm 1 [Changuel, 2014] UMM CMUM TRF

Conclusions and Future Works

▷ **Summary of presented works**

- A new set of equivalent conditions for synchronization of identical linear systems
- A consensus viewpoint on an existing quality-fair PI-based media delivery control scheme
- Equivalent conditions above provide PI gain tuning technique
- Simulation results confirm effectiveness and provide assessment of previous tuning

▷ **Future Directions**

- Extend the theoretical results to the case with complex eigenvalues
- Allow for static nonlinearities to improve the effectiveness with different streams
- Propose an alternative decentralized MANE (theoretical extension is straightforward)

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