Clegg integrators and First Order Reset Elements:
theoretical and experimental results for a class of hybrid dynamical systems

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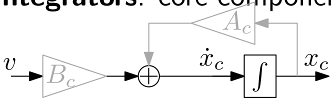
Outline

1. Clegg integrators and First Order Reset Elements (FORE) and an overview of hybrid dynamical systems
2. Exponential stability of reset control systems
3. Set-point regulation of linear plants using adaptive FORE
4. Some additional hybrid applications

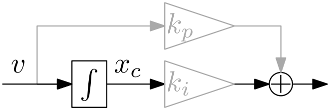
An analog integrator and its Clegg extension (1956)

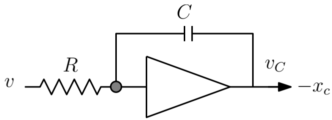
Integrators: core components of dynamical control systems



$$\dot{x}_c = A_c x_c + B_c v$$

Example: PI controller



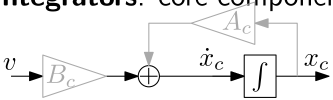


$$\dot{x}_c = \frac{1}{RC} v$$

- In an analog integrator, the state information is stored in a capacitor:

An analog integrator and its Clegg extension (1956)

Integrators: core components of dynamical control systems



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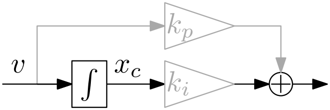
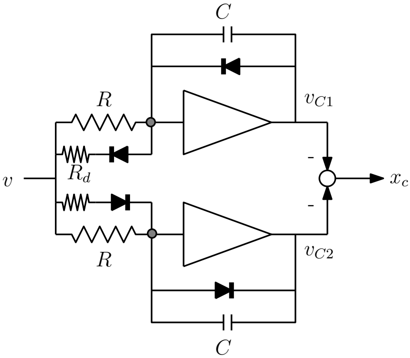


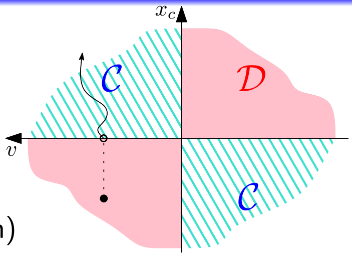


- Clegg's integrator (1956):
  - *feedback diodes*: the **positive** part of x_c is all and only coming from the **upper** capacitor (and viceversa)
  - *input diodes*: when $v \leq 0$ the upper capacitor is reset and the lower one integrates (and viceversa) $[R_d \ll 1]$
- As a consequence $\Rightarrow$ v and x_c **never have opposite signs**

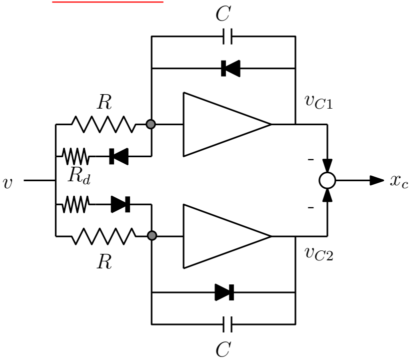
Hybrid dynamics may flow or jump

Hybrid Clegg integrator:

$$\dot{x}_c = \frac{1}{RC} v, \qquad \text{allowed when } x_c v \geq 0,$$
$$x_c^+ = 0, \qquad \text{allowed when } x_c v \leq 0,$$



- Flow set $\mathcal{C}$: where x_c may flow (1st eq'n)
- Jump set $\mathcal{D}$: where x_c may jump (2nd eq'n)



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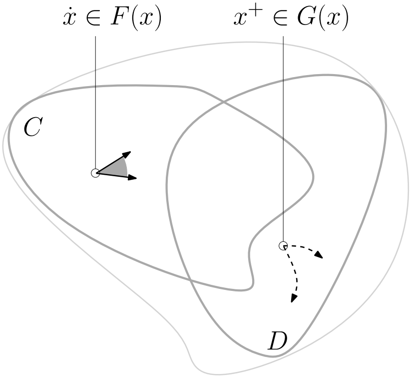
- As a consequence $\Rightarrow$ v and x_c **never have opposite signs**

Hybrid dynamical systems review: **dynamics**

$\mathcal{H} = (\mathcal{C}, \mathcal{D}, F, G)$

- $n \in \mathbb{N}$ (state dimension)
- $\mathcal{C} \subseteq \mathbb{R}^n$ (flow set)
- $\mathcal{D} \subseteq \mathbb{R}^n$ (jump set)
- $F : \mathcal{C} \rightrightarrows \mathbb{R}^n$ (flow map)
- $G : \mathcal{D} \rightrightarrows \mathbb{R}^n$ (jump map)

$$\mathcal{H} : \begin{cases} \dot{x} \in F(x), & x \in \mathcal{C} \\ x^+ \in G(x), & x \in \mathcal{D} \end{cases}$$



