Dynamic allocation of input-redundant control systems: theory and applications

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Problem Data

▷ A linear plant with *weak* or *strong* input redundancy

- **Weak**: means that equilibria can be induced by different input patterns
- **Strong**: means that transients can be induced by different input patterns

$$\begin{aligned} \dot{x} &= Ax + Bu + B_d d \\ y &= Cx + Du + D_d d, \end{aligned}$$

Def'n: A plant is input-redundant if one of the following two conditions is satisfied

- it is *strongly input-redundant* from u if it satisfies $\mathrm{Ker}\left(\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]\right) \neq \emptyset$; denote

$$B_\perp \text{ such that } \mathrm{Im}(B_\perp) = \mathrm{Ker}\left(\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]\right);$$

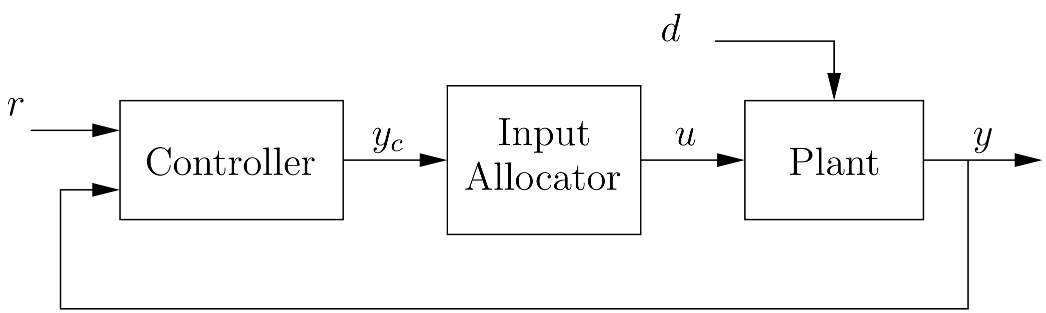
- it is *weakly input-redundant* from u to y if $P^\star := \lim_{s \to 0}(C(sI - A)^{-1}B + D)$ is finite and satisfies $\mathrm{Ker}(P^\star) \neq \emptyset$; denote

$$B_\perp \text{ such that } \mathrm{Im}(B_\perp) = \mathrm{Ker}(P^\star).$$

Key idea

▷ Assume that a controller has been designed disregarding input redundancy

$$\begin{aligned}\dot{x}_c &= A_c x_c + B_c y + B_r r \\ y_c &= C_c x_c + D_c y + D_r r,\end{aligned}$$



▷ Design an input allocator which

- exploits strong redundancy to achieve *fast reallocation* during transients
- exploits weak redundancy to achieve *slow reallocation* at the steady-state

▷ The allocator measures the controller output and adds a compensating signal

- Choose that signal as $B_\perp w$ for some w
- Pick w as the output of a pool of integrators (dynamic solution)

Linear solution - strong redundancy

$$\begin{aligned} \dot{w} &= -KB_\perp^T \bar{W}(u - u_0) \\ u &= y_c + B_\perp w, \end{aligned}$$

▷ K diagonal allows to promote/penalize different redundant directions

▷ $\bar{W}$ diagonal allows to promote/penalize different actuators

Th'm: If $K > 0$ and $B_\perp^T \bar{W} B_\perp > 0$ then internal stability and output response y unaffected by allocator

▷ Role of K: changes convergence speed but not the steady-state input:

$$u^\star = u_0 + \left(I - B_\perp (B_\perp^T \bar{W} B_\perp)^{-1} B_\perp^T \bar{W}\right) y_c^\star$$

which is the optimizer of $\min_w J(u) := (u - u_0)^T \bar{W}(u - u_0)$ (where $u = y_c^\star + B_\perp w$ is the steady-state plant input

▷ Role of $\bar{W}$: changes the steady-state input allocation

▷ u_0 is a useful drift term (will remove next for simplicity)

Example 1

▷ Randomly generated exponentially stable plant

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|ccc} -0.157 & -0.094 & 0.87 & 0.253 & 0.743 \\ -0.416 & -0.45 & 0.39 & 0.354 & 0.65 \\ \hline 0 & 1 & 0 & 0 & 0 \end{array}\right].$$

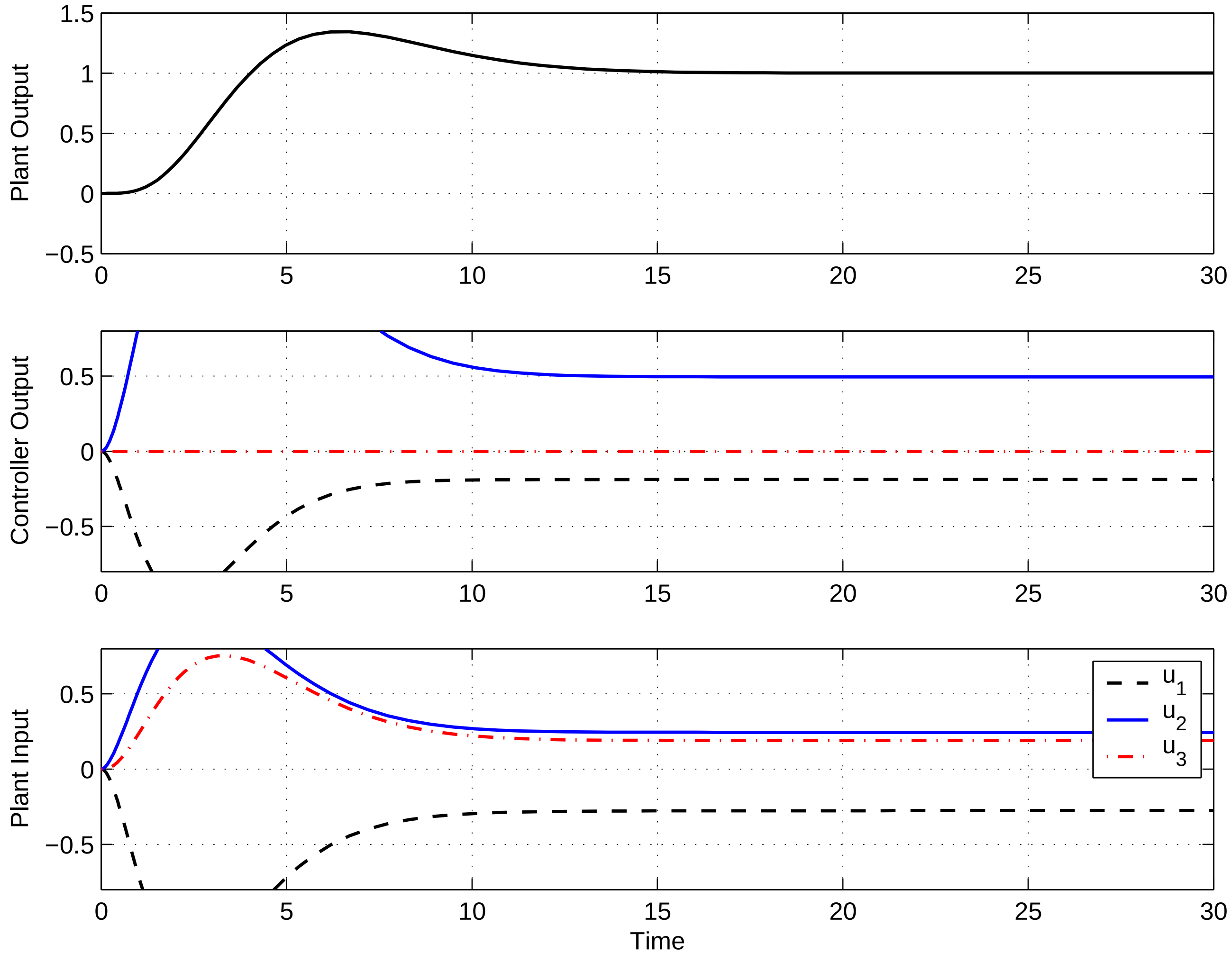
▷ Plant is strongly input redundant (one direction) and weakly input redundant (two directions) - will use it during the rest of the talk

▷ Controller design:

- negative error feedback interconnection;
- inserting an integrator;
- stabilizing LQG controller only using first two input channels

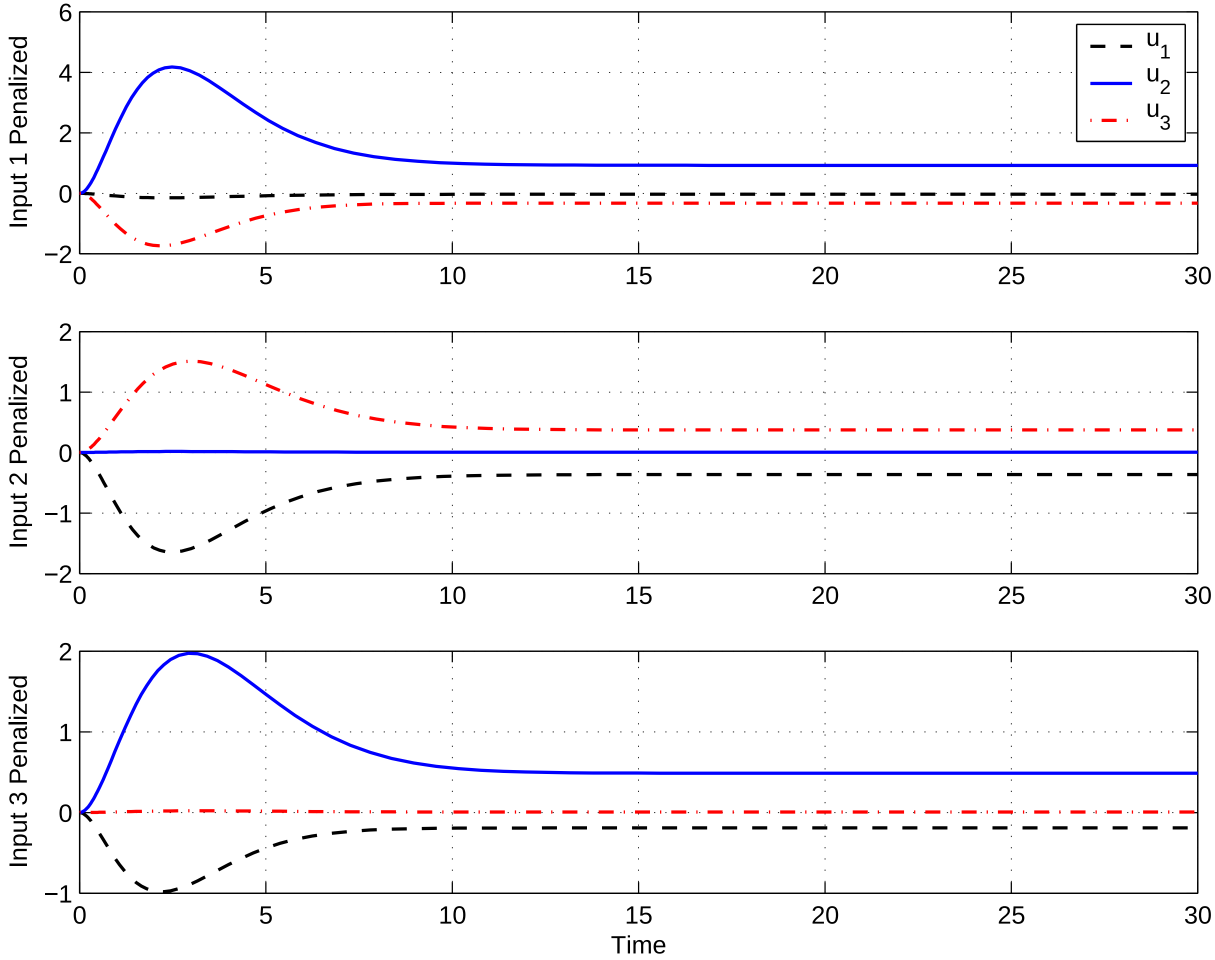
Example 1 (simulation)

▷ Responses using $K = 10I$ and $\bar{W} = I$



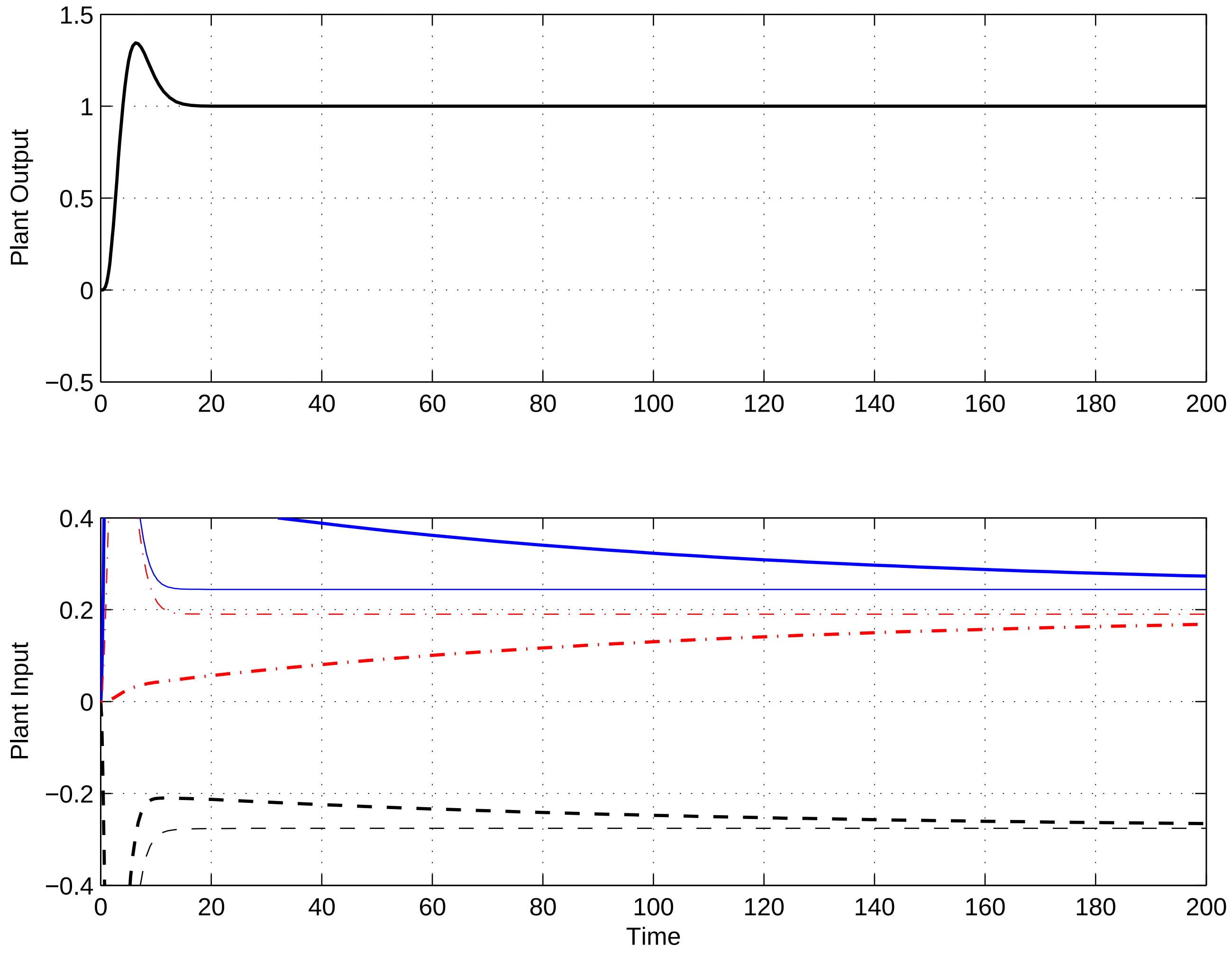
Example 1 (changing $\bar{W}$)

▷ Using $\bar{W} = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then $\bar{W} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and finally $\bar{W} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 100 \end{bmatrix}$



Example 1 (changing K)

▷ Using $K = 10$ (solid) and $K = 0.01$ (dash-dotted)



Linear solution - weak redundancy

$$
\begin{aligned}
\dot{w} &= -\rho K B_\perp^T \bar{W} u \\
u &= y_c + B_\perp w,
\end{aligned}
$$

▷ K diagonal allows to promote/penalize different redundant directions

▷ $\bar{W}$ diagonal allows to promote/penalize different actuators

Th'm: If $K > 0$ and $B_\perp^T \bar{W} B_\perp > 0$ then internal stability and steady-state output response y unaffected by allocator for small enough ρ

▷ Proof uses two time scale arguments

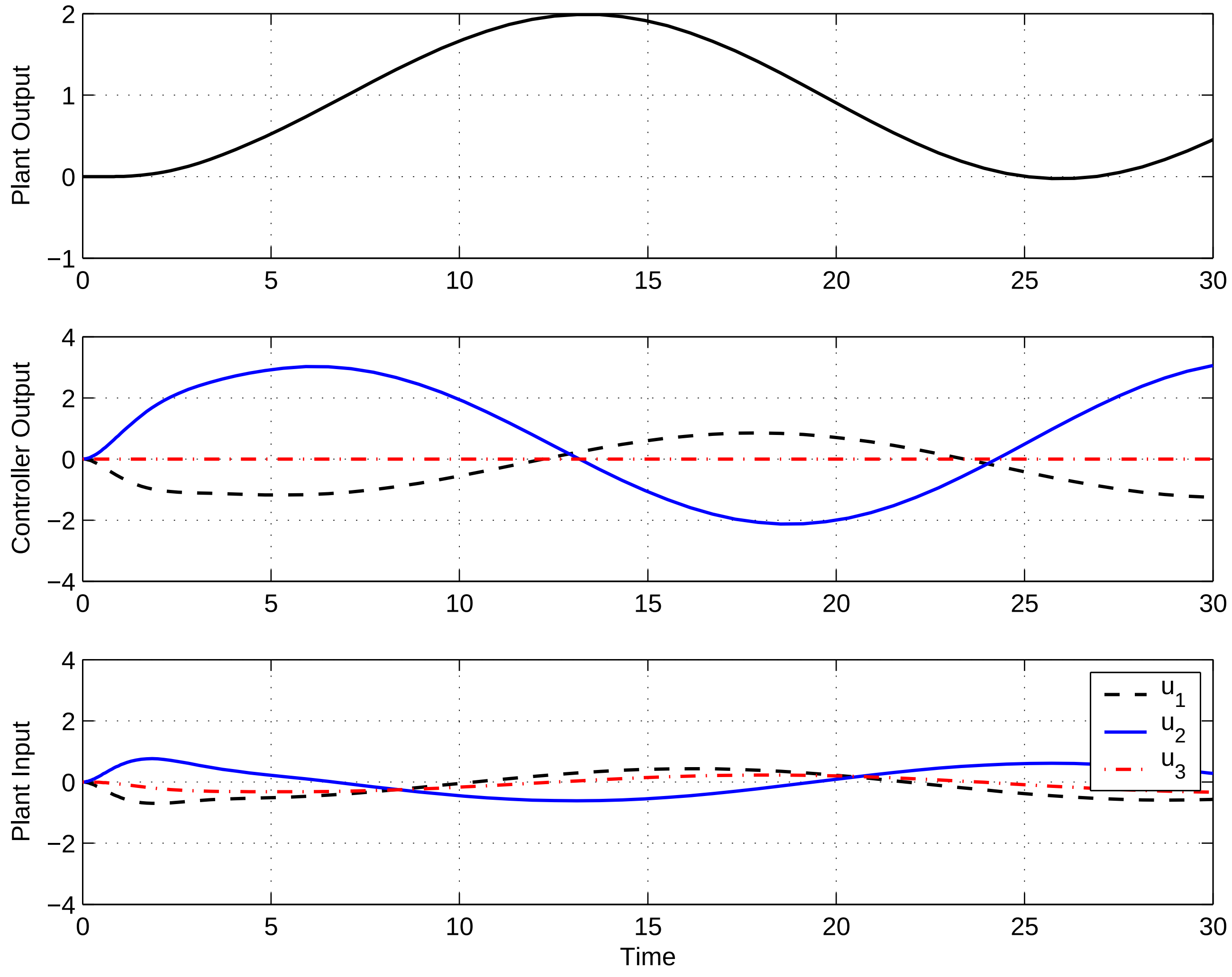
▷ Same design procedures as before for K and $\bar{W}$

▷ Very useful when wanting signals to slowly drift in certain directions

▷ Can mix strong and weak redundant directions selecting the columns of $B_\perp$

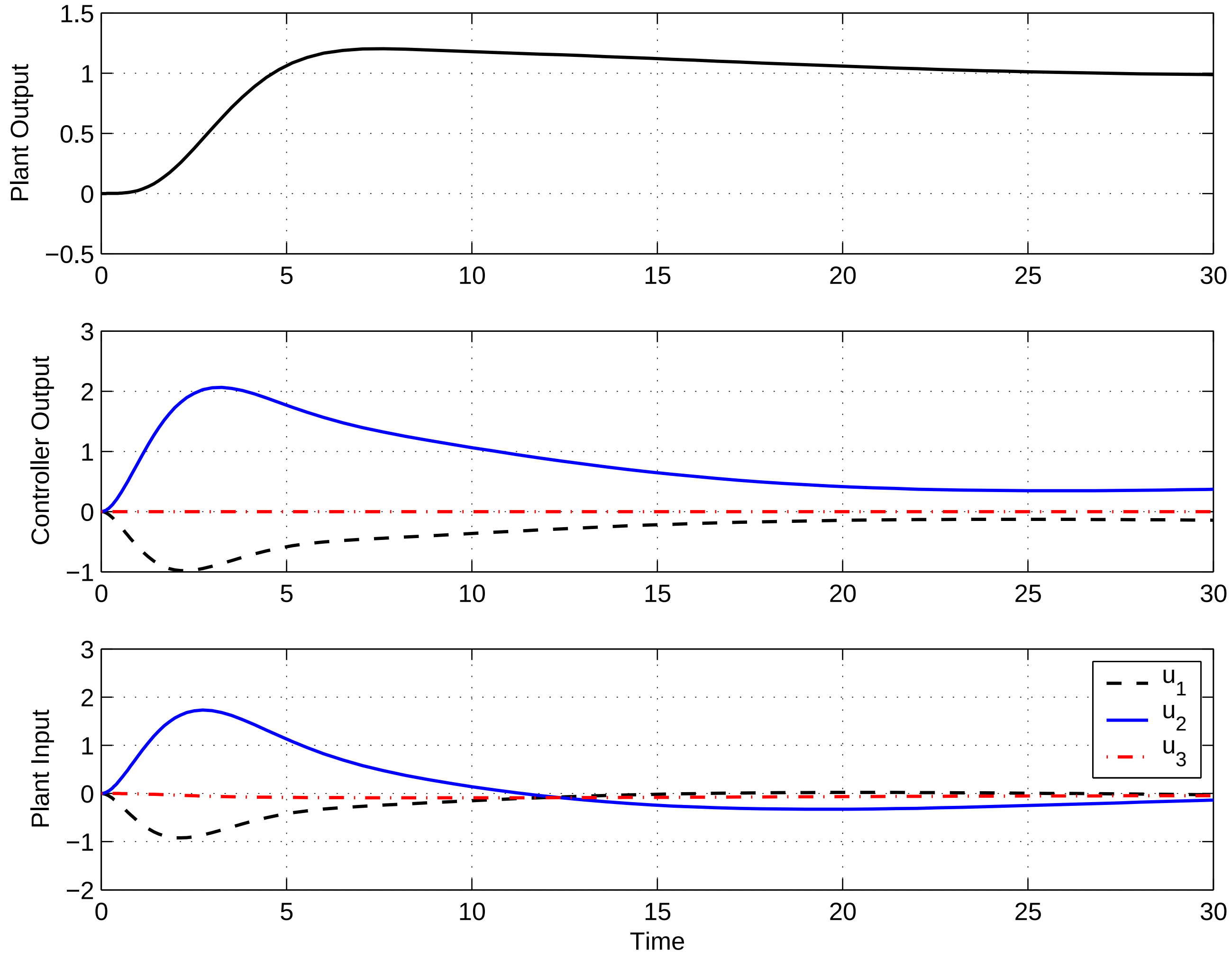
Example 1 (revisited)

- ▷ Responses using $K = I$ and $\bar{W} = I$ (instability!)



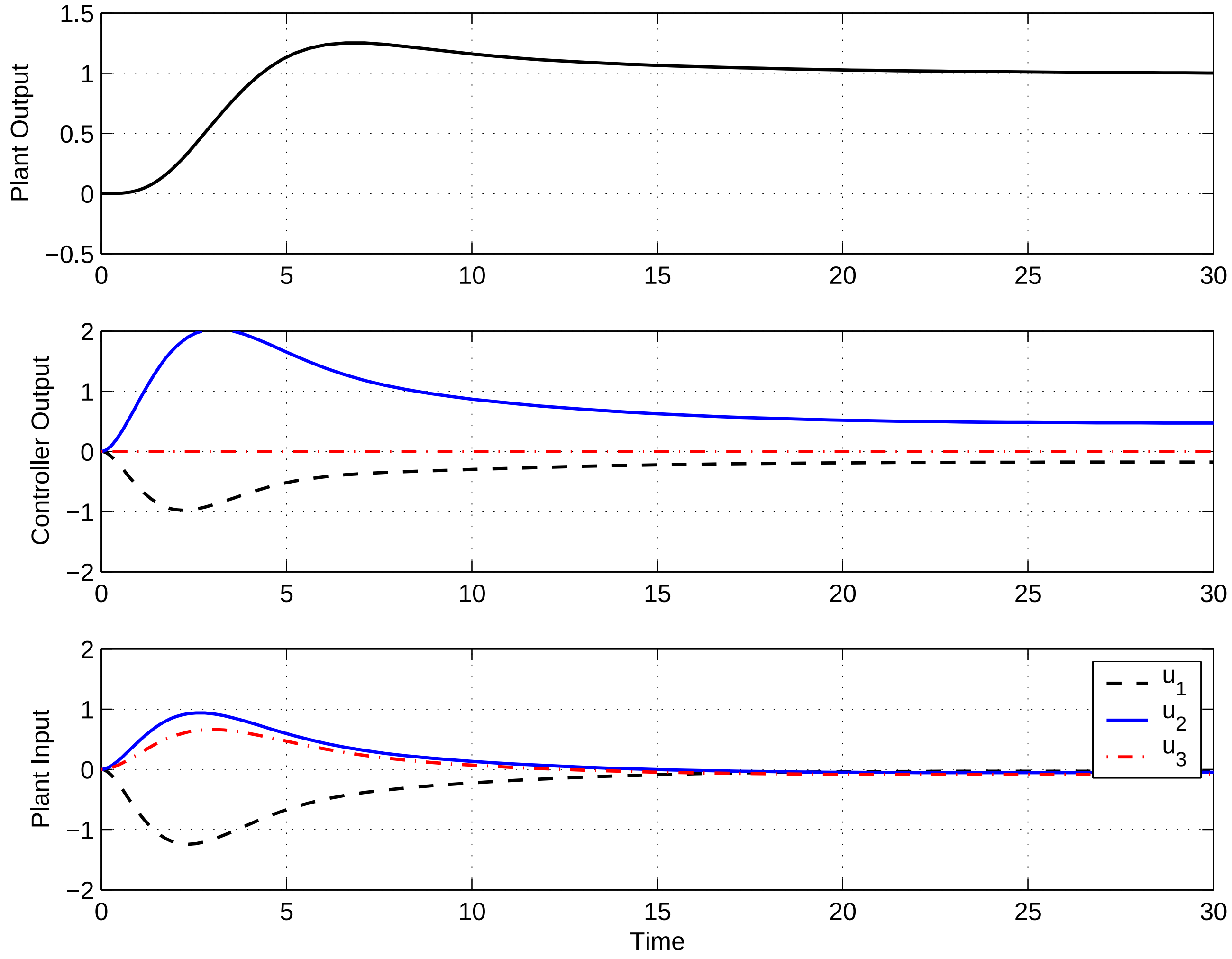
Example 1 (revisited better)

▷ Responses using $K = 0.1I$ and $\bar{W} = I$



Example 1 (revisited even better)

▷ Responses using $K = \left[\begin{smallmatrix} 100 & 0 \\ 0 & 0.1 \end{smallmatrix}\right]$ and $\bar{W} = I$



Nonlinear solution - magnitude saturation

▷ Key idea is to make W nonlinear $\Rightarrow$ penalize more and more each actuator as it approaches its magnitude saturation limit

$$W(y_u) = \Big(\mathrm{diag}((1+\epsilon)M - \mathrm{abs}(\mathrm{sat}_M(y_u)))\Big)^{-1}$$

▷ Nonlinear allocation aims at keeping each input far from its saturation limits

$$\begin{aligned} \dot{w} &= -\rho K B_\perp^T W(y_u) y_u \\ y_u &= y_c + B_\perp w \end{aligned}$$

▷ Deal with saturation using existing tools: **anti-windup compensation**

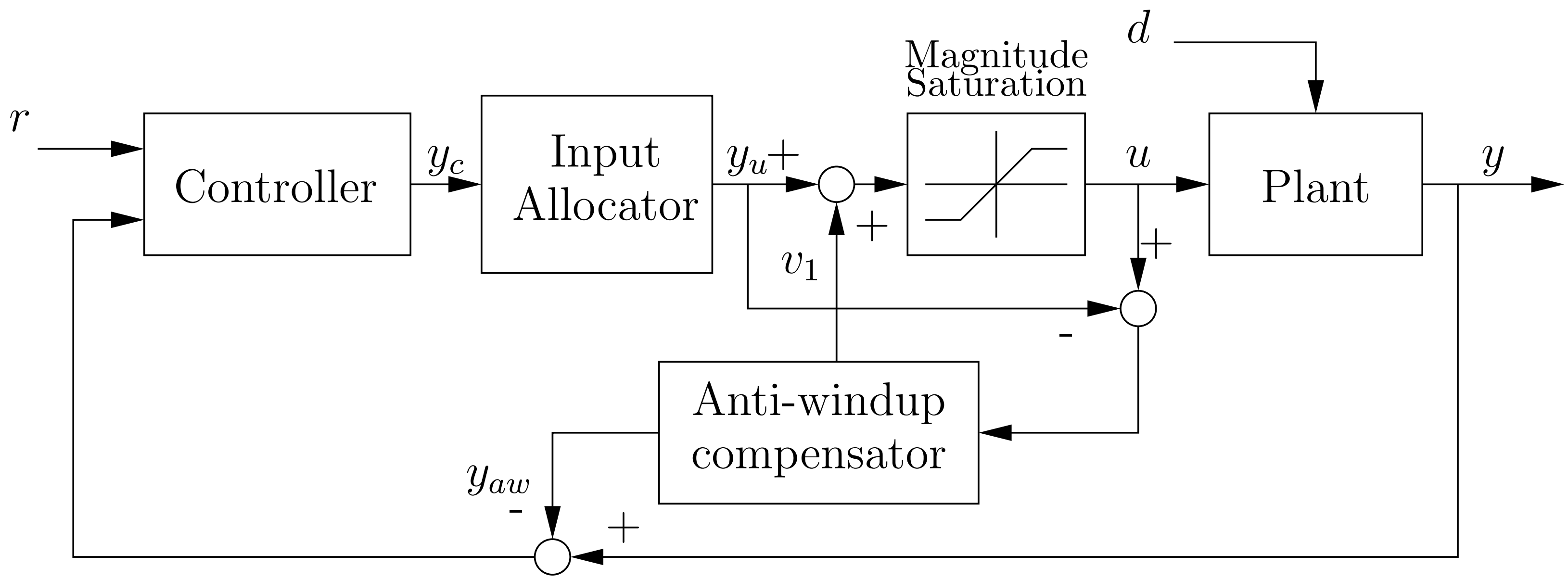
▷ Rough idea: rely on nonlinear state feedback $v_1 = k(x)$ ensuring that for a family of so-called *feasible functions* $y_u(\cdot)$, system

$$\dot{x}_{aw} = A x_{aw} + B\left(\mathrm{sat}_M(y_u + k(x_{aw})) - y_u\right)$$

is $\mathcal{L}_2$ stable from $y_u - \mathrm{sat}_M(y_u)$ to x_{aw}

Nonlinear solution - magnitude saturation (cont'd)

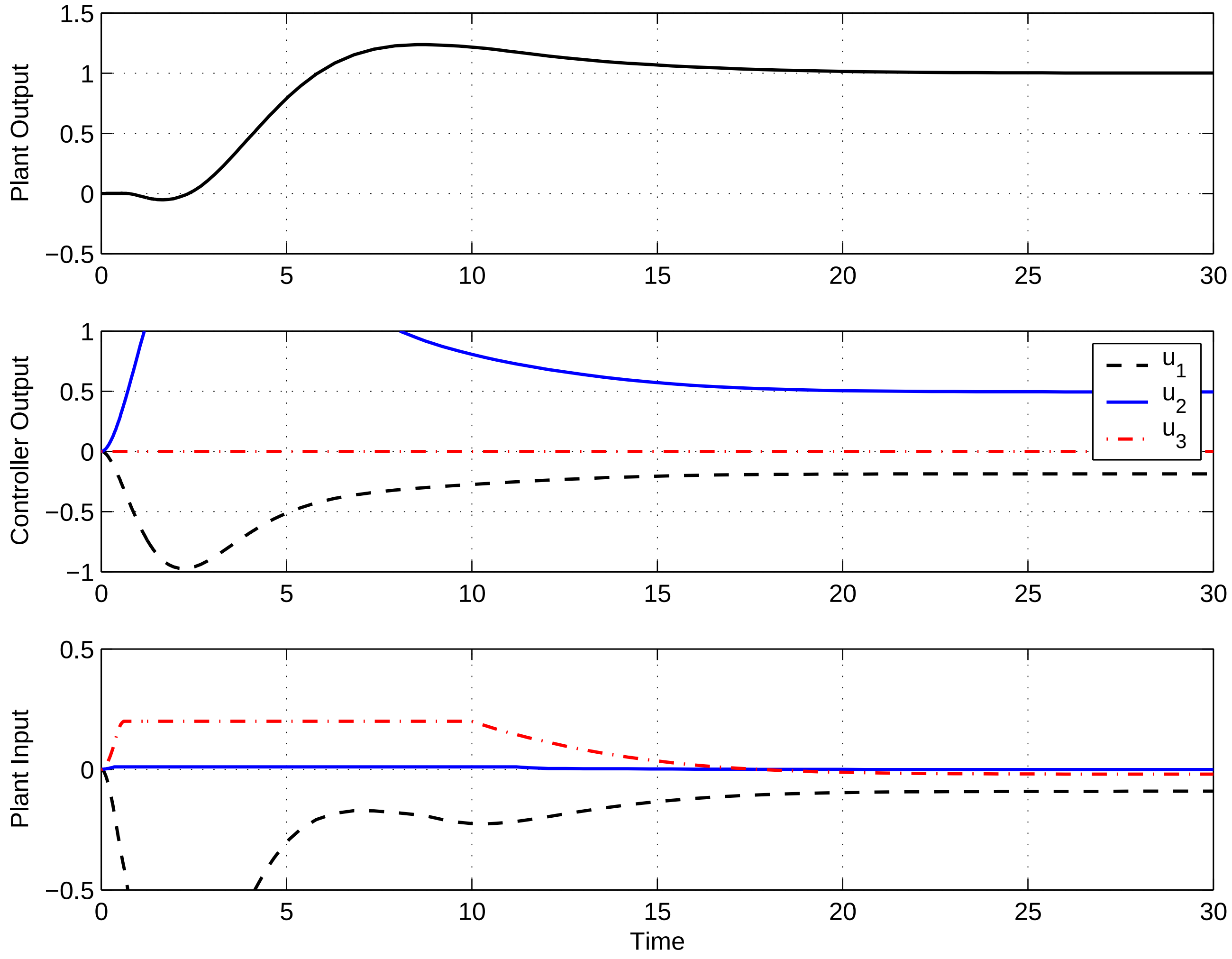
Th'm: The nonlinear system with allocator is GES before saturation. Moreover, for any feasible function $y_u(\cdot)$ the overall scheme (with saturation) recovers in an $\mathcal{L}_2$ sense the response without saturation



▷ **Interpretation**: *anti-windup deals* with saturation during transients; *dynamic allocation* avoids saturation at the steady-state

Example 1 (revisited with magnitude saturation)

▷ Input usage after allocation $[9.5\ 3.37\ 7]\%$ (note $u_2^* \approx 0.5 \gg m_2 = 0.01$)



Nonlinear solution - magnitude and rate saturation

▷ Magnitude allocator ($K, W(\cdot)$) augmented with rate allocator (K_r, W_r) only acting at transients

▷ Overall solution has an always well-posed algebraic loop

$$\begin{aligned}
\dot{w} &= -KB_\perp^T W(y_u)y_u - K_r B_\perp^T W_r \mathrm{dz}_R(W_r(y_{c,d} + B_\perp \dot{w})) \\
y_u &= y_c + B_\perp w \\
W(y_u) &= \big(\mathrm{diag}((1+\epsilon)M - \mathrm{abs}(\mathrm{sat}_M(y_u)))\big)^{-1},
\end{aligned}$$

▷ Algebraic loop can be replaced by arbitrarily fast strictly proper dynamics

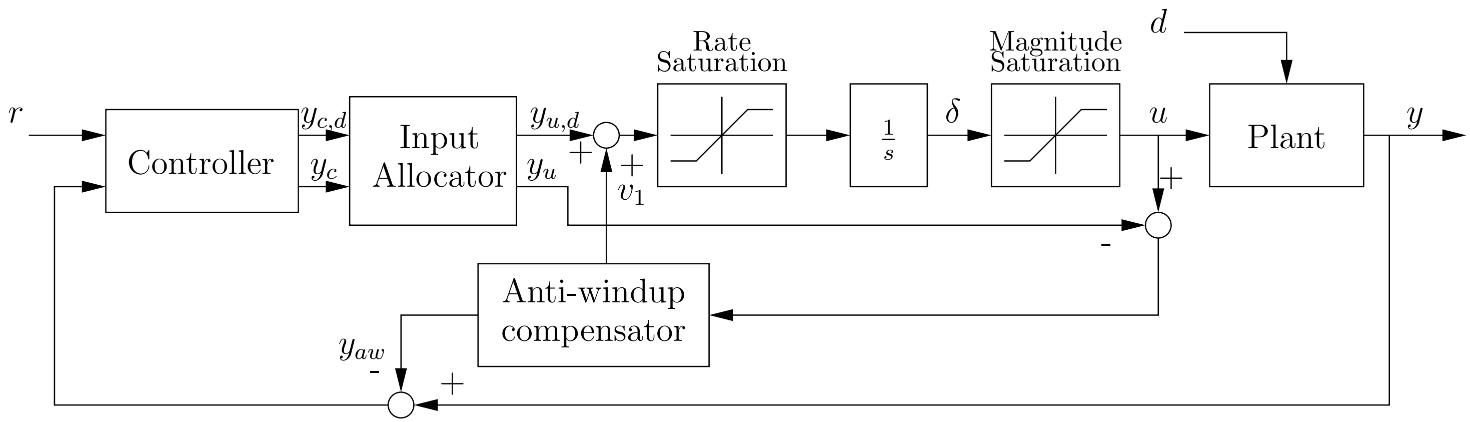
▷ Anti-windup action generalizes to ensuring that for a family of so-called *feasible functions* $y_u(\cdot)$, system

$$\begin{aligned}
\dot{x}_{aw} &= Ax_{aw} + B(\mathrm{sat}_M(\delta_{aw} + y_u) - y_u) \\
\dot{\delta}_{aw} &= \mathrm{sat}_R\left(y_{u,d} + k_r\left(\left[\begin{smallmatrix} x_{aw} \\ \delta_{aw} \end{smallmatrix}\right]\right)\right) - y_{u,d}
\end{aligned}$$

is $\mathcal{L}_2$ stable from $\left[\begin{smallmatrix} y_u - \mathrm{sat}_{M-\varepsilon}(y_u) \\ y_{u,d} - \mathrm{sat}_{R-\varepsilon}(y_{u,d}) \end{smallmatrix}\right]$ to (x_{aw}, δ_{aw}).

Nonlinear solution - magnitude and rate saturation (cont'd)

Th'm The nonlinear system with allocator is semiglobally ES before saturation. Moreover, for any feasible function $y_u(\cdot)$ the overall scheme (with saturation) recovers in an $\mathcal{L}_2$ sense the response without saturation



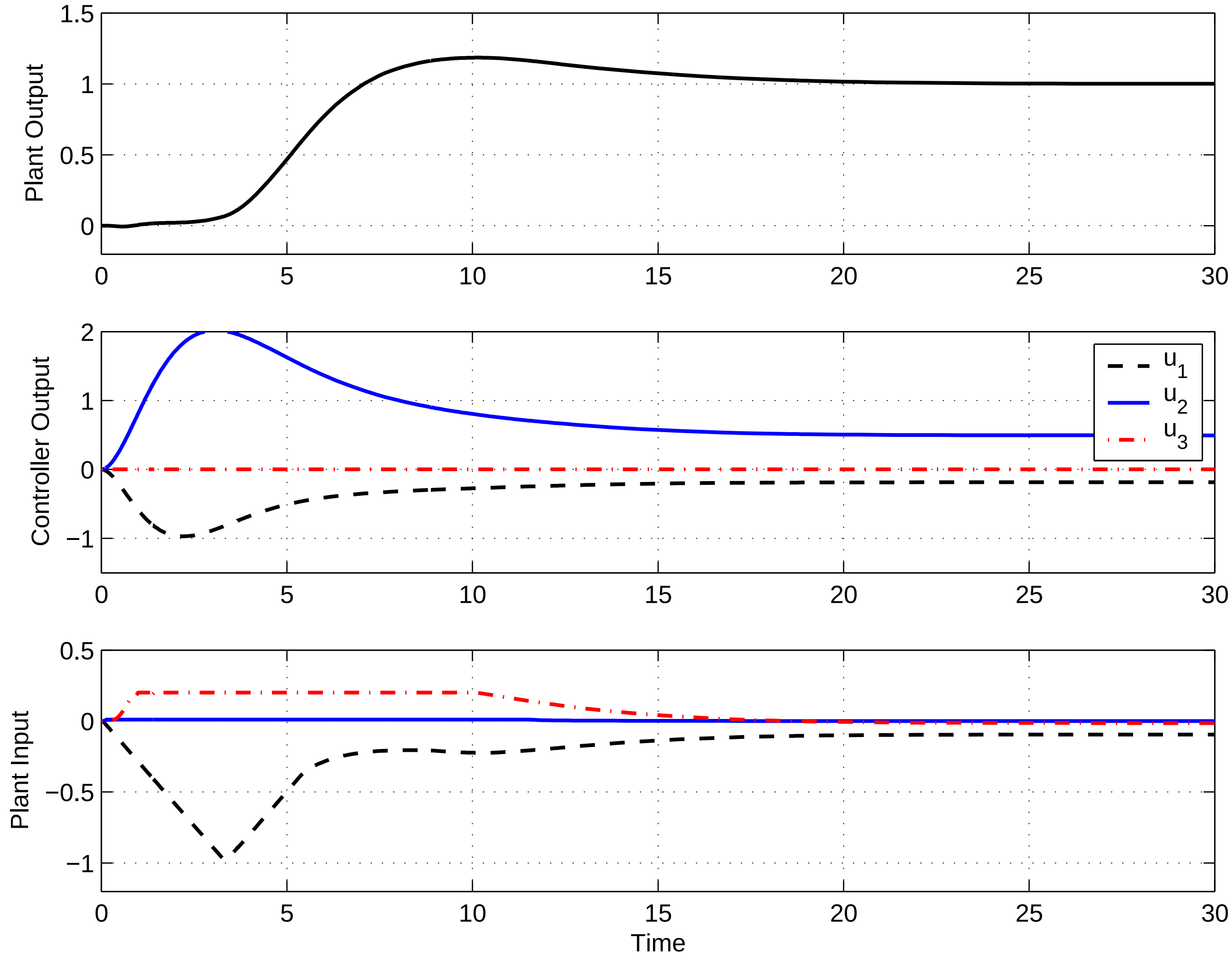
▷ **Interpretation**: the two allocators are independent as long as the magnitude one is slow enough

▷ Once again AW deals with (rate and magnitude) saturation during transients while allocator affects transients (rate) and steady state (mag)

▷ Future research: combined recipes for AW and allocator to optimize transients

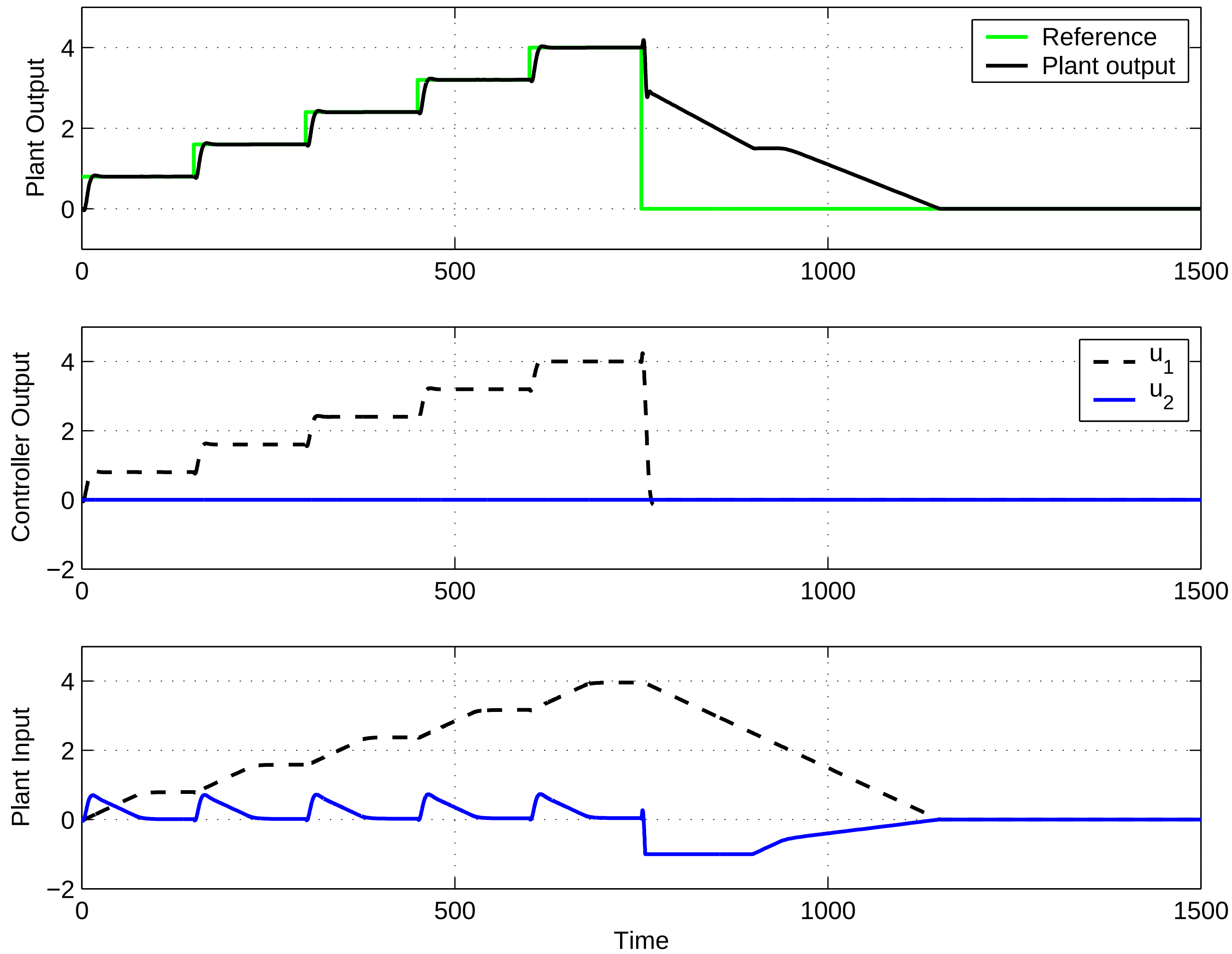
Example 1 (revisited with magnitude and rate saturation)

▷ Magnitude and rate saturation levels are $\begin{bmatrix} M \\ R \end{bmatrix} = \begin{bmatrix} 1 & 0.01 & 0.2 \\ 0.3 & 10 & 1 \end{bmatrix}$



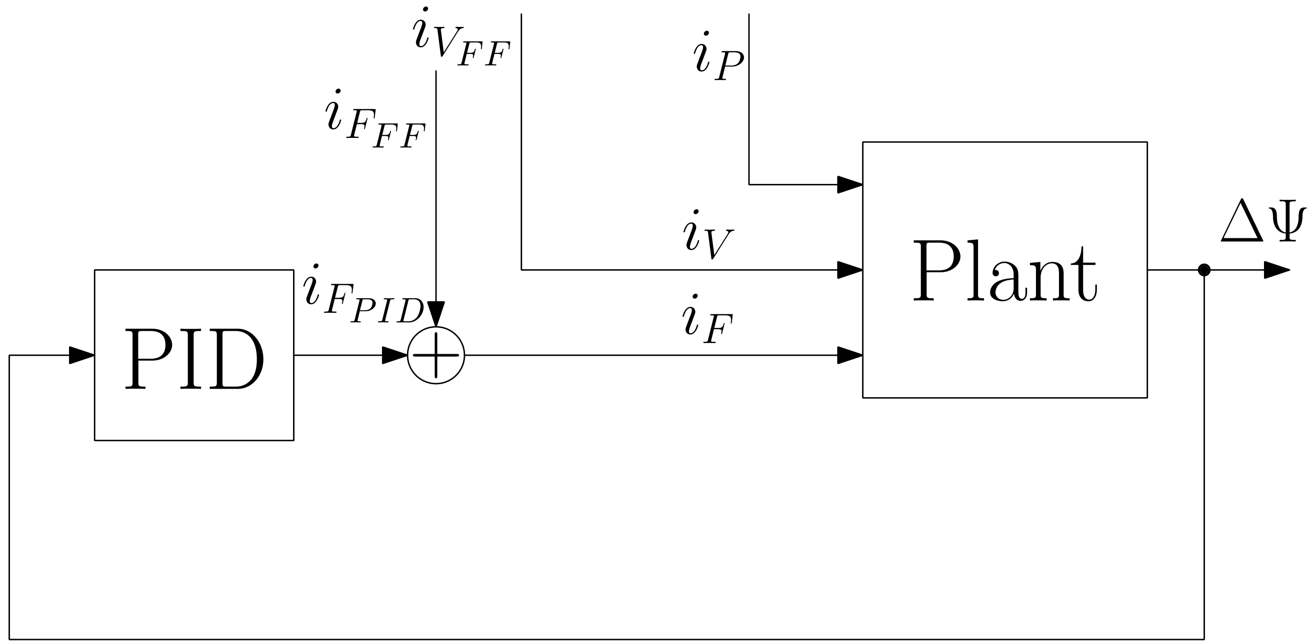
Example 2

▷ Plant is ES. Magnitude and rate saturation levels are $\begin{bmatrix} M \\ R \end{bmatrix} = \begin{bmatrix} 100 & 1 \\ 0.1 & 100 \end{bmatrix}$



Application: plasma position and elongation control

▷ Frascati Tokamak Upgrade: $\Delta\Psi$ = plasma horiz. position, I_p = plasma current



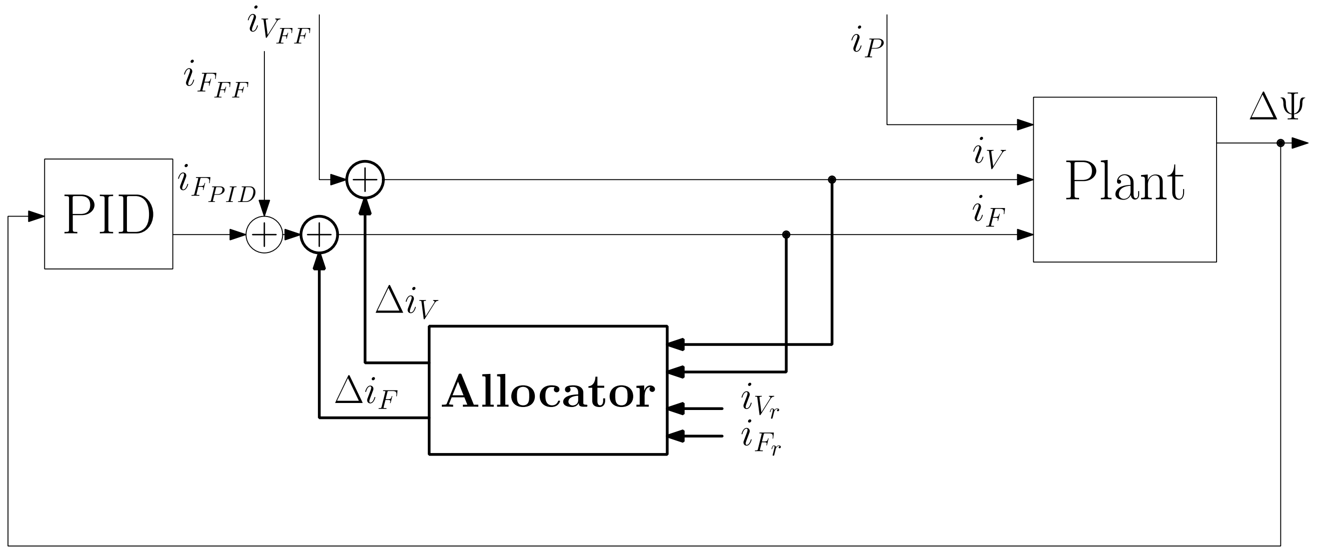
▷ V coil: very slow and powerful; F coil: fast and squeezes the plasma

▷ Goal: Want to use the F coil to perform two actions:

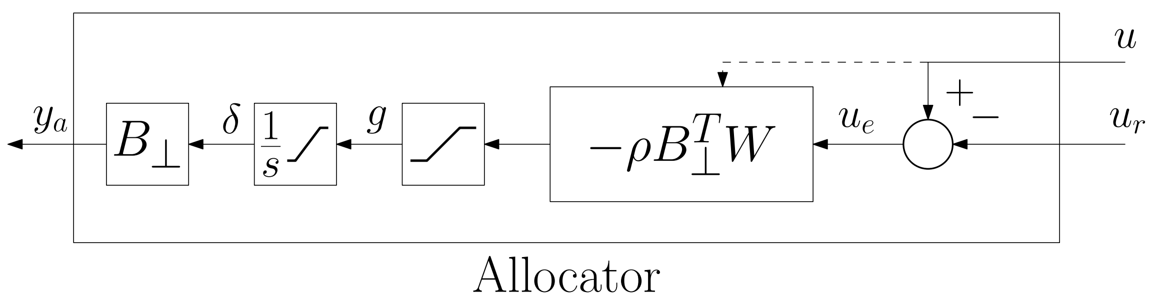
- high bandwith disturbance rejection on $\Delta\Psi$
- low bandwith elongation regulation

Solution with allocator

▷ Transfer (slowly) the control authority from F to V using the dynamic allocator

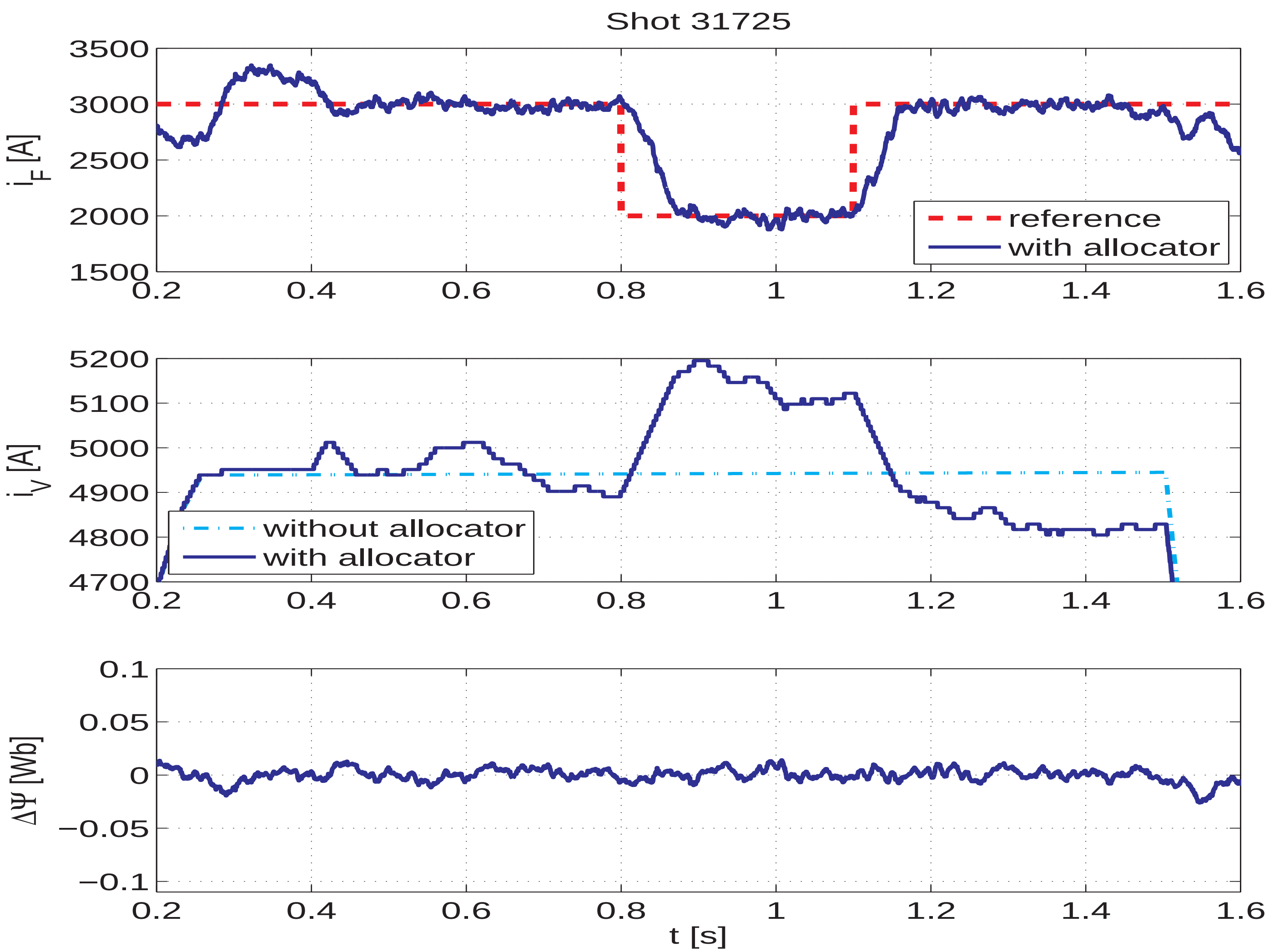


▷ Zoom of the allocator block (note the drift term $u_0 = u_r$ which is now a reference signal for I_F)



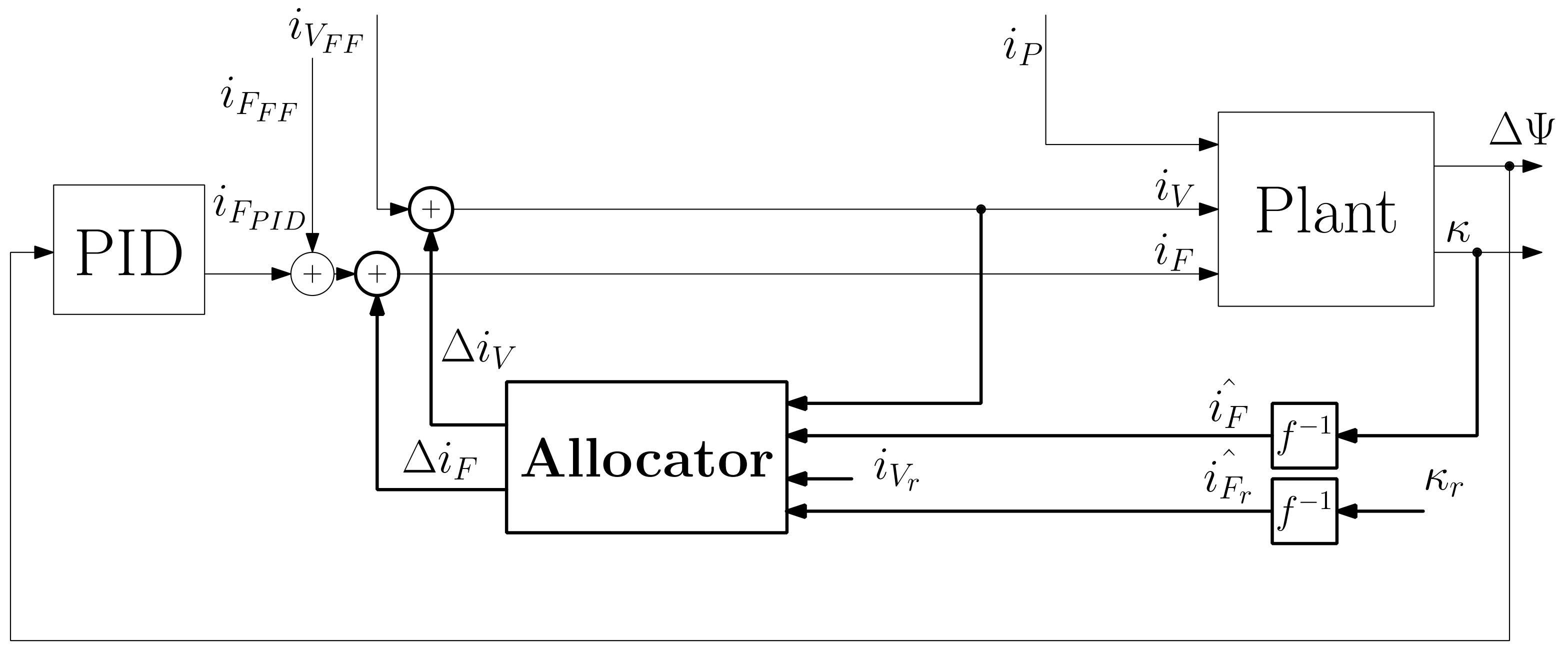
Experiments: F current regulation

▷ F current is slowly regulated without affecting $\Delta\Psi$



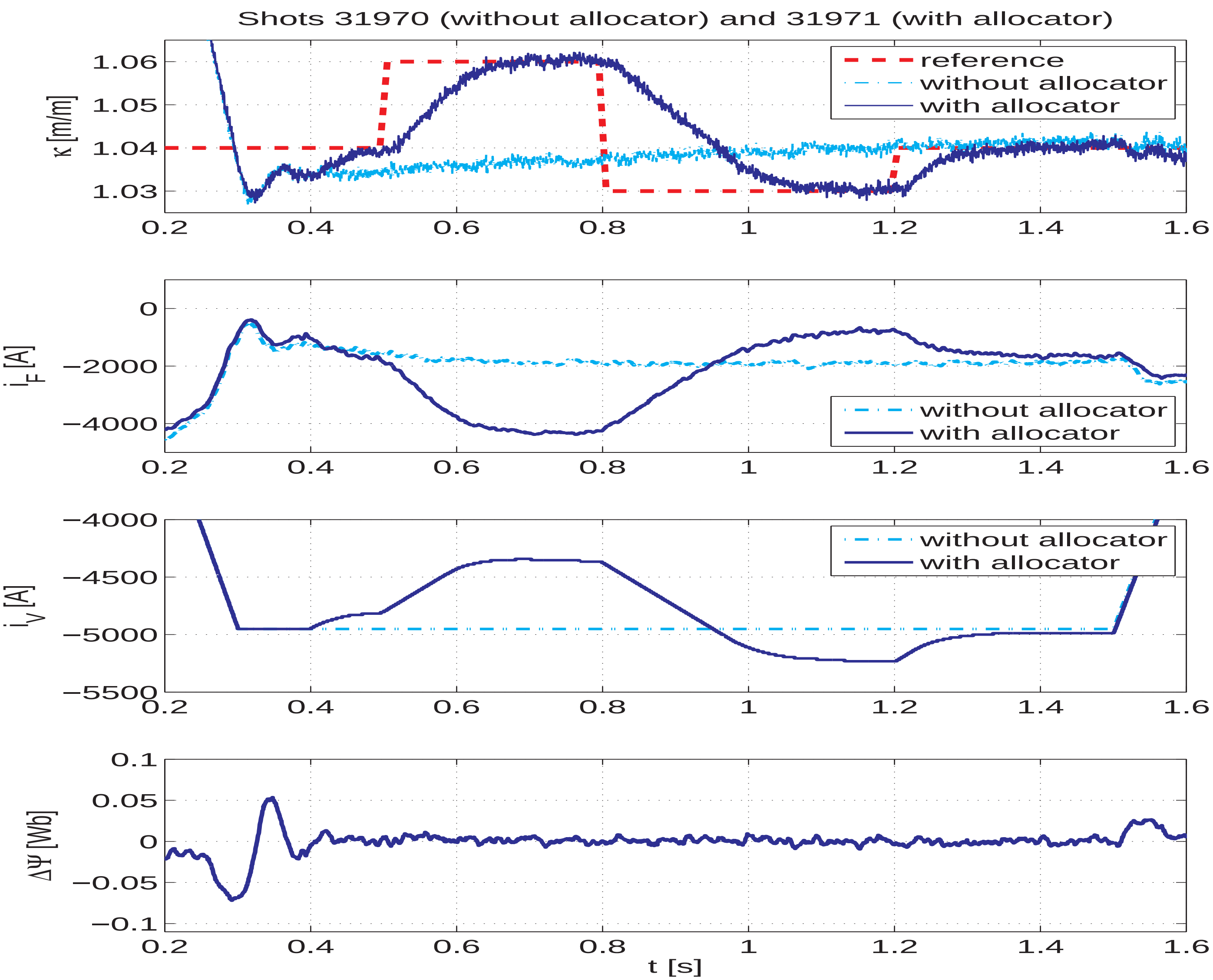
From current to elongation regulation

▷ An approximately known nonlinear static map f relates I_F to the elongation e



▷ Invert the map f to perform elongation regulation

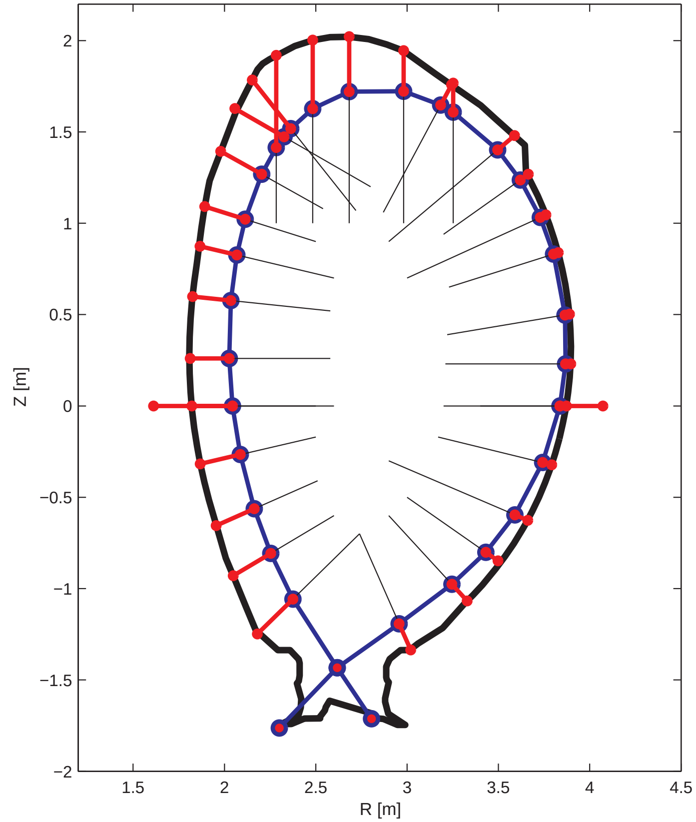
Experiments: Elongation regulation



Joint European Torus (JET) plasma shape control

- We want to control the plasma shape on a poloidal cross section.
- Shape is described by a finite number of geometrical parameters called **gaps**.
- Gaps are defined as the distances between the plasma boundary and the first wall along certain segments.
- Gaps values are evaluated from magnetic sensor measurements by estimation algorithms.
- We want to control:

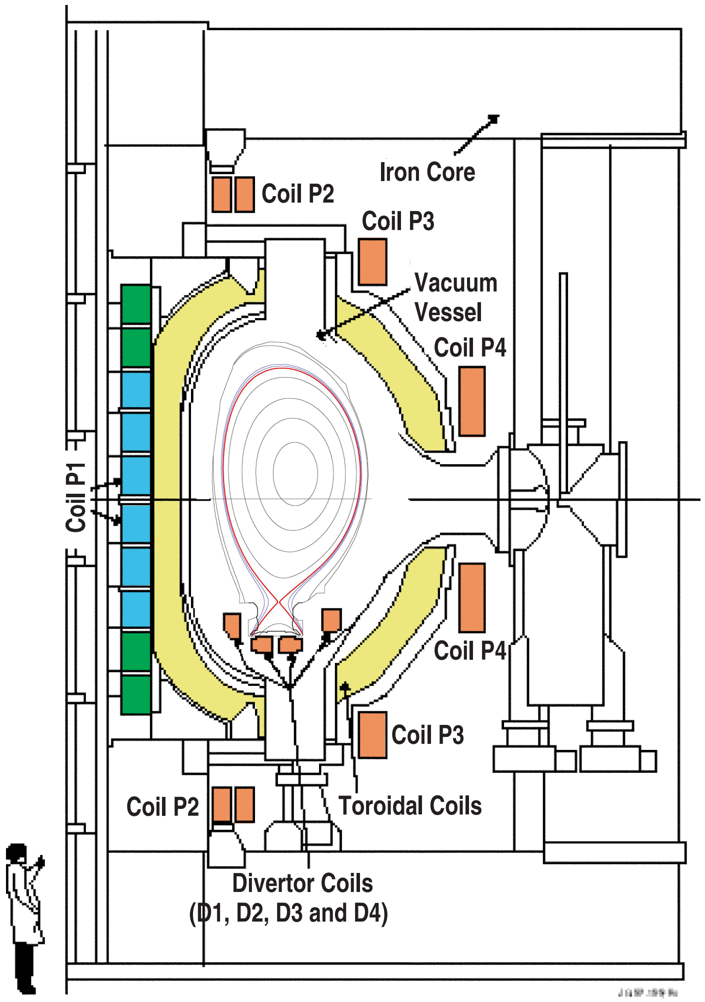
  32 outputs.



JET plasma shape control

- JET has 8 poloidal field (PF) coils available as actuators for plasma shape control.
- JET PF coils are connected to form 9 circuits.
- Control inputs represented by currents flowing in the circuits.
- Inputs available:

  9 control inputs.



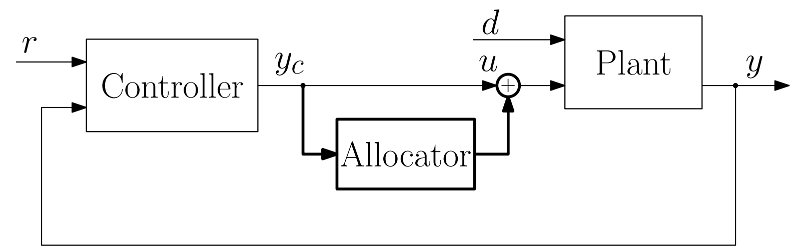
JET plasma shape control

- **Nr. controlled outputs** $>$ **Nr. control inputs**, so not all desired reference shapes can be obtained exactly.
- Current solution, XSC (eXtreme Shape Controller), is a **linear compensator** which minimizes the steady state error $\|r - y\|_2$; XSC is designed considering a **linearized model** (CREATE-L) of the plasma shape response around an equilibrium configuration.
- **Problem**: input saturations are not taken into account.
- Input saturations can cause losses in terms of: performance, disturbance rejection capability, stability.
- **Proposed solution**: add an **input dynamic allocator block** between the linear controller and the plant for **saturation avoidance**.

Allocator for input redundant plants

- Essential features of the dynamic allocator seen before

$$\dot{w} = -\rho K B_\perp^T \bar{W}(u - u_0)$$
$$u = y_c + B_\perp w$$



- K diagonal allows to promote/penalize different redundant directions
- $\bar{W}$ diagonal allows to promote/penalize different actuators
- ρ positive scalar gives convergence speed
- The interconnected system converges to a value $u^\star$ which minimizes the function $J = (u - u_0)^T \bar{W}(u - u_0)$ under the constraint $u = y_c + B_\perp w$.

Allocator for input redundant plants

- In strongly input redundant plants ($\ker(B) \cap \ker(D) \neq \emptyset$) choosing $B_\perp$ so that $\mathrm{Im}(B_\perp) = \ker(B) \cap \ker(D)$ the allocator action results invisible at the plant output.

- In weakly input redundant plants ($\ker(P^\star) \neq \emptyset$,with $P^\star := P(0)$ and $P(s) = C(sI - A)^{-1}B + D$) choosing $B_\perp$ so that $\mathrm{Im}(B_\perp) = \ker(P^\star)$ the allocator action perturbs the plant output just in the transient, but not at steady state.

Input allocation for non redundant plants

- Many plants are not even weakly redundant, namely

$$\ker(P^\star) = \emptyset \quad \Longleftrightarrow \quad \mathrm{rank}(P^\star) = n_u$$

  this is a generic situation for “square” and “tall” plants, i.e. whenever $n_u \leq n_y$

- In this case, input allocation **inevitably affects** both the transient and the **steady state output** response

- A **trade off** arises between **desirable input modifications**, aimed at keeping the input inside a favorable region, and the correspondingly induced **undesired output modifications**, which should be kept as small as possible

Cost function and new allocator

- We introduce a more general **cost function** [before]

$$J(u^\star, \delta y^\star) \quad [(u-u_0)^T \bar{W}(u-u_0)]$$

measuring the trade-off between the modified steady state value of the plant input $u^\star$ and the associated output modification $\delta y^\star$ with respect to the original $y^\star$ (the superscript $\star$ denotes the steady state value).

- The new allocator is described by the relations [before] :

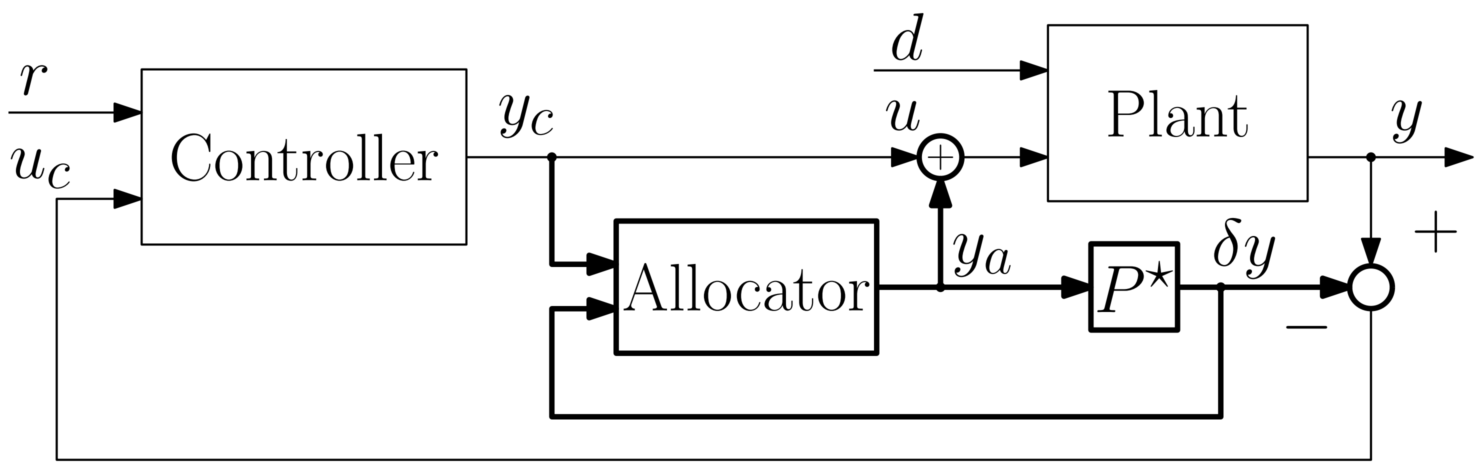
$$\begin{aligned} \dot{w} &= -\rho K \left(\nabla J \left[\begin{smallmatrix} I \\ P^\star \end{smallmatrix} \right] B_0 \right)^T \\ u &= y_c + B_0 w \end{aligned} \quad \left[\begin{aligned} \dot{w} &= -\rho K B_\perp^T \bar{W}(u-u_0) \\ u &= y_c + B_\perp w \end{aligned} \right]$$

where $w \in \mathbb{R}^{n_w}$ represents the allocator state, ρ is a positive scalar, K is a symmetric positive definite matrix and B_0 is a suitable full column rank matrix, generalizing the matrix $B_\perp$.

Allocator interconnection

This new allocator should be interconnected to the unconstrained closed-loop via the equations

$$\begin{aligned} u_c &= y - P^\star B_0 w \\ u &= y_c + B_0 w. \end{aligned}$$



▷ The signal $P^\star y_a$ ensures that the allocator does not “fight” against the controller at the steady-state (use two time scales again in proof)

Allocator parameters and convergence theorem

▷ In the allocator equation given before:

$$\begin{aligned} \dot{w} &= -\rho K \left(\nabla J \left[\begin{smallmatrix} I \\ P^\star \end{smallmatrix} \right] B_0 \right)^T \\ u &= y_c + B_0 w \end{aligned}$$

- The matrix B_0 is selected considering that each of its columns corresponds to an **"allocation direction"**, which will be dynamically weighted by the corresponding component of w.
- The selection of K as a diagonal positive definite matrix allows the designer to specify some fixed **relative weights among the directions** given by B_0.
- The parameter ρ specifies the allocator **convergence speed**.

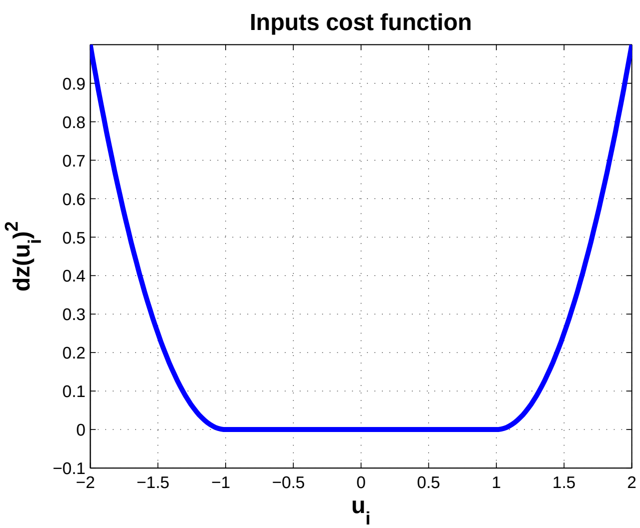
Th'm Under some mild technical assumptions, the allocator is such that under constant inputs, $(u, \delta y)$ converge to the minimum of J.

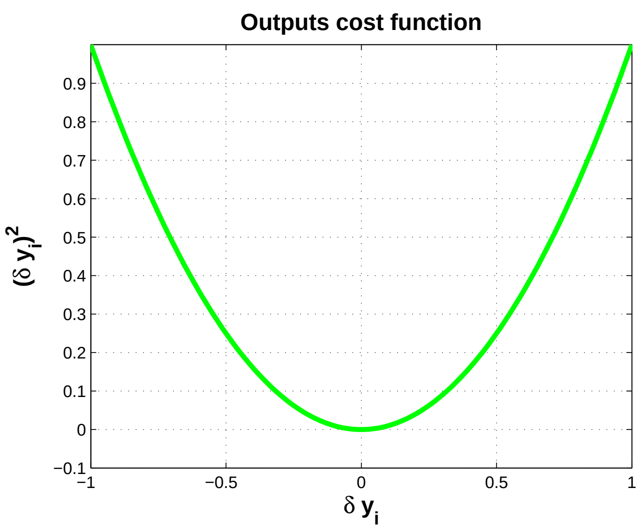
Example of a cost function

A possible selection of the cost function is

$$J(u, \delta y) = \sum_{i=1}^{n_u} a_i \mathrm{dz}(u_i)^2 + \sum_{i=1}^{n_y} b_i (\delta y_i)^2$$

where $\mathrm{dz}(u_i) = \mathrm{sign}(u_i)\max\{0, |u_i| - 1\}$, $a_i \geq 0$, $i = 1, \ldots, n_u$ and $b_i > 0$ $i = 1, \ldots, n_y$.





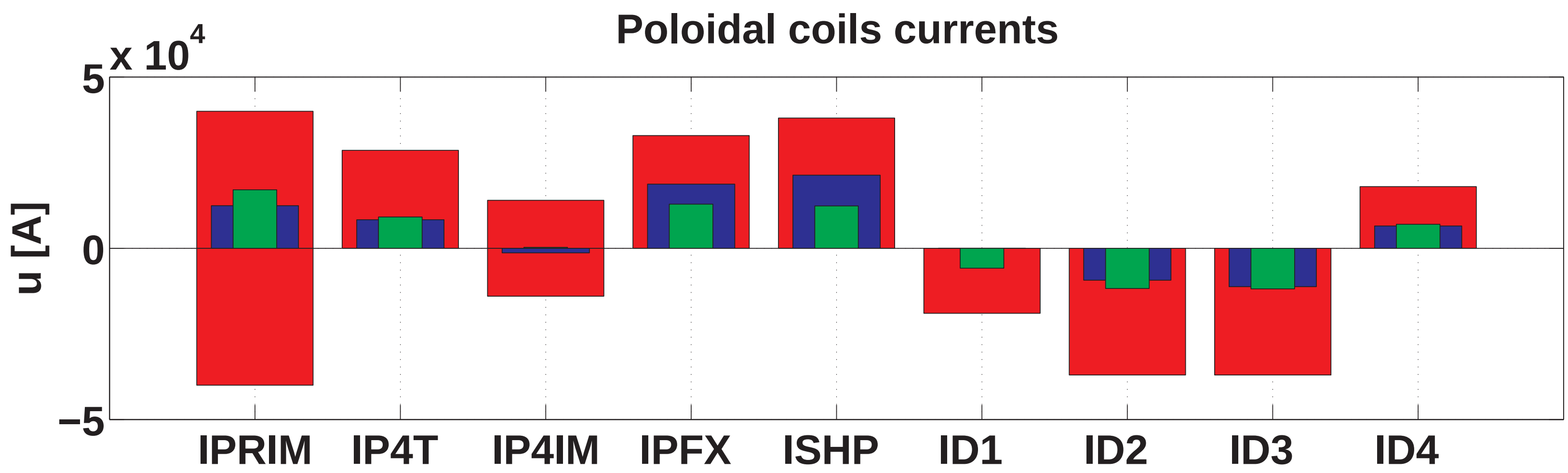
Alternative non symmetric choices are possible (see paper).

Choice of the matrix B_0

- If B_0 is chosen as a full rank square matrix $B_0 \in \mathbb{R}^{n_u \times n_u}$, the allocator can give a contribution in every direction of the input space.
- We can decide to trade some allocation degrees of freedom for ensuring that ν selected **outputs** will remain **unchanged at steady state**.
- In the same spirit, we can decide to trade some allocation degrees of freedom for ensuring that μ selected **inputs** will remain **unchanged at every time**.
- The maximum number of outputs or inputs we can maintain unchanged is given by:

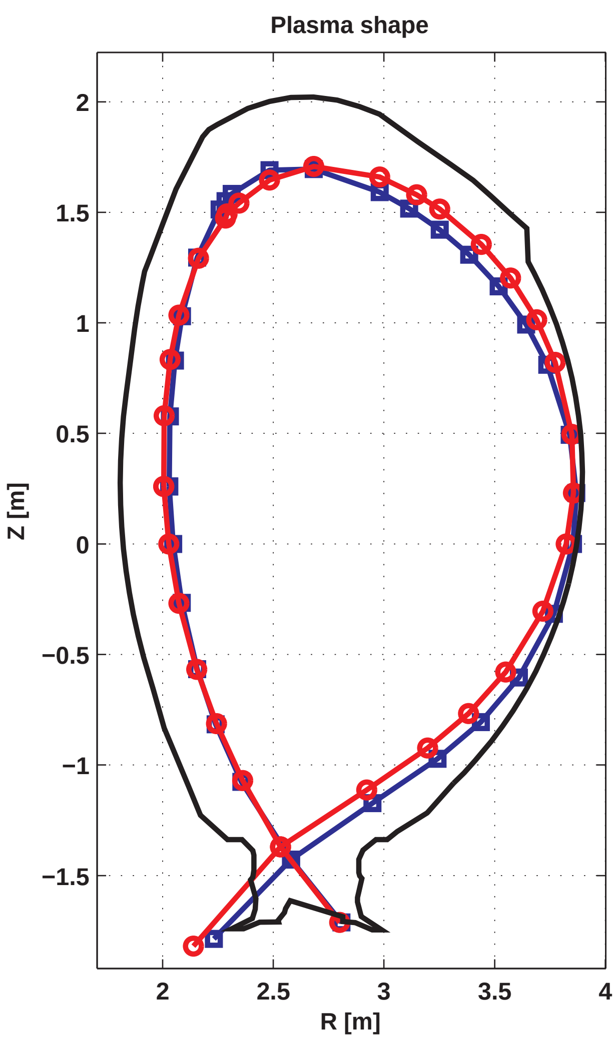
$$\nu + \mu < n_u.$$

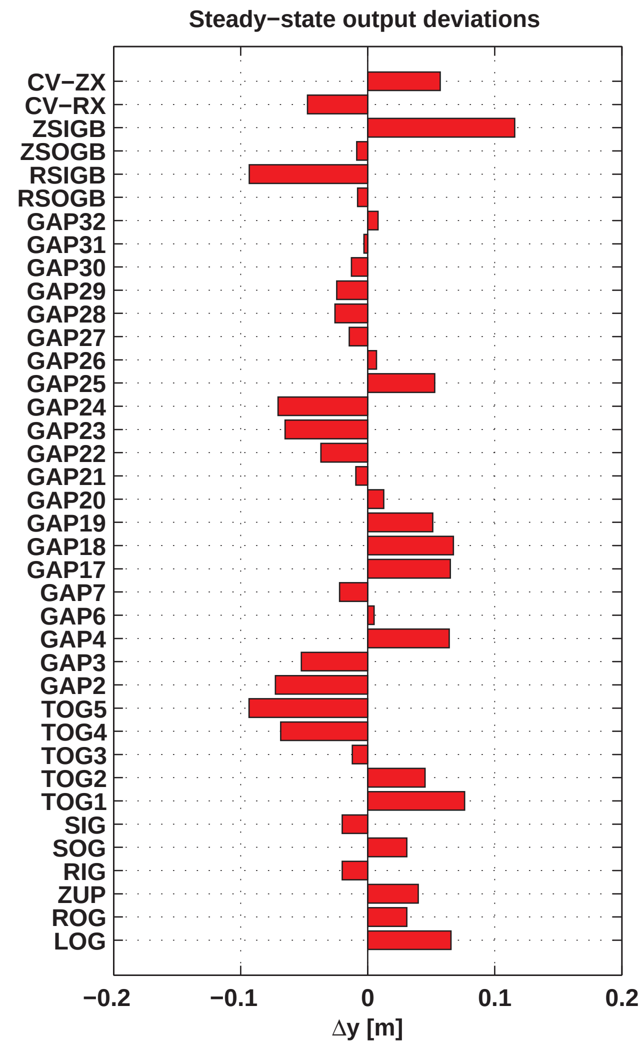
Open loop simulations - Test 1



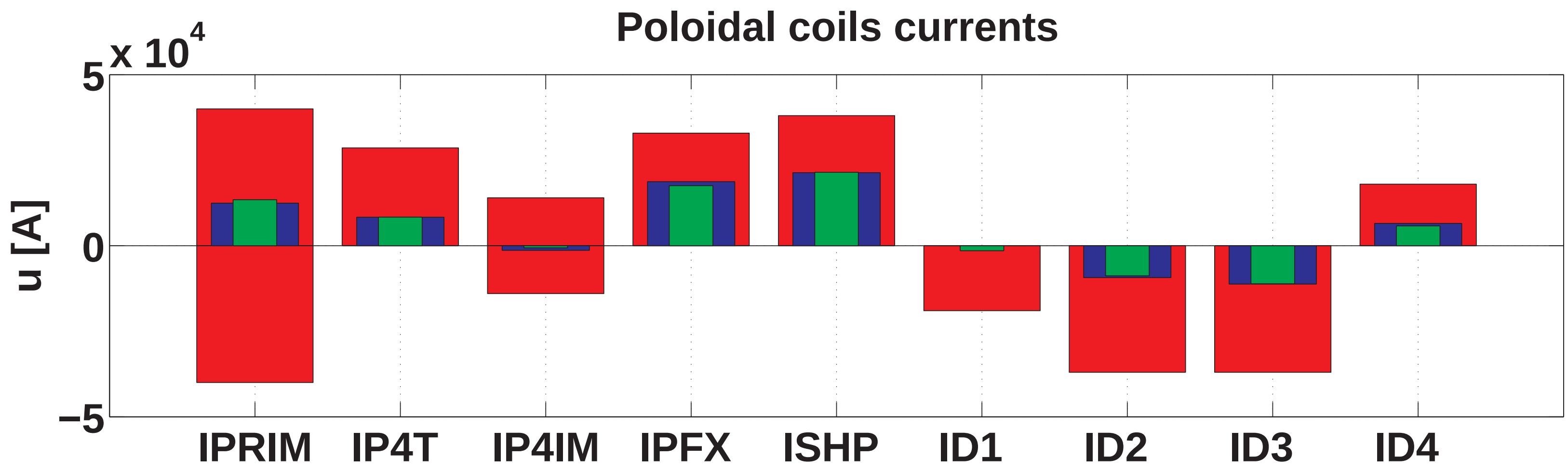
- Input ranges (red), controller output (blue), steady state allocated input (green).
- ID1 is moved away from saturation after the reallocation.
- The output (red shape) is consequently modified with respect to the nominal one (blue shape), but the error (red bars) is maintened quite small.

Open loop simulations - Test 1



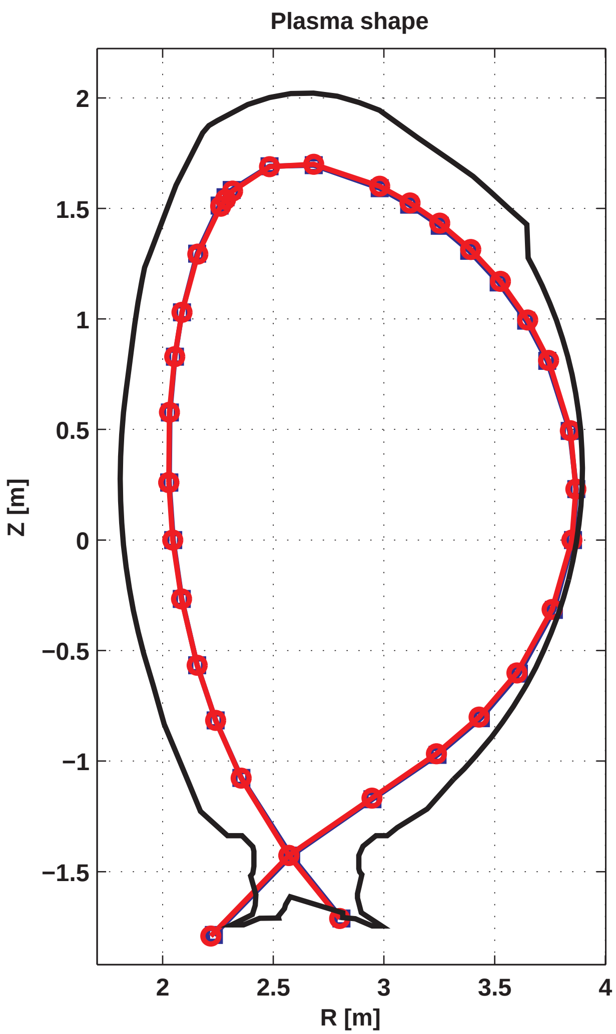


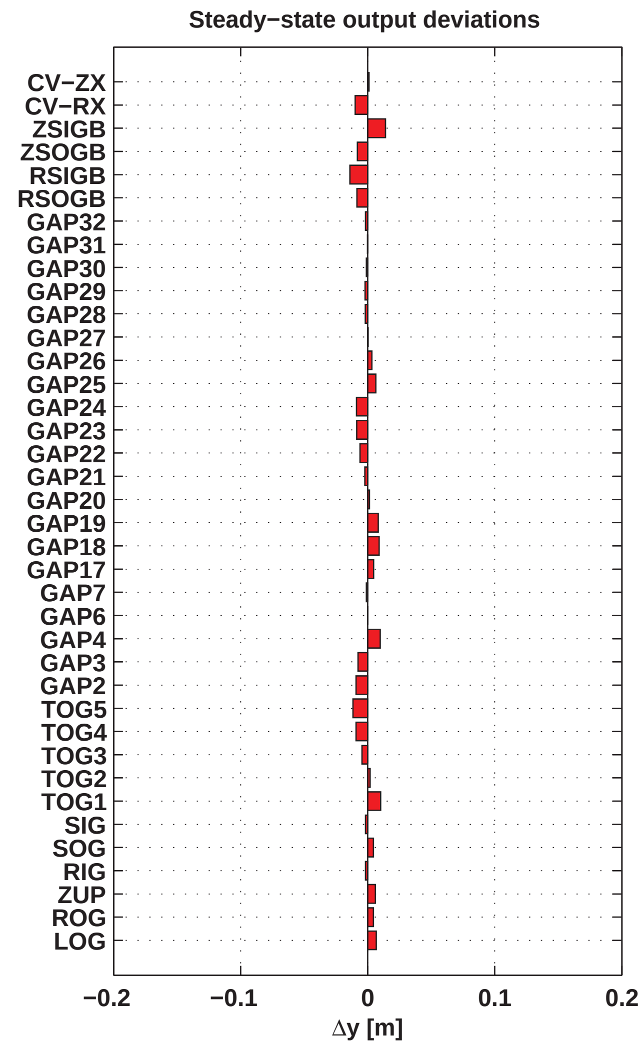
Open loop simulations - Test 2



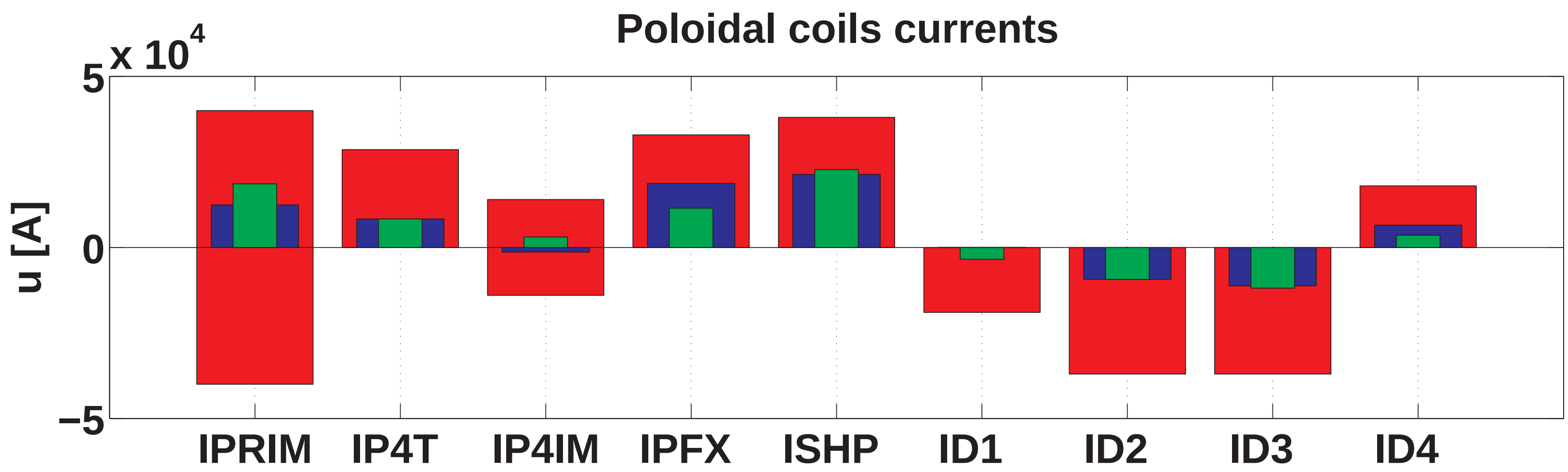
- By increasing the output penalties ($b_i = 3 \cdot 10^8$) the resulting output steady state error can be reduced.
- On the other hand, the distance of ID1 from thes aturation now is smaller.

Open loop simulations - Test 2



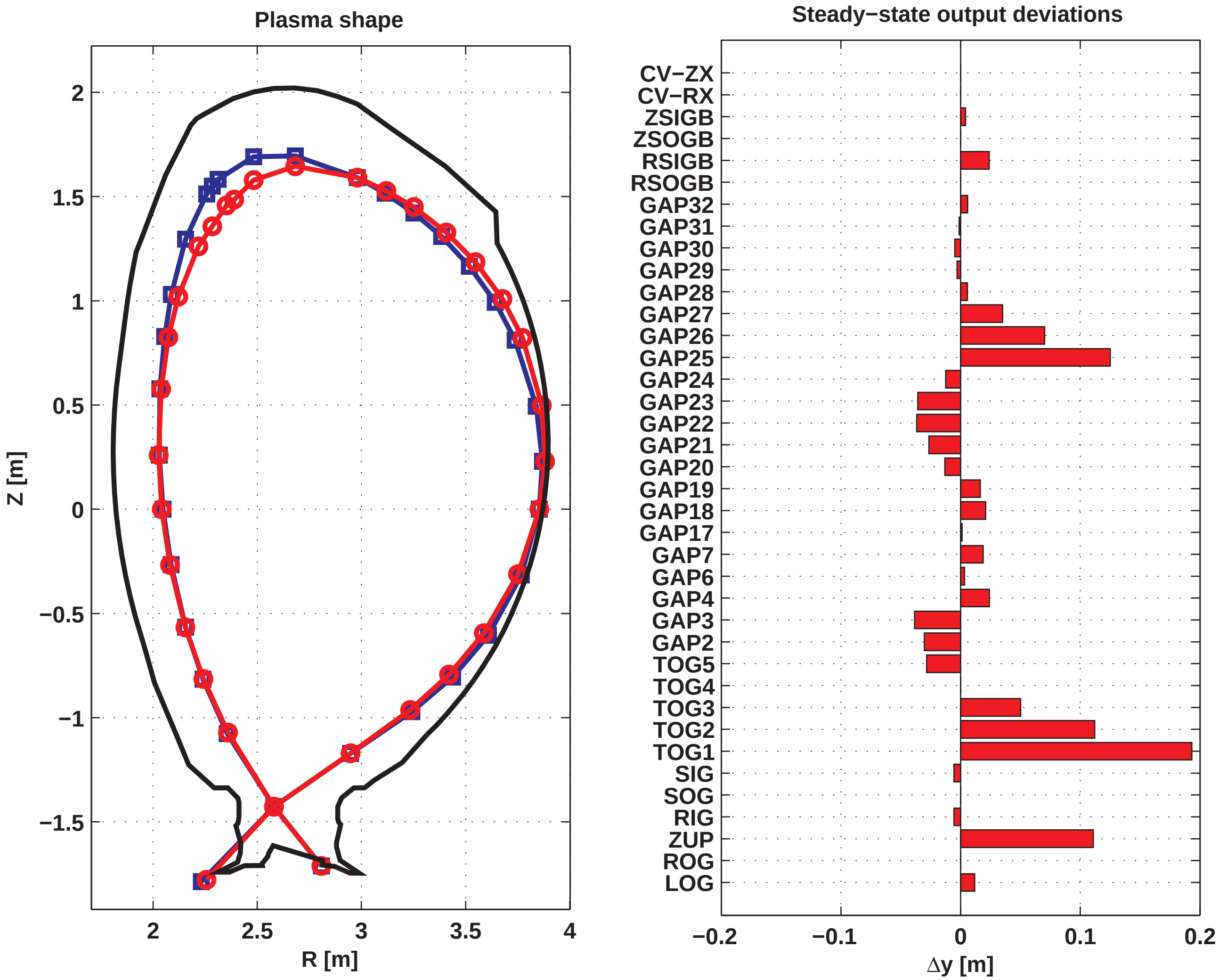


Open loop simulations - Test 3



- The output penalties are the same of Test 1 ($b_i = 3 \cdot 10^6$).
- The matrix B_0 is changed in order to fix 5 outputs (CV-RX, CV-ZX, ZSOGB, RSIGB and RSOGB, i.e. X-point and strike points) and one input (IP4T current).
- Note that the fixed quantities actually take the nominal values.
- ID1 is kept far from saturation, while some output errors increas.

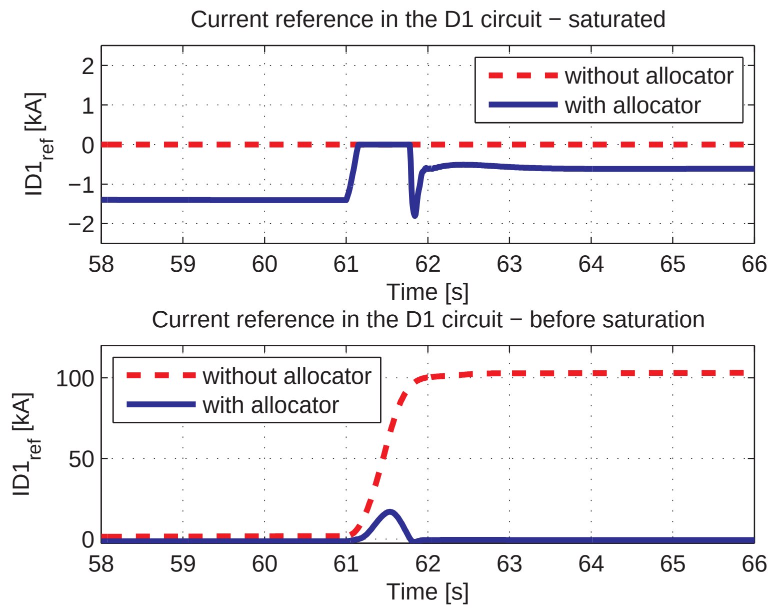
Open loop simulations - Test 3



Closed loop simulations

Shape references move from a configuration to a new one:

- constant until time $t_1 = 61\ s$,
- ramp up in the interval $[t_1, t_2]$,
- constant again after $t_2 = 61.5\ s$.



Without the allocator, the controller commands $100\ kA$ of current (lower figure, red), way beyond the range [-19 kA, 0 kA].

