

# Output feedback tracking in billiards using mirrors without smoke

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# Outline

The impact dynamics

Issues with linear controller/observer

Hybrid tracking/observer laws for polyhedral billiards

Extension to convex billiards

Special polyhedral billiards

Output feedback tracking

Conclusions

# Impulsive or jump dynamics: impacts

Polyhedral region

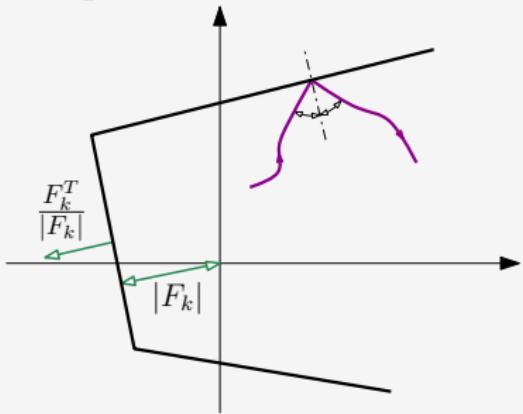
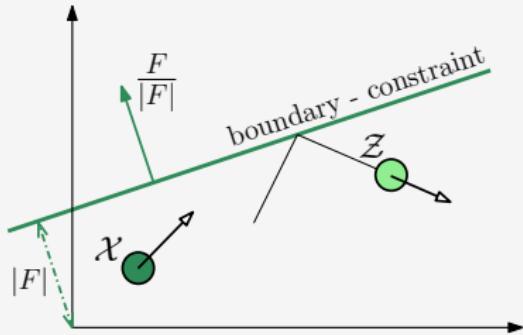
$$\mathcal{F} := \{s \in \mathbb{R}^4 \mid \forall q \in \mathcal{Q}, \langle F_q, s_p - s_o \rangle \leq 1\}$$

Dynamic boundary

$$\mathcal{J} := \{s \in \mathcal{F} \mid \exists q \in \mathcal{Q}, \langle F_q, s_p - s_o \rangle = 1, \langle F_q, s_v \rangle \geq 0\}$$

Reset at impacts (impulsive phenomena)

$$s^+ = \begin{bmatrix} s_p \\ M(F_q)s_v \end{bmatrix}$$



# Continuous or flow dynamics: free motion

Reference mass dynamics

$$\mathcal{Z} : \begin{cases} \dot{z}_p = z_v \\ \dot{z}_v = \alpha \\ y = Cz \quad (= z_p) \end{cases}$$

Controlled mass dynamics

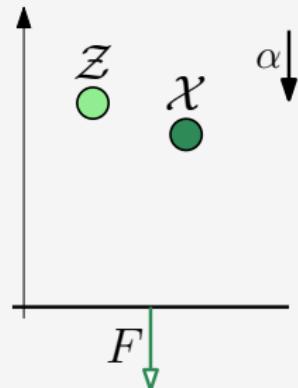
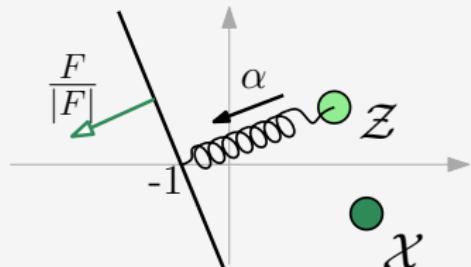
$$\mathcal{X} : \dot{x} = Ax + Bu_c$$

Observer dynamics

$$\hat{\mathcal{X}} : \dot{\hat{x}} = A\hat{x} + u_o$$

$\alpha \in \mathbb{R}^n$  measured,

$$A = \left[ \begin{array}{c|c} 0 & I \\ \hline 0 & 0 \end{array} \right], \quad B = \left[ \begin{array}{c} 0 \\ I \end{array} \right],$$



# Linear feedback does not guarantee stability/convergence

Formulation: asymptotic stability of

$$\mathcal{A}_o = \{(x, z) \mid x = z\}$$

Linear feedback:

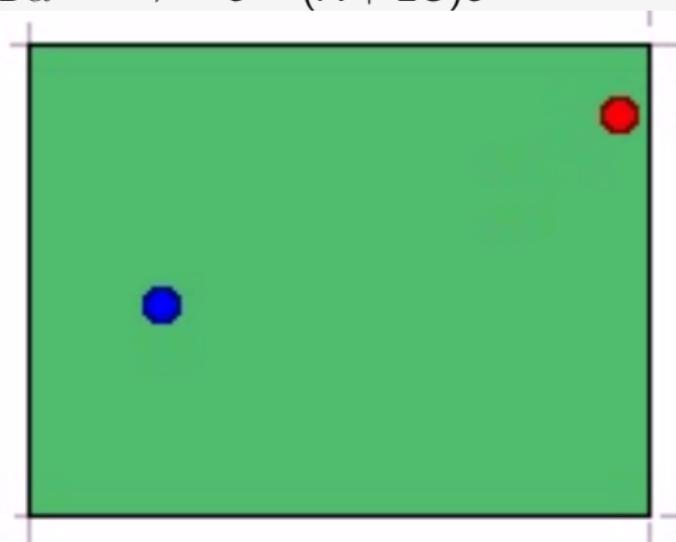
$$\begin{aligned} u_c &= K(x - z) + \alpha & \Rightarrow \dot{e} &= (A + BK)e \\ u_o &= L(x_p - z_p) + B\alpha & \Rightarrow \dot{e} &= (A + LC)e \end{aligned}$$

Choose  $K$  and  $L$  s.t.

$$\begin{aligned} V &= (x - z)^T P (x - z) \\ &=: |x - z|_P^2 \\ \dot{V} &< -\gamma V, \gamma > 0 \end{aligned}$$

$\Rightarrow \mathcal{A}_o$  asymptotically stable  
(without impacts).

Impacts: **stability, convergence!**



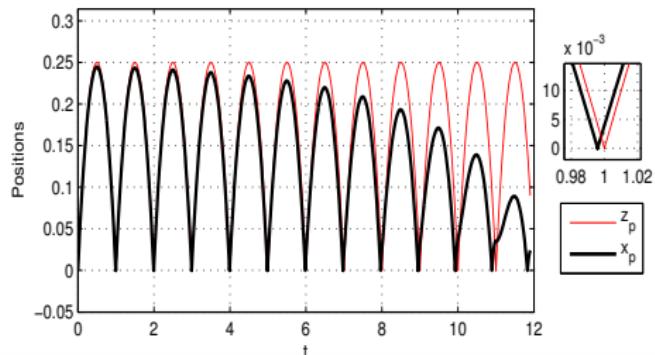
# Bouncing ball tracking is a simple example of instability

Initial conditions and gains:

$$z_0 = [0 \ v]^T$$

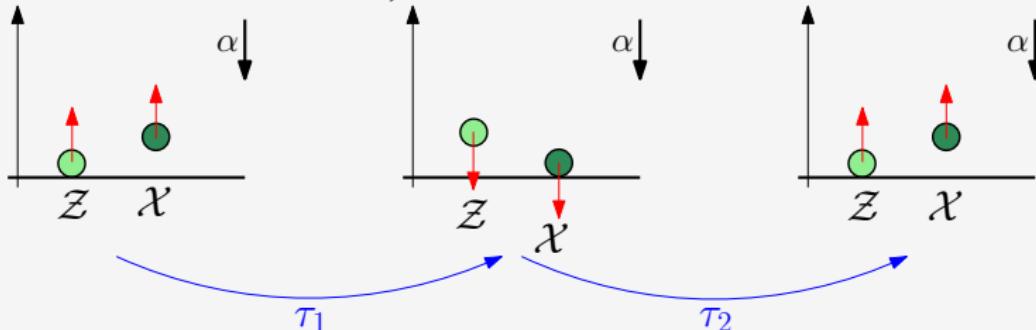
$$x_0 = z_0 + \varepsilon$$

$$K = [-4 \ -4]$$

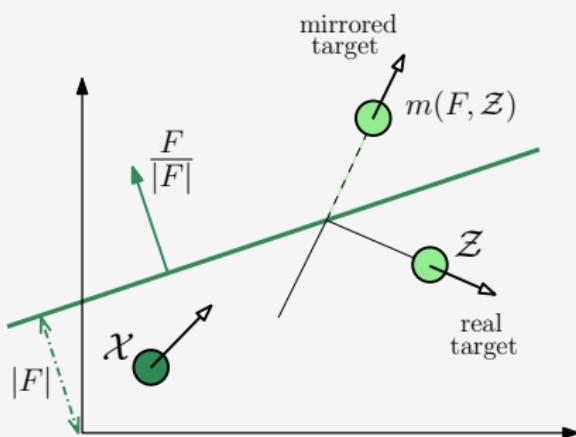


$e = x - z$  at the  $k$ th impact of  $\mathcal{Z}$  is given approximately by

$$\left( \begin{bmatrix} -1 & 0 \\ (8+2\frac{\alpha}{v}) & -1 \end{bmatrix} e^{\begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix} \frac{2v}{\alpha}} \right)^k \varepsilon, \quad \Rightarrow \quad \text{Unstable for } \frac{v}{\alpha} \leq 0.613.$$



# A possible solution: mirroring through the boundary



Tracking controller:

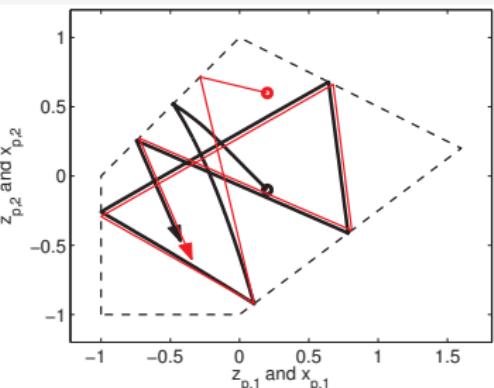
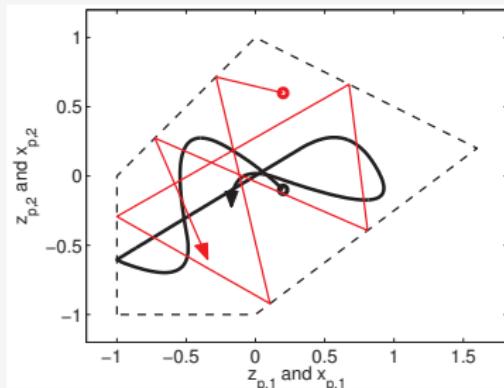
$$m(q, z) : \mathcal{Q} \times \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$$

Observer:

$$m_o(q, z_p) : \mathcal{Q} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$$

$$m_o(q, z_p) = Cm(q, z)$$

- ▶  $q = 0$  real target  $\mathcal{Z}$ ;
- ▶  $q = i$  mirrored target  $\mathcal{Z}$  through  $F_i$ ;



# Hybrid solution generalizes linear controller/observer

Linear:

$$\mathcal{A}_o = \{(x, z) \mid x = z\}$$

$$u_c = K(x - z) + \alpha$$

$$u_o = L(x_p - z_p) + B\alpha$$

$$e = x - z$$

$$e_p = x_p - z_p$$

$$= C(x - z)$$

$$\dot{e} = (A + BK)e$$

$$\dot{e} = (A + LC)e$$

$$V = |x - z|_P^2$$

$$\dot{V} \leq -\gamma V$$

Hybrid:

$$\mathcal{A} = \{(x, z) \mid x = m(q, z), q \in \mathcal{Q}\} \cap \mathcal{F} \times \mathcal{F}$$

$$u_c = K(x - m(q, z)) + M(q)\alpha$$

$$u_o = L(x_p - m_o(q, z_p)) + M_e(q)B\alpha$$

$$q = \underset{q^* \in \mathcal{Q}}{\operatorname{argmin}} |x - m(q^*, z)|_P$$

$$e = x - m(q, z)$$

$$e_p = x_p - m_o(q, z)$$

$$= C(x - m(q, z))$$

$$\dot{e} = (A + BK)e$$

$$\dot{e} = (A + LC)e$$

$$V = |x - m(q, z)|_P^2 = \min_{q^* \in \mathcal{Q}} |x - m(q^*, z)|_P^2$$

$$\dot{V} \leq -\gamma V$$

$$V^+ \leq V$$

# Key steps in the proof of asymptotic tracking/estimation

Hybrid:

$$\begin{aligned}|x_p - z_p| &\leq |x - m(q, z)| \\&\leq R|(x, z)|_{\mathcal{A}} \leq R|x - m(q, z)|\end{aligned}$$

$$\Rightarrow (x, z) \in \mathcal{A} \text{ implies } x_p = z_p$$

argmin well defined in  $\mathcal{A} + \varepsilon \mathbb{B}$

$z$  replaced by reflection  $m(q, z)$

$q$  selects suitable reflection and adds necessary degree of freedom

Error dynamics is linear!

Use identities like

$$|x - m(q, z)|_P = |m(q, x) - z|_P,$$

$$|m(q, x) - m(q, z)|_P = |x - z|_P$$

Rely on average dwell time logic

$$\mathcal{A} = \{(x, z) \mid x = m(q, z), q \in \mathcal{Q}\} \cap \mathcal{F} \times \mathcal{F}$$

$$u_c = K(x - m(q, z)) + M(q)\alpha$$

$$u_o = L(x_p - m_o(q, z_p)) + M_e(q)B\alpha$$

$$q = \operatorname{argmin}_{q^* \in \mathcal{Q}} |x - m(q^*, z)|_P$$

$$e = x - m(q, z)$$

$$e_p = x_p - m_o(q, z)$$

$$= C(x - m(q, z))$$

$$\dot{e} = (A + BK)e$$

$$\dot{e} = (A + LC)e$$

$$V = |x - m(q, z)|_P^2 = \min_{q^* \in \mathcal{Q}} |x - m(q^*, z)|_P^2$$

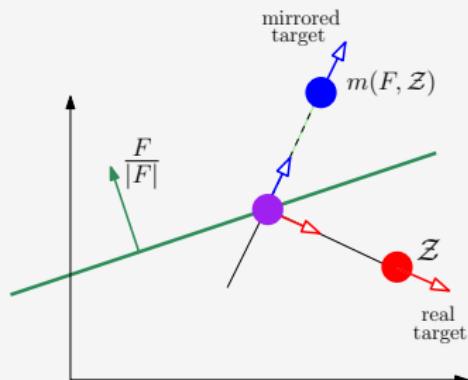
$$\dot{V} \leq -\gamma V$$

$$V^+ \leq V$$

# $V$ does not increase at impacts: one boundary ( $\mathcal{Z}$ impacts)

$$\begin{aligned} V &= |x - m(q, z)|_P^2 \\ &= \min_{q^* \in Q} |x - m(q^*, z)|_P^2 \end{aligned}$$

- 1) Recognize  
 $z^+ = m(1, z)$  when  $\mathcal{Z}$  impacts  $F$ ,  
 $x^+ = m(1, x)$  when  $\mathcal{X}$  impacts  $F$ .



mirroring of the mirrored image!

- 2) Since  $z^+ = m(1, z)$  then  
 $m(1, z^+) = m(1, m(1, z)) = z$

- 3) If  $P = [ \begin{smallmatrix} p_1 & p_2 \\ p_2 & p_3 \end{smallmatrix} ] \otimes I_2$  and  
 $K = [ \begin{smallmatrix} k_1 & k_2 \end{smallmatrix} ] \otimes I_2$ , then  
 $|m(1, x) - m(1, z)|_P = |x - z|_P$   
 $|x - m(1, z)|_P = |m(1, x) - z|_P$

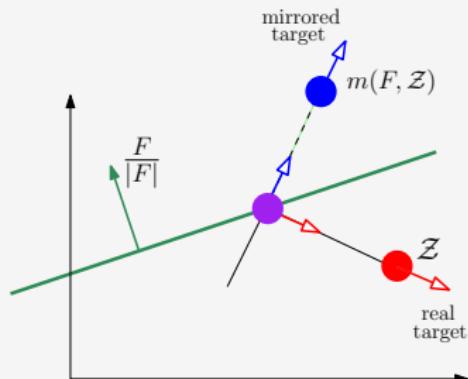
4)  $\mathcal{Z}$  impacts  $F$ :

$$\begin{aligned} \min_{q^* \in \{0,1\}} |x^+ - m(q^*, z^+)|_P &= \min_{q^* \in \{0,1\}} |x - m(q^*, m(1, z))|_P \\ &= \min\{|x - m(0, m(1, z))|_P, |x - m(1, m(1, z))|_P\} \\ &= \min\{|x - m(1, z))|_P, |x - z|_P\} \\ V^+ &= V \end{aligned}$$

# $V$ does not increase at impacts: one boundary ( $\mathcal{X}$ impacts)

$$\begin{aligned} V &= |x - m(q, z)|_P^2 \\ &= \min_{q^* \in Q} |x - m(q^*, z)|_P^2 \end{aligned}$$

- 1) Recognize  
 $z^+ = m(1, z)$  when  $\mathcal{Z}$  impacts  $F$ ,  
 $x^+ = m(1, x)$  when  $\mathcal{X}$  impacts  $F$ .



mirroring of the mirrored image!

- 2) Since  $z^+ = m(1, z)$  then  
 $m(1, z^+) = m(1, m(1, z)) = z$

- 3) If  $P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \otimes I_2$  and  
 $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix} \otimes I_2$ , then  
 $|m(1, x) - m(1, z)|_P = |x - z|_P$   
 $|x - m(1, z)|_P = |m(1, x) - z|_P$

5)  $\mathcal{X}$  impacts  $F$ :

$$\begin{aligned} \min_{q^* \in \{0,1\}} |x^+ - m(q^*, z^+)|_P &= \min_{q^* \in \{0,1\}} |m(1, x) - m(q^*, z)|_P \\ &= \min\{|m(1, x) - m(0, z)|_P, |m(1, x) - m(1, z)|_P\} \\ &= \min\{|m(1, x) - z|_P, |m(1, x) - m(1, z)|_P\} \\ &= \min\{|x - m(1, z))|_P, |x - z|_P\} \\ V^+ &= V \end{aligned}$$

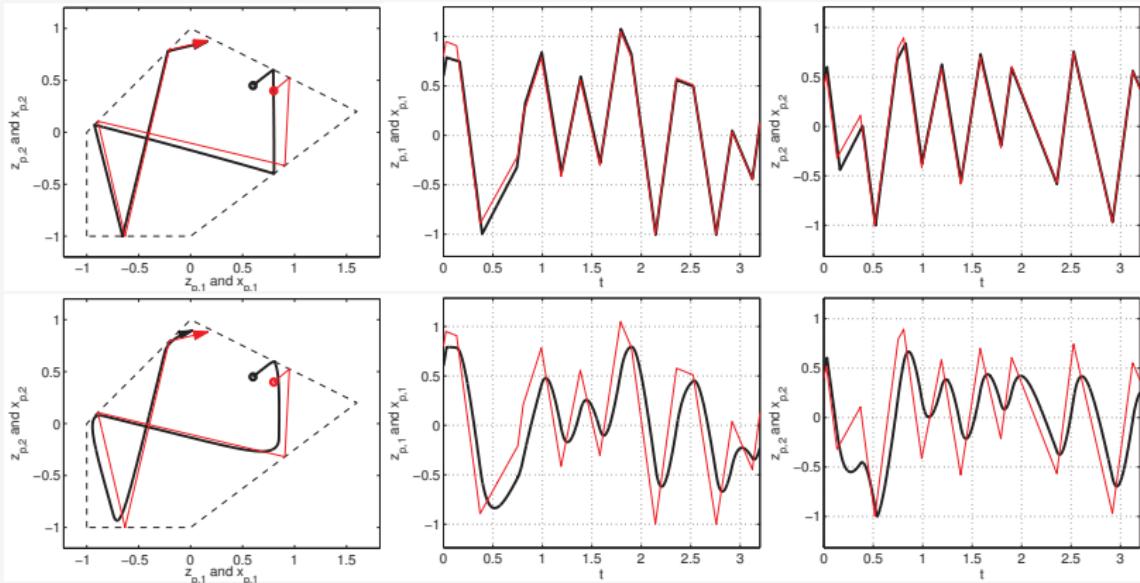
# Simulation: state feedback tracking hybrid vs linear

Billiard:

$$[ F_1 \mid F_2 \mid F_3 \mid F_4 \mid F_5 ] = \left[ \begin{array}{c|c|c|c|c} 0 & -1 & -1 & \frac{1}{2} & \frac{3}{4} \\ -1 & 0 & 1 & 1 & -1 \end{array} \right]$$

Parameters:

$$P = \left[ \begin{array}{c|c} 0.4I & 0.1I \\ 0.1I & 0.2I \end{array} \right] \text{ and } K = -[ 10I \mid 11I ],$$



Local tracking in **convex billiards**

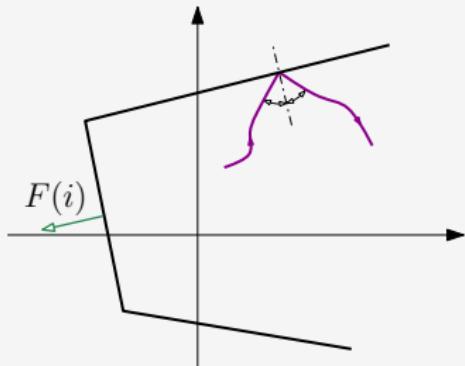
# System dynamics generalized to a convex billiard

Polyhedral region  $\mathcal{F}_p \subseteq \mathbb{R}^2$   
given by  $F : \mathcal{Q} \rightarrow \mathbb{R}^2$ ,

$$F(q)^T(s_p - \eta_o) \leq 1, \forall s_p \in \mathcal{F}_p, q \in \mathcal{Q}$$
$$\exists q : F(q)^T(s_p - \eta_o) = 1, \forall s_p \in \partial \mathcal{F}_p.$$

Given  $s = [s_p^T \ s_v^T]$ , impacts occur  
when  $\exists q$  such that

$$F(q)^T(s_p - \eta_o) = 1$$
$$F(q)^T(s_v - \eta_o) \geq 0$$



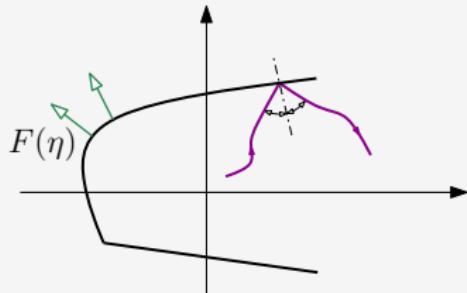
Convex region  $\mathcal{F}_p \subseteq \mathbb{R}^2$   
given by  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$F(\eta)^T(s_p - \eta_o) \leq 1, \forall s_p \in \mathcal{F}_p, \eta \in \partial \mathcal{F}_p$$
$$F(\eta)^T(\eta - \eta_o) = 1, \forall \eta \in \partial \mathcal{F}_p.$$

Given  $s = [s_p^T \ s_v^T]$ , impacts occur  
when

$$F(s_p)^T(s_p - \eta_o) = 1$$
$$F(s_p)^T(s_v - \eta_o) \geq 0$$

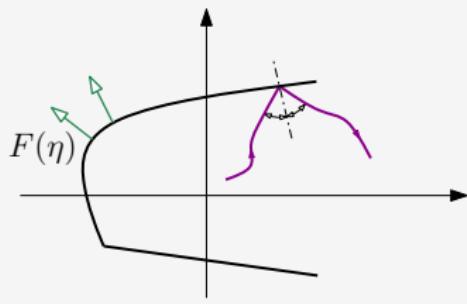
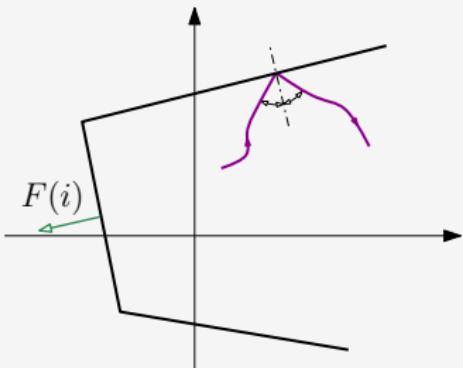
$F(\cdot)$  is piecewise locally Lipschitz.



# System dynamics generalized to a convex billiard

Reset at impacts:

$$s^+ = \begin{bmatrix} s_p \\ M(F(s_p))s_v \end{bmatrix}$$



## Possible choices of $q^*$ are updated at each impact

Augment the dynamics with  $\eta \in \partial\mathcal{F} \subset \mathbb{R}^2$ :

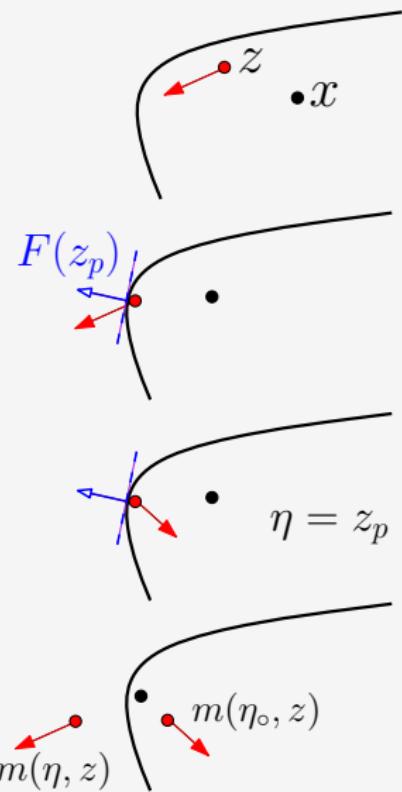
- $\eta^+ = z_p$  when  $\mathcal{Z}$  impacts
- $\eta^+ = x_p$  when  $\mathcal{X}$  impacts

Define  $m(\eta, z)$  as

- $m(\eta, z) = \text{mirroring of } z \text{ w.r.t. } F(\eta)$ ,
- $m(\eta_o, z) = z$

$$q = \underset{\eta^* \in \{\eta, \eta_o\}}{\operatorname{argmin}} |x - m(q^*, z)|_P$$

$$u = K(x - m(q, z)) + M(F(q))\alpha$$



## Lyapunov function may now increase at jumps (nasty!)

Augment the dynamics with  $\eta \in \partial\mathcal{F} \subset \mathbb{R}^2$ :

- $\eta^+ = z_p$  when  $\mathcal{Z}$  impacts
- $\eta^+ = x_p$  when  $\mathcal{X}$  impacts

Define  $m(\eta, z)$  as

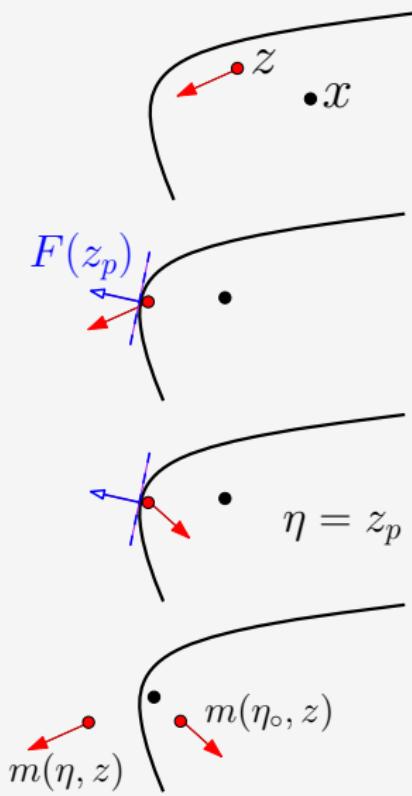
- $m(\eta, z) = \text{mirroring of } z \text{ w.r.t. } F(\eta)$ ,
- $m(\eta_o, z) = z$

$$V = \min_{q^* \in \{\eta, \eta_o\}} |x - m(q^*, z)|_P^2$$

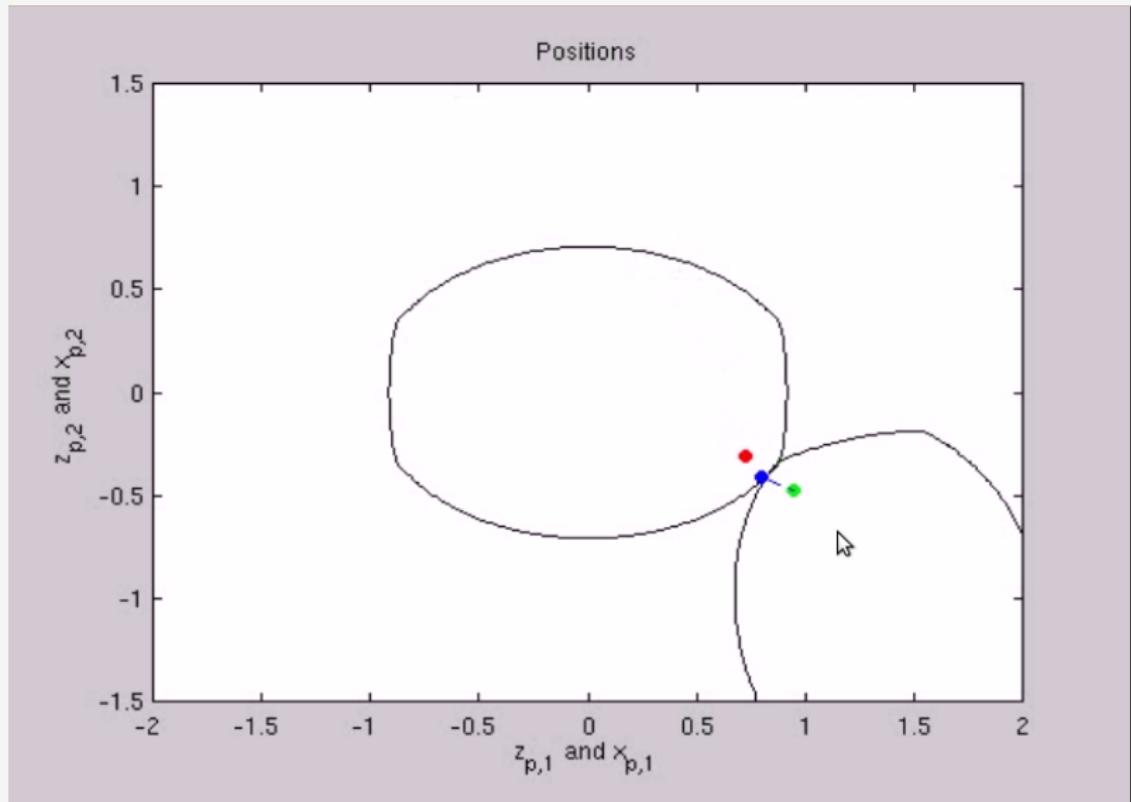
$$\dot{V} \leq -\gamma V$$

$$V^+ \leq V + e(L, x, z)$$

asymptotic stability if there is sufficient flow

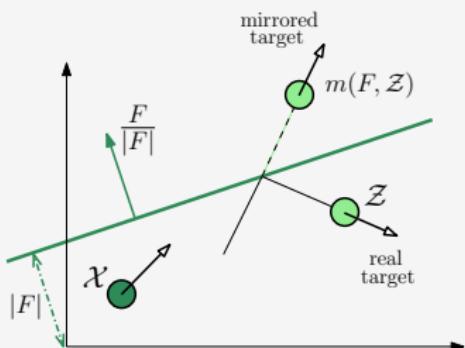


## Simulation: max of two quadratics defines boundary



Global tracking/observer for **special polyhedral billiards**

# Global tracking algorithm for **special polyhedral billiards**



Previous approach:

$$m(q, z) : \mathcal{Q} \times \mathbb{R}^4 \rightarrow \mathbb{R}^4$$

- ▶  $q = 0$  real target  $\mathcal{Z}$ ;
- ▶  $q = i$  mirrored target  $\mathcal{Z}$  through  $F_i$ ;

New ingredients:

- ▶  $q \in \mathcal{Q}$  (no  $F_i$ ) composition of mirroring;  
extra targets
- ▶ instead of  $q = \operatorname{argmin}_{q^* \in \mathcal{Q}} |x - m(q^*, z)|_P$ ,  
 $\dot{q} = 0$  (constant during flows),  
 $q^+ = ?$  (update at jumps)  
 $\operatorname{argmin}$  is well defined in  $\mathcal{A} + \varepsilon \mathbb{B}$ .

# Global tracking algorithm: **single boundary**

$$\mathcal{Q} = \{0, 1\}$$

Global exponential stability of:

$$e = x - m(q, z)$$

Using:

$$u_c = K(x - m(q, z)) + M(q)\alpha$$

$$u_o = L(x_p - m_o(q, z_p)) + M_e(q)B\alpha$$

$$q^+ = \underset{q^* \in \{0,1\}}{\operatorname{argmin}} |x^+ - m(q^*, z^+)|_P$$

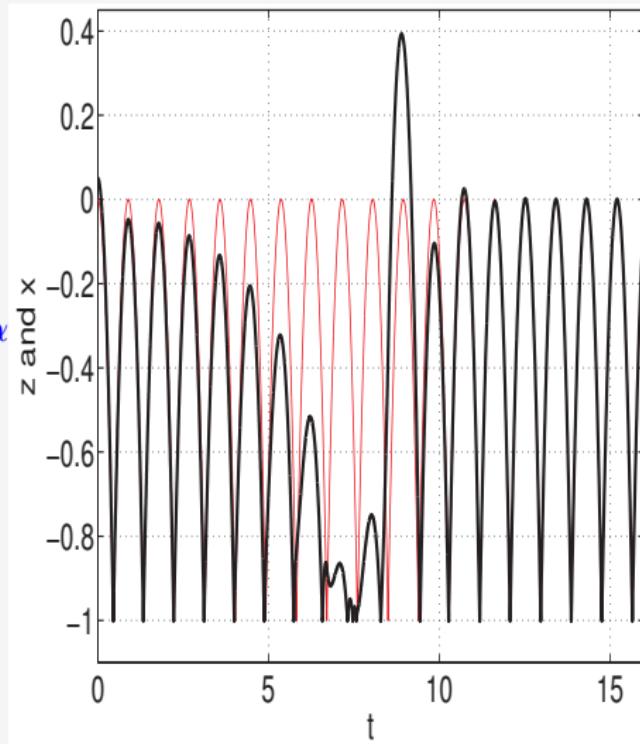
$$\text{expl. solut'n: } q^+ = 1 - q$$

Key facts:

$$|m(q, x) - z|_P = |x - m(q, z)|_P,$$

$$m(q^+, z^+) = m(q, z), \text{ (z impacts)},$$

$$m(q^+, x^+) = m(q, x), \text{ (x impacts)}$$



# Global tracking algorithm: two parallel boundaries

$$\mathcal{Q} = \{0, 1, 2\}$$

Global exponential stability of:

$$e = x - m(q, z)$$

Using:

$$u_c = K(x - m(q, z)) + M(q)\alpha$$

$$u_o = L(x_p - m_o(q, z_p)) + M_e(q)B\alpha$$

$$q^+ \approx \underset{q^* \in \{0,1,2\}}{\operatorname{argmin}} |x^+ - m(q^*, z^+)|_P$$

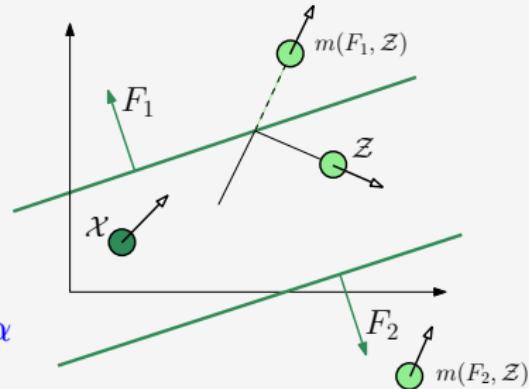
replaced by logic  $q^+ = \delta(q, i) \rightarrow$

Key facts:

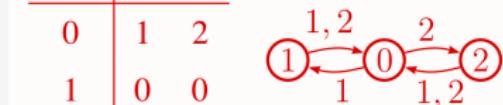
$$|x^+ - m(q^+, z^+)|_P \leq |x - m(q, z)|_P$$

with  $P$  diagonal ( $\dot{V} \leq -|Ce|^2$ )

Use LaSalle (for hybrid systems)



q \ i	1	2
0	1	2
1	0	0
2	0	0



# Global tracking algorithm: two orthogonal boundaries

$$\mathcal{Q} = \{0, 1, 2, 3\}$$

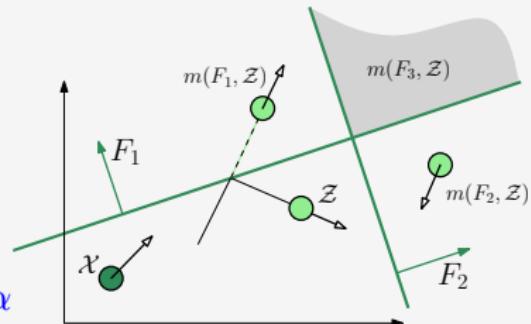
Global exponential stability of:

$$e = x - m(q, z)$$

Using:

$$u_c = K(x - m(q, z)) + M(q)\alpha$$

$$u_o = L(x_p - m_o(q, z_p)) + M_e(q)B\alpha$$



$$q^+ \approx \underset{q^* \in \{0,1,2,3\}}{\operatorname{argmin}} |x^+ - m(q^*, z^+)|_P$$

replaced by logic  $q^+ = \delta(q, i) \rightarrow$

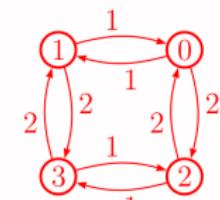
Key facts:

$$m(1, m(2, z)) = m(2, m(1, z)) =: m(3, z)$$

$$m(1, m(3, z)) = m(2, z)$$

$$m(2, m(3, z)) = m(1, z)$$

$q \setminus i$	1	2
0	1	2
1	0	3
2	3	0
3	2	1



# Global tracking algorithm: rectangular billiards

$$\mathcal{Q} = \{0, \dots, 8\}$$

Global exponential stability of:

$$e = x - m(q, z)$$

Using:

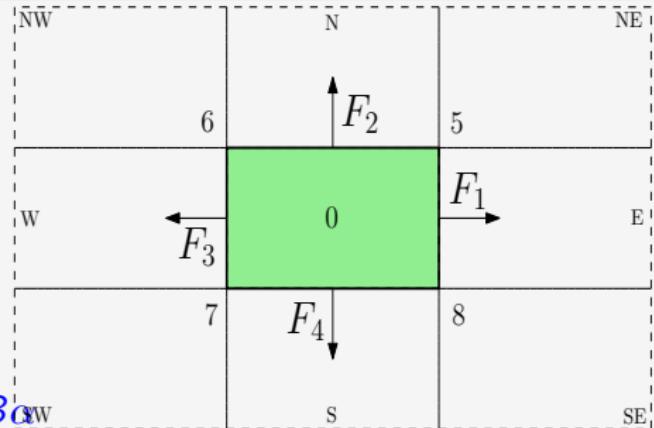
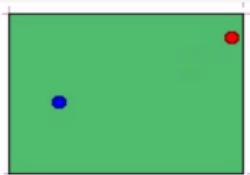
$$u_c = K(x - m(q, z)) + M(q)\alpha$$

$$u_o = L(x_p - m_o(q, z_p)) + M_e(q)B\alpha$$

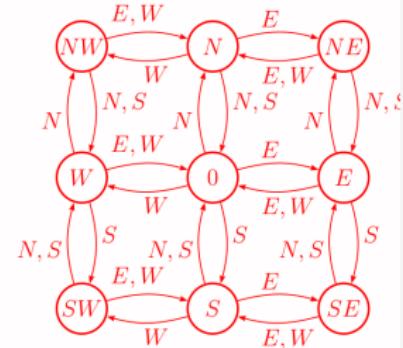
$$q^+ \approx \underset{q^* \in \{0,1,2,3\}}{\operatorname{argmin}} |x^+ - m(q^*, z^+)|_P$$

replaced by logic  $q^+ = \delta(q, i) \rightarrow$

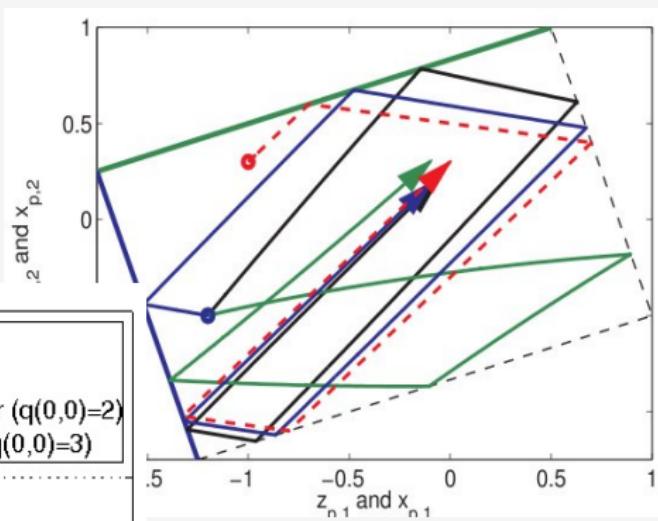
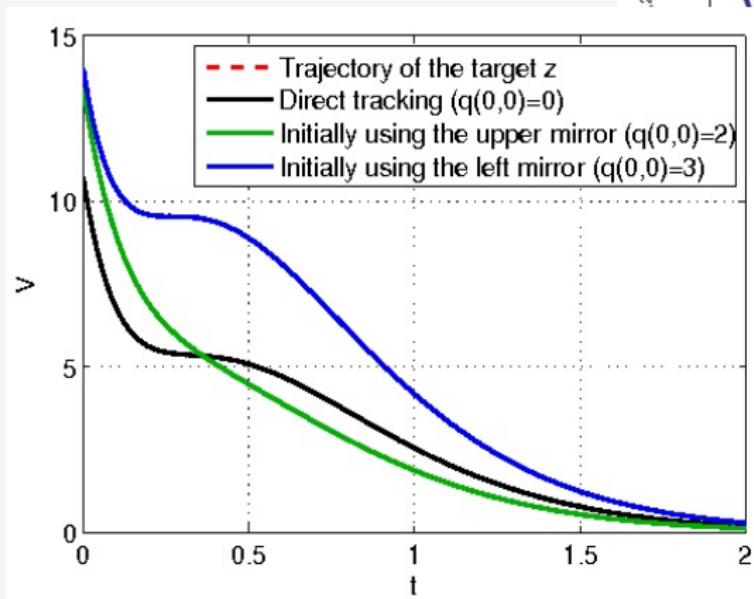
Key facts: Combine prev. results



$q \setminus i$	1	2	3	4
0	1	2	3	4
1	0	5	0	8
2	5	0	6	0
3	0	6	0	7
4	8	0	7	0
5	2	1	2	1
6	2	3	2	3
7	4	3	4	3
8	4	1	4	1



# Rectangular billiard: different initial conditions of $q$



Combining controller and observer for **Output feedback tracking**

# Controlled mass provides position measurement only

Flow dynamics:

$$\mathcal{Z} : \dot{z} = Az + B\alpha \quad \mathcal{X} : \begin{cases} \dot{x} = Ax + Bu_c \\ y = Cx \end{cases} \quad \hat{\mathcal{X}} : \begin{cases} \dot{\hat{x}} = A\hat{x} + u_o \\ \dot{q} = 0 \\ \dot{\hat{q}} = 0 \end{cases}$$

Exogenous

Controlled

Observer

Jump dynamics: Change of  $V$  at jumps easily extends

$$z^+ \in \bigcup_{i \in \mathcal{M}(z)} m(F_i, z)$$

$$x^+ = x$$

$$\hat{x}^+ = \hat{x}$$

$$q^+ = \delta(q, i)$$

$$\hat{q}^+ = \hat{q}$$

with  $z \in \mathcal{J}$

$$z^+ = z$$

$$x^+ = \bigcup_{i \in \mathcal{M}(x)} m(F_i, x)$$

$$\hat{x}^+ = \hat{x}$$

$$q^+ = \delta(q, i)$$

$$\hat{q}^+ = \delta(\hat{q}, i)$$

with  $x \in \mathcal{J}$

$$z^+ = z$$

$$x^+ = x$$

$$\hat{x}^+ = \bigcup_{i \in \mathcal{M}(\hat{x})} m(F_i, \hat{x})$$

$$q^+ = q$$

$$\hat{q}^+ = \delta(\hat{q}, i)$$

with  $(\hat{x}_p, \hat{x}_v + u_{o,p}) \in \mathcal{J}$

Controller and Observer inputs:

$$u_c = K(m(\hat{q}, \hat{x}) - m(q, z)) + M(q)\alpha$$

$$u_o = L(C\hat{x} - Cm(\hat{q}, x)) + M_e(\hat{q})Bu_c$$

## Choice of the error variables is a key to proving GES

Flow dynamics:

$$\mathcal{Z} : \dot{z} = Az + B\alpha \quad \mathcal{X} : \begin{cases} \dot{x} = Ax + Bu_c \\ y = Cx \end{cases} \quad \hat{\mathcal{X}} : \begin{cases} \dot{\hat{x}} = A\hat{x} + u_o \\ \dot{q} = 0 \\ \dot{\hat{q}} = 0 \end{cases}$$

Error dynamics:  $e_1 = x - m(q, z)$ ,  $e_2 = \hat{x} - m(\hat{q}, x)$

$$\begin{aligned}\dot{e}_2 &= A\hat{x} + LC(\hat{x} - m(\hat{q}, x)) + M_e(\hat{q})Bu_c - M_e(\hat{q})(Ax + Bu_c) \\ &= A\hat{x} + LCe_2 + M_e(\hat{q})Bu_c - AM_e(\hat{q})x - M_e(\hat{q})Bu_c \\ &= (A + LC)e_2\end{aligned}$$

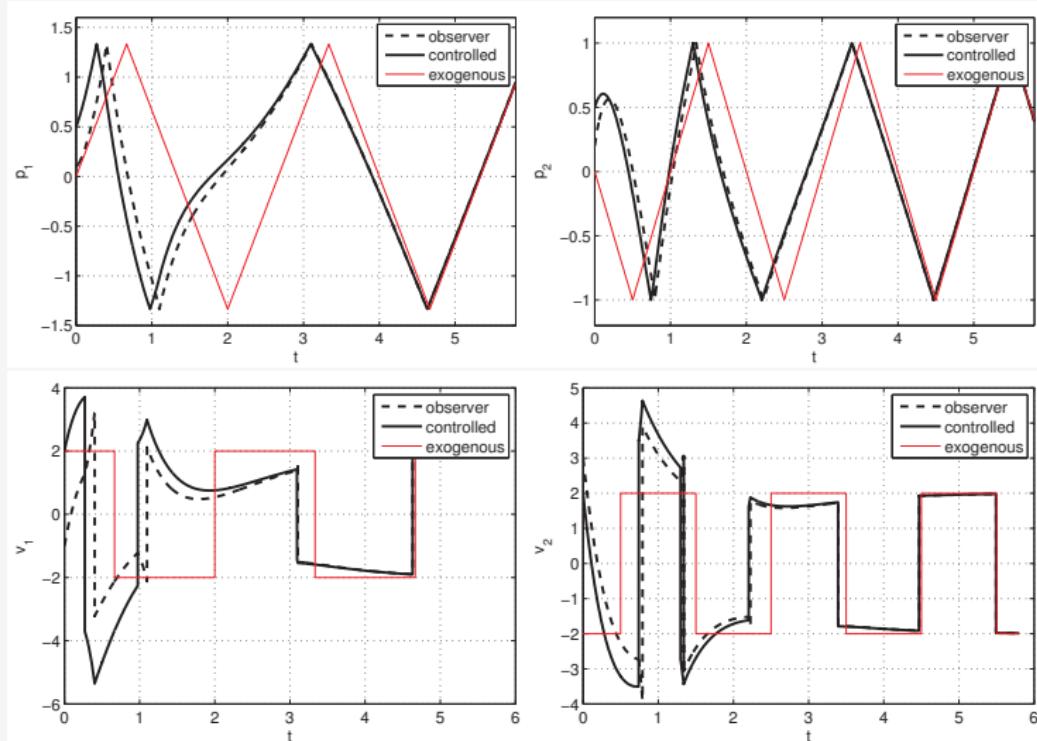
$$\begin{aligned}\dot{e}_1 &= Ax + BK(m(\hat{q}, \hat{x}) - m(q, z)) + BM(q)\alpha - M_e(q)(Az + B\alpha) \\ &= Ax + BK(m(\hat{q}, \hat{x}) - m(q, z)) + BM(q)\alpha - AM_e(q)z - BM(q)\alpha \\ &= Ae_1 + BK(m(\hat{q}, \hat{x}) - x + x - m(q, z)) \\ &= (A + BK)e_1 + BK(m(\hat{q}, \hat{x}) - x) \\ &= (A + BK)e_1 + BK M_e(\hat{q})(\hat{x} - m(\hat{q}, x)) = (A + BK)e_1 + BK M_e(\hat{q})e_2\end{aligned}$$

Controller and Observer inputs:

$$\begin{aligned}u_c &= K(m(\hat{q}, \hat{x}) - m(q, z)) + M(q)\alpha \\ u_o &= LC(\hat{x} - m(\hat{q}, x)) + M_e(\hat{q})Bu_c\end{aligned}$$

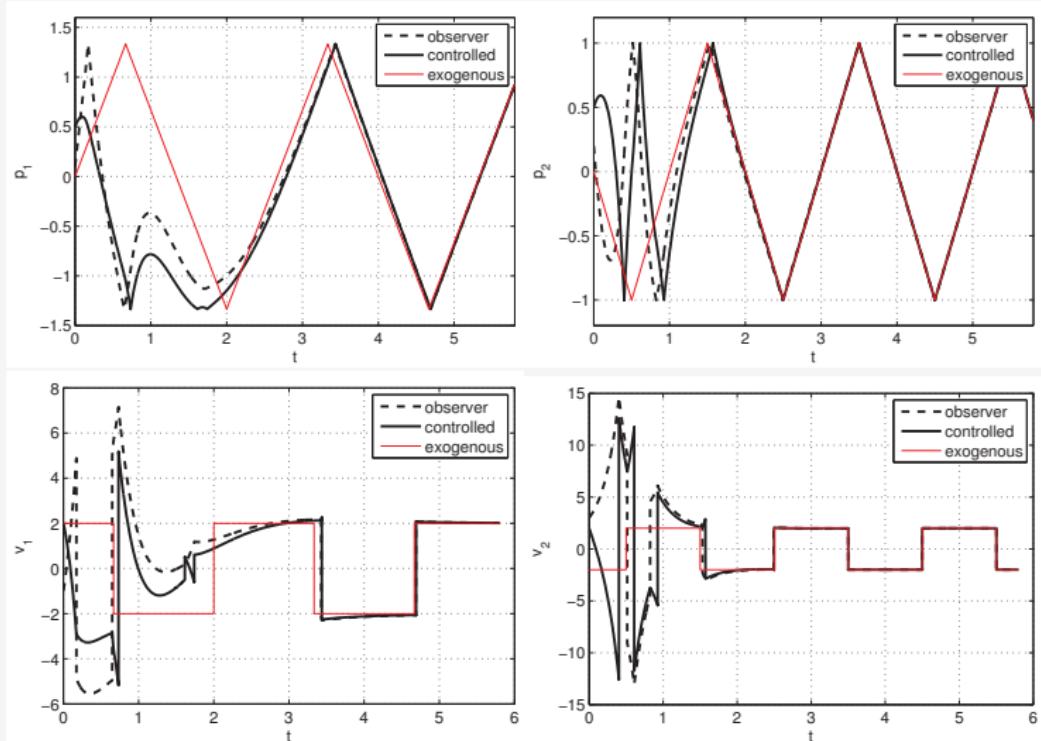
# Example: rectangular billiard once again

Starting from the initial conditions  $q_0 = 0, \dot{q}_0 = 0$



# Transient is affected by initial condition of $q, \dot{q}$

Starting from the initial conditions  $q_0 = 3, \dot{q}_0 = 8$



# Conclusions

## Summary:

- ▶ Tracking and Observation
  - (i) **global exponential results** for single wall, two parallel walls, two orthogonal walls, rectangular polyhedra, equilateral triangles, two walls that meet at special acute angles ( $\theta$  s.t  $\ell\theta = \pi$ ,  $\ell \in \mathbb{N}_{\geq 3}$ ).
  - (ii) **local results** for general polyhedra and convex piecewise Lipschitz boundaries
  - (iii) **separation principle** by suitably choosing feedback signals and error dynamics
- ▶ **No high-gain** required (any stabilizing gain  $K$  in the linear sense guarantees exponential stability)
- ▶ **Robustness to delays in impact detection** - semiglobal practical results.

## Extensions:

- ▶  $\mathbb{R}^2 \rightarrow \mathbb{R}^n$
- ▶ Impacts with dissipation
- ▶ More general continuous dynamics