Hybrid dynamical systems review: **continuous dynamics**

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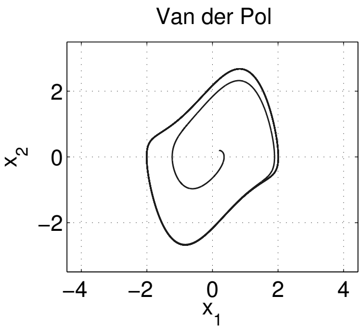
$$\mathcal{H} : \begin{cases} \dot{x} \in F(x), & x \in \mathcal{C} \\ x^+ \in G(x), & x \in \mathcal{D} \end{cases}$$

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + x_2(1 - x_1^2) \end{cases}$$



Hybrid dynamical systems review: **discrete dynamics**

$\mathcal{H} = (\mathcal{C}, \mathcal{D}, F, G)$

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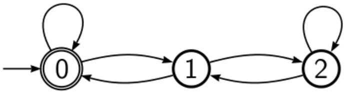
- $\mathcal{D} \subseteq \mathbb{R}^n$ (jump set)
- $F : \mathcal{C} \rightrightarrows \mathbb{R}^n$ flow map)
- $G : \mathcal{D} \rightrightarrows \mathbb{R}^n$ (jump map)

$$\mathcal{H} : \begin{cases} \dot{x} \in F(x), & x \in \mathcal{C} \\ x^+ \in G(x), & x \in \mathcal{D} \end{cases}$$

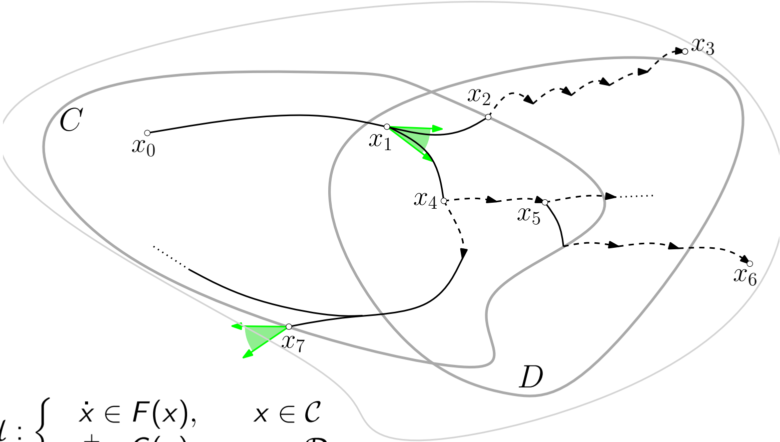
$$x^+ \in \begin{cases} \{0,1\} & \text{if } x = 0 \\ \{0,2\} & \text{if } x = 1 \\ \{1,2\} & \text{if } x = 2 \end{cases}$$



A possible sequence of states from $x_0 = 0$ is:

$$(0 \cdot 1 \cdot 2 \cdot 1)^i \quad i \in N$$

Hybrid dynamical systems review: **trajectories**

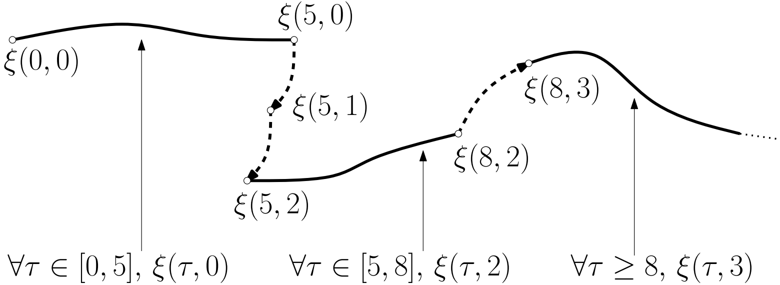


$$\mathcal{H}: \begin{cases} \dot{x} \in F(x), & x \in \mathcal{C} \\ x^+ \in G(x), & x \in \mathcal{D} \end{cases}$$

Hybrid dynamical systems review: **hybrid time**

The motion of the state is parameterized by two parameters:

- $t \in \mathbb{R}_{\geq 0}$, takes into account the elapse of time during the continuous motion of the state;
- $j \in \mathbb{Z}_{\geq 0}$, takes into account the number of jumps during the discrete motion of the state.



Hybrid dynamical systems review: **hybrid time**

$E \subseteq \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is a **compact hybrid time domain** if

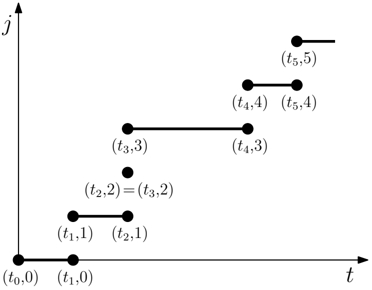
$$E = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}] \times \{j\})$$

where $0 = t_0 \leq t_1 \leq \cdots \leq t_J$.

E is a **hybrid time domain** if for all $(T, J) \in \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$

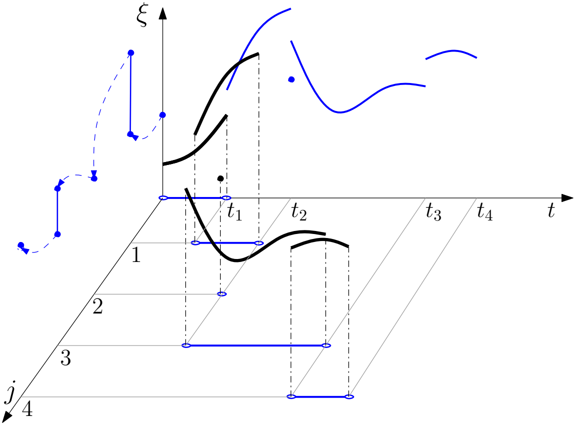
$$E \cap ([0, T] \times \{0, 1, \ldots, J\})$$

is a compact hybrid time domain.



Hybrid dynamical systems review: **solution**

- Formally, a solution satisfies the flow dynamics when flowing and satisfies the jump dynamics when jumping



Hybrid dynamical systems review: **Lyapunov theorem**

Theorem Given the Euclidean norm $|x| = \sqrt{x^T x}$ and a hybrid system

$$\mathcal{H}: \begin{cases} \dot{x} = f(x), & x \in \mathcal{C} \\ x^+ = g(x), & x \in \mathcal{D}, \end{cases}$$

aassume that function $V: \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfies for some scalars c_1, c_2 positive and c_3 positive:

$$\begin{aligned} c_1|x|^2 \leq V(x) \leq c_2|x|^2, & \qquad \forall x \in \mathcal{C} \cup \mathcal{D} \cup G(\mathcal{D}) \\ \langle \nabla V(x), f(x) \rangle \leq -c_3|x|^2, & \qquad \forall x \in \mathcal{C}, \\ V(g(x)) - V(x) \leq -c_3|x|^2, & \qquad \forall x \in \mathcal{D}, \end{aligned}$$

then the origin is uniformly globally exponentially stable (UGES) for $\mathcal{H}$, namely there exist $K, \lambda > 0$ such that all solutions satisfy

$$|\xi(t,j)| \leq K e^{\lambda(t+j)} |\xi(0,0)|, \quad \forall (t,j) \in \operatorname{dom} \xi$$

Note: Lyapunov conditions comprise **flow** and **jump** conditions.

Note: UGAS is characterized in terms of hybrid time (t, j)

Hybrid dynamical systems review: **Lyapunov theorem**

Theorem Given a closed set $\mathcal{A} \subset \mathbb{R}^n$ and a hybrid system

$$\mathcal{H} : \begin{cases} \dot{x} \in F(x), & x \in \mathcal{C} \\ x^+ \in G(x), & x \in \mathcal{D}, \end{cases}$$

aassume that function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfies for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and ρ positive definite:

$$\begin{aligned} &\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}}), && \forall x \in \mathcal{C} \cup \mathcal{D} \cup G(\mathcal{D}) \\ &\langle \nabla V(x), f \rangle \leq -\rho(|x|_{\mathcal{A}}), && \forall x \in \mathcal{C}, f \in F(x), \\ &V(g) - V(x) \leq -\rho(|x|_{\mathcal{A}}), && \forall x \in \mathcal{D}, g \in G(x) \end{aligned}$$

then $\mathcal{A}$ is uniformly globally asymptotically stable (UGAS) for $\mathcal{H}$, namely there exists $\beta \in \mathcal{KL}$ such that all solutions satisfy

$$|\xi(t,j)|_{\mathcal{A}} \leq \beta(|\xi(0,0)|_{\mathcal{A}}, t+j), \quad \forall (t,j) \in \mathrm{dom}\ \xi$$

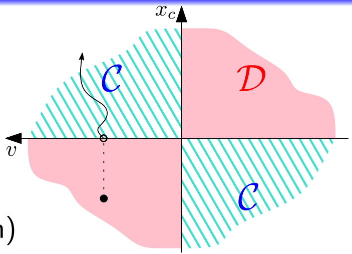
Note: Lyapunov conditions comprise **flow** and **jump** conditions.

Note: UGAS is characterized in terms of hybrid time (t,j)

Hybrid dynamics and the Clegg integrator (recall)

Hybrid Clegg integrator:

$$\begin{aligned} \dot{x}_c &= \frac{1}{RC} v, && \text{allowed when } x_c v \geq 0, \\ x_c^+ &= 0, && \text{allowed when } x_c v \leq 0, \end{aligned}$$



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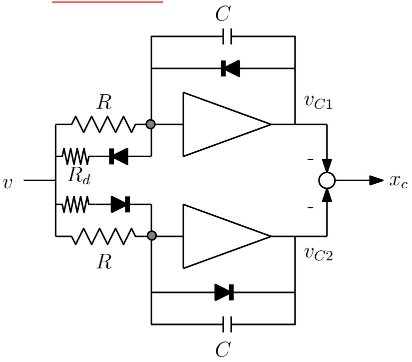

- Clegg's integrator (1956):
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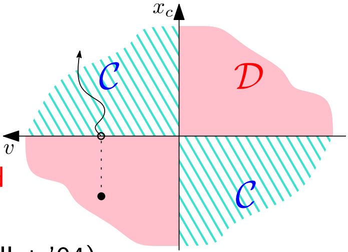
- As a consequence $\Rightarrow$ v and x_c **never have opposite signs**

Hybrid dynamics of the Clegg integrator (revisited)

Hybrid Clegg integrator:

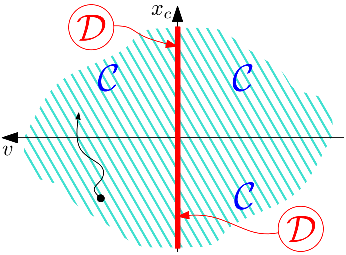
$$\dot{x}_c(t,j) = (RC)^{-1} v(t,j), \quad x_c(t,j)v(t,j) \geq 0,$$
$$x_c(t,j+1) = 0, \quad x_c(t,j)v(t,j) \leq 0,$$



- Flow set $\mathcal{C} := \{(x_c, v) : x_c v \geq 0\}$ is closed
- Jump set $\mathcal{D} := \{(x_c, v) : x_c v \leq 0\}$ is closed
- Stability is robust! (Teel 2006–2012)

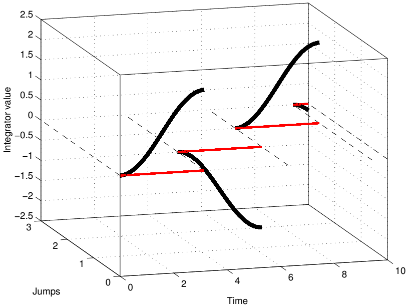
Previous models (Clegg '56, Horowitz '73, Hollot '04):

$$\begin{aligned} \dot{x}_c &= (RC)^{-1} v, && \text{if } v \neq 0, \\ x_c^+ &= 0, && \text{if } v = 0, \end{aligned}$$



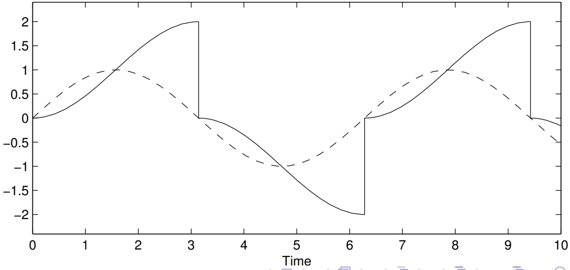
- Imprecise: solutions $\exists$ s.t. $x_c v < 0$, but Clegg's x_c and v **always** have same sign!
- Unrobust: $\mathcal{C}$ is almost all $\mathbb{R}^2$ (arbitrary small noise disastrous)
- Unsuitable: Adds extra solutions $\Rightarrow$ Lyapunov results too conservative!

Example: Clegg response to a sine input



- Input selected as $v(t,j) = \sin(t)$ (dashed line below)
- Solution (bold black) as a function of the hybrid time domain (red)
- State x_c is reset upon entering the 2nd and 4th quadrants (in this case $\equiv$ at the zero crossing)

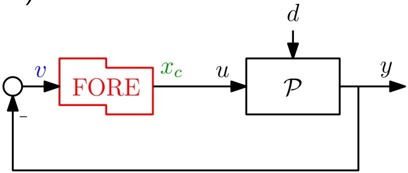
- Solid: projection of x_c on the ordinary time axis t
- Dash: projection of u on the ordinary time axis t



Stabilization using hybrid jumps to zero

First Order Reset Element (Horowitz '74):

$$\dot{x}_c = a_c x_c + b_c v, \qquad x_c v \geq 0,$$
$$x_c^+ = 0, \qquad x_c v \leq 0,$$



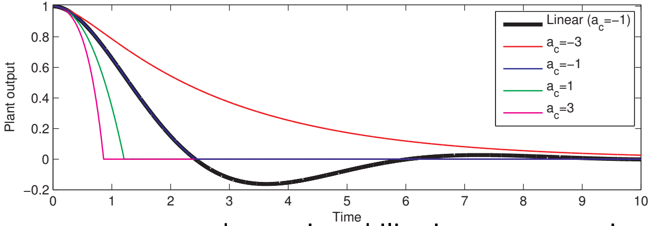
Theorem If $\mathcal{P}$ is linear, minimum phase and relative degree one, **then** a_c, b_c or (a_c, b_c) large enough $\Rightarrow$ global exponential stability

Theorem In the planar case, γ_{dy} shrinks to zero as parameters grow

Simulation uses:

$$\mathcal{P} = \frac{1}{s}$$

$$b_c = 1$$



Interpretation: Resets remove overshoots, instability improves transient

Piecewise quadratic Lyapunov function construction

- Given $N \geq 2$ (number of sectors)
- Patching angles:

$$-\theta_\epsilon = \theta_0 < \theta_1 < \cdots < \theta_N = \frac{\pi}{2} + \theta_\epsilon$$

- Patching hyperplanes ($C_p = [0 \ \cdots \ 0 \ 1]$)

$$\Theta_i = \left[\ 0_{1\times(n-2)} \quad \sin(\theta_i) \quad \cos(\theta_i) \ \right]^T$$

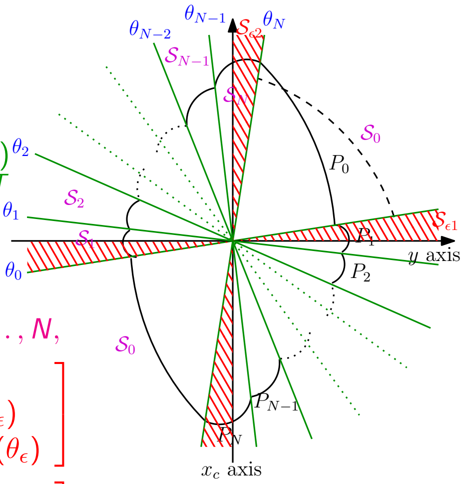
- Sector matrices:

$$S_0 := \Theta_0 \Theta_N^T + \Theta_N \Theta_0^T$$

$$S_i := -(\Theta_i \Theta_{i-1}^T + \Theta_{i-1}\Theta_i^T), \quad i = 1, \ldots, N,$$

$$S_{\epsilon 1} := \begin{bmatrix} 0_{(n-2)\times(n-2)} & 0 & 0 \\ 0 & 0 & \sin(\theta_\epsilon) \\ 0 & \sin(\theta_\epsilon) & -2\cos(\theta_\epsilon) \end{bmatrix}$$

$$S_{\epsilon 2} := \begin{bmatrix} 0_{(n-2)\times(n-2)} & 0 & 0 \\ 0 & -2\cos(\theta_\epsilon) & \sin(\theta_\epsilon) \\ 0 & \sin(\theta_\epsilon) & 0 \end{bmatrix}$$



Hybrid closed-loop:

$$\begin{aligned} \dot{x} &= A_F x + B_d d, && x \in \mathcal{C} \\ x^+ &= A_J x, && x \in \mathcal{D} \end{aligned}$$

Piecewise quadratic Lyapunov theorem

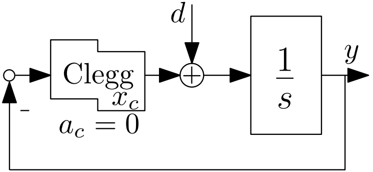
Theorem: If the following LMIs in the green unknowns (where $Z = [I_{n-2}\ 0_{(n-2)\times 2}]$) are feasible:

$$
\begin{aligned}
&(Flow) && \begin{bmatrix} A_F^T P_i + P_i A_F + \tau_{Fi} S_i & P_i B_d & C^T \\ \star & -\gamma I & 0 \\ \star & \star & -\gamma I \end{bmatrix} < 0, i = 1, \dots, N, \\
&(Jump) && A_J^T P_1 A_J - P_0 + \tau_J S_0 \leq 0 \\
&(Cont'ty) && \Theta_{i\perp}^T (P_i - P_{i+1}) \Theta_{i\perp} = 0, \quad i = 0, \dots, N-1, \\
&(Cont'ty) && \Theta_{N\perp}^T (P_N - P_0) \Theta_{N\perp} = 0 \\
&(Overlap) && A_J^T P_1 A_J - P_1 + \tau_{\epsilon 1} S_{\epsilon 1} \leq 0 \\
&(Overlap) && A_J^T P_1 A_J - P_N + \tau_{\epsilon 2} S_{\epsilon 2} \leq 0 \\
&(Origin) && \begin{bmatrix} Z(A_F^T P_0 + P_0 A_F) Z^T & Z P_0 B_d & Z C^T \\ \star & -\gamma I & 0 \\ \star & \star & -\gamma I \end{bmatrix} < 0,
\end{aligned}
$$

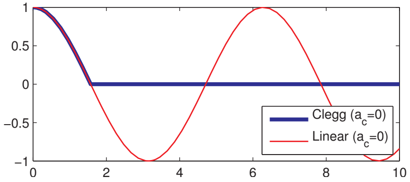
then global exponential stability + finite $\mathcal{L}_2$ gain γ_{dy} from d to y

Example 1: Clegg ($a_c = 0$) connected to an integrator

- Block diagram:



- Output response (overcomes linear systems limitations)



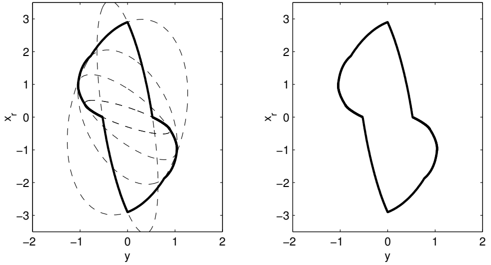
- Quadratic Lyapunov functions are unsuitable

- Gain γ_{dy} estimates ($N = \#$ of sectors)

N	2	4	8	50
gain γ_{dy}	2.834	1.377	0.914	0.87

- A lower bound: $\sqrt{\frac{\pi}{8}} \approx 0.626$

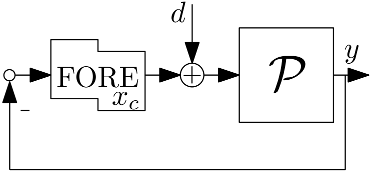
- Lyapunov func'n level sets for $N = 4$



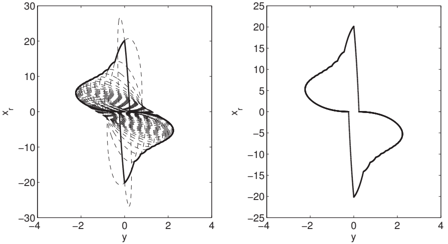
- $P_1, \ldots, P_4$ cover 2nd/4th quadrants
- P_0 covers 1st/3rd quadrants

Example 2: FORE (any a_c) and linear plant (Hollot et al.)

- Block diagram ($\mathcal{P} = \frac{s+1}{s(s+0.2)}$)



- $a_c = 1$: level set with $N = 50$



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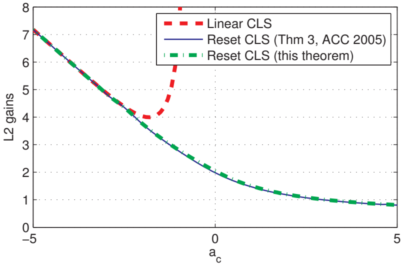
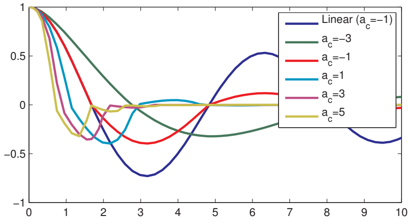


- Time responses



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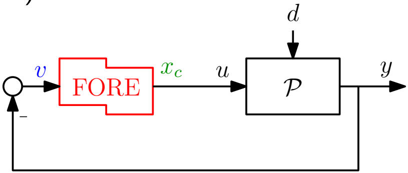

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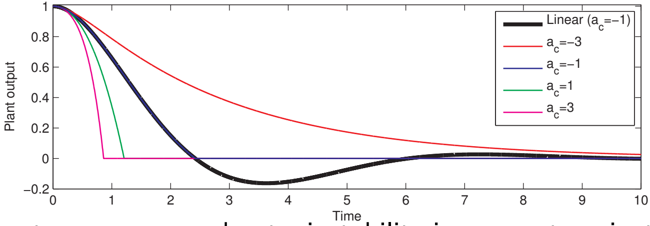
$$\mathcal{P} = \frac{1}{s}$$

$$b_c = 1$$



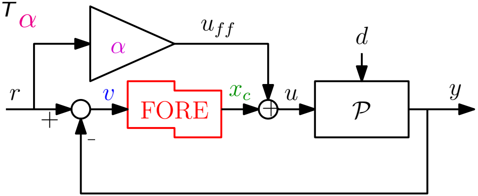
Interpretation: Resets remove overshoots, instability improves transient

Set point adaptive **regulation** using hybrid jumps to zero

- Parametric feedforward $u_{ff} = \Psi(r)^T \alpha$

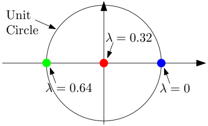
$$\begin{cases} \dot{x}_c = a_c x_c + b_c v, \\ \dot{\alpha} = 0, \end{cases} \quad x_c v \geq 0,$$

$$\begin{cases} x_c^+ = 0, \\ \alpha^+ = \alpha + \lambda \frac{\Psi(r)}{|\Psi(r)|^2} x_c, \end{cases} \quad x_c v \leq 0,$$

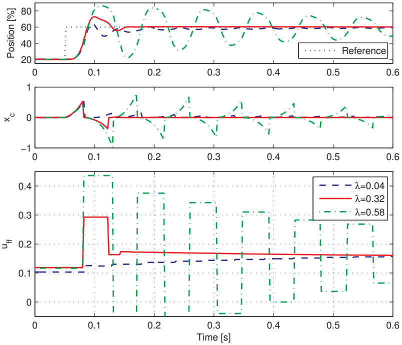


Theorem: If FORE stabilizes with $r = 0$, then for constant r, $y \to r$

Lemma: Tuning of λ using discrete-time rules (Ziegler-Nichols)

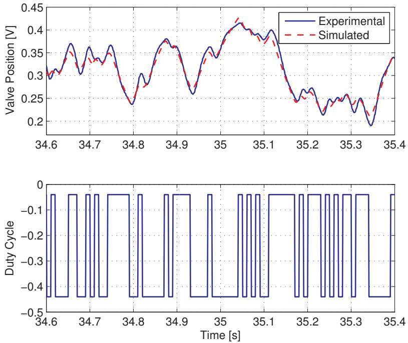


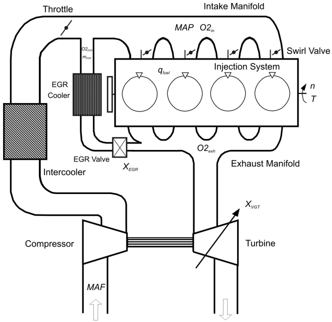
Example: EGR Experiment (next slide)



Fast regulation of EGR valve position in Diesel engines

- EGR: Recirculates Exhaust Gas in Diesel engines
- Subject to strong disturbances $\Rightarrow$ need aggressive controllers (recall exp. unstable transients)

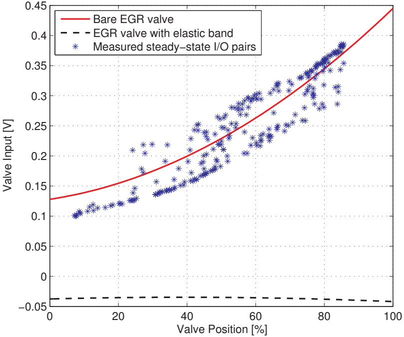




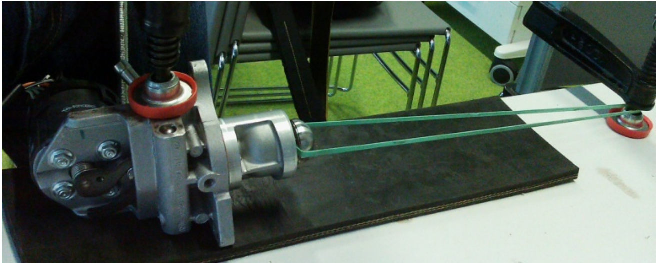
- Identified valve transfer function:

$$P(s) = \frac{2200}{(s + 164.4)(s + 10.69)}.$$

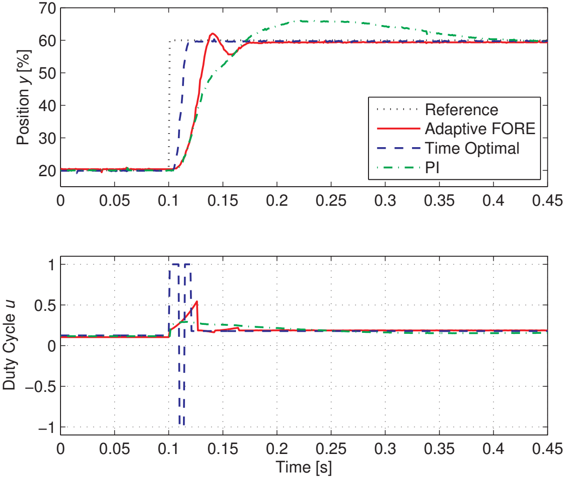
Feedforward: α converges to suitable parametrization



- $\star$: steady-state input/output pairs (stiction!!)
- Red Solid: $u_{ff} = \Psi^T(r)\alpha^*$, with α^* steady-state for α
- Black dashed: $u_{ff} = \Psi^T(r)\bar{\alpha}^*$ when pulling the valve with an elastic band



Laboratory experiments close to time-optimal

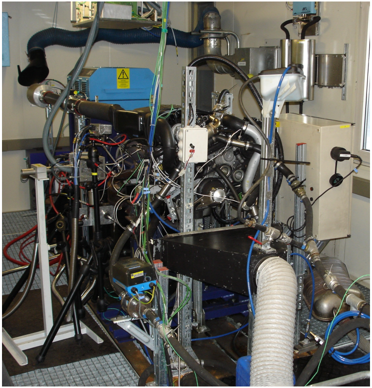


- Time-optimal:
  unrobust, obtained via trial and error
- PI:
  Tuned using standard MATLAB tools
- Adaptive FORE:
  Response after $\alpha \to \alpha^* = (0.128,\ 0.087,\ 0.115)$

- Note the exponentially diverging voltage:
  aggressive action for disturbance rejection on the real engine

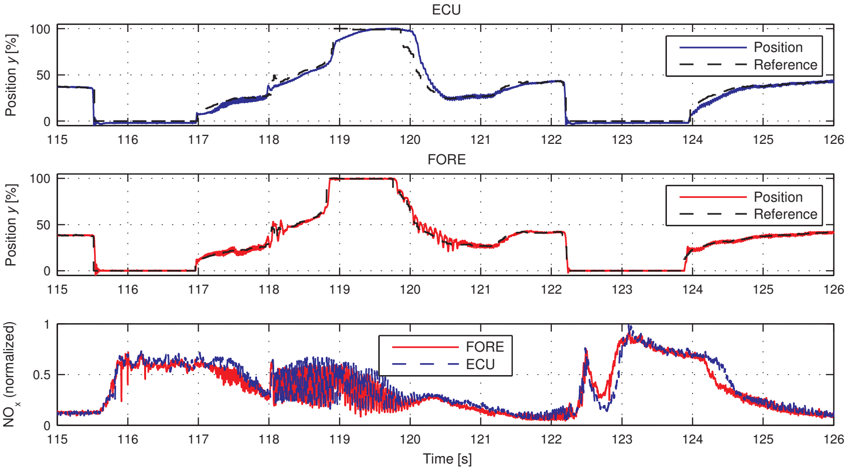
Experiments on Diesel engine testbench (JKU)

Experimental testbench at the Johannes Kepler Universitet (Linz, Austria)



- **Specs**: 2 liter, 4 cylinder passenger car turbocharged Diesel engine
- **Compared**: to factory EGR valve controller coded in ECU (gain scheduled PI with feedforward)
- **Test cycle**: Urban part of *New European Driving Cycle*
- **Relevance**: Faster EGR positioning $\Rightarrow$ Reduced NO_x emissions

Adaptive FORE provides substantial performance increase



- Mean squared error: ECU $= 6.68$ (100%), FORE $= 1.53$ (23 %)
- Improvement most important with EGR almost closed ($t \approx 117,\ 124$)

From discrete + continuous to hybrid dynamical systems

$$\underbrace{\text{Continuous dynamical system } \frac{dx(t)}{dt} = f(x(t)),\ \forall t \in \mathbb{R}_{\geq 0} \qquad \text{Discrete dynamical system } x(k+1) = g(x(k)),\ \forall k \in \mathbb{Z}_{\geq 0} \text{ (possible discrete variables)}}$$

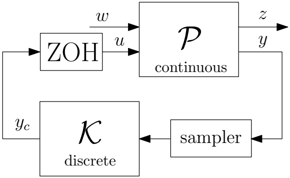
$$\Downarrow$$

Hybrid dynamical system

$$\begin{cases} \dfrac{dx(t,k)}{dt} = f(x(t,k)), & x(t,k) \in \mathcal{C} \subset \mathbb{R}^n \\ x(t,k+1) = g(x(t,k)), & x(t,k) \in \mathcal{D} \subset \mathbb{R}^n \end{cases}$$

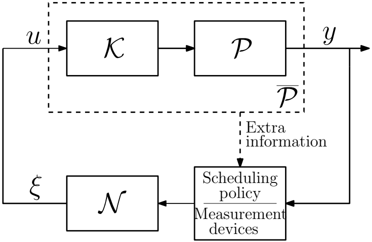
- Continuous time domain $t \in \mathbb{R}_{\geq 0}$ and Discrete time domain $k \in \mathbb{Z}_{\geq 0}$ merged into Hybrid time domain $(t,k) \in \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$
- Solution x can "flow" if $\in \mathcal{C}$, can "jump" if $\in \mathcal{D}$
- Fundamental stability results now available (Converse Lyapunov theorems, ISS, invariance principle, $\mathcal{L}_p$ stability) [Teel '04 $\to$ '12]

Hybrid systems tools utilized in diverse scenarios



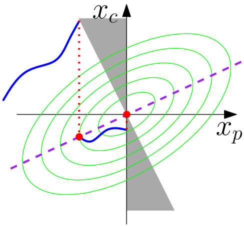
Sampled-data control design with saturation

- uniform sampling time assumed
- jumps correspond to sampling actions



Lazy sensors for reduced transmission rate

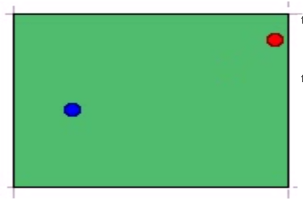
- reduce sample transmission over networks
- a special case of event-triggered sampling
- jumps occur at samples transmissions



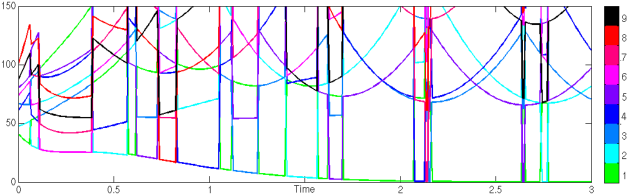
Nonlinear stabilization via hybrid loops

- enforces jumps of the controller state in some sets
- so as to guarantee decrease of suitable functions
- useful, e.g., for overshoot reduction with SISO plants
- enables design of hybrid $\mathcal{H}_\infty$ controllers

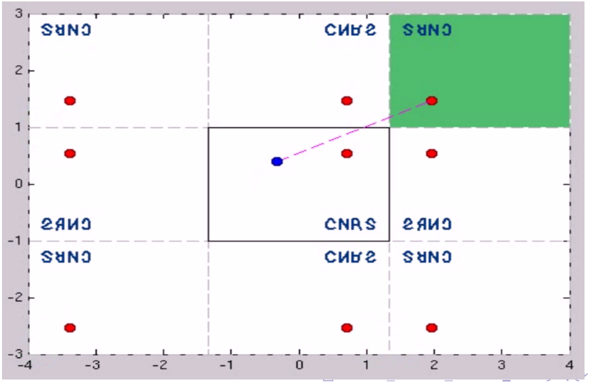
Impacting systems: billiard ball x tracks billiard ball z



Lyapunov functions $V_i(x, z)$ along hybrid trajectory



Linear Feedback
Hybrid Feedback



Idea: Follow the "closest" ball among all the target balls z mirrored by the walls

Use Hybrid Lyapunov function

$$V(x, z) = \min_{i \in \{0,\dots,9\}} \underbrace{|x - m_i(z)|_P}_{V_i(x,z)}$$

where

$m_0(z) = z$ (real ball)

$m_k(z) = M_k z + c_k$, $k = 1, \dots, 8$ (mirrored balls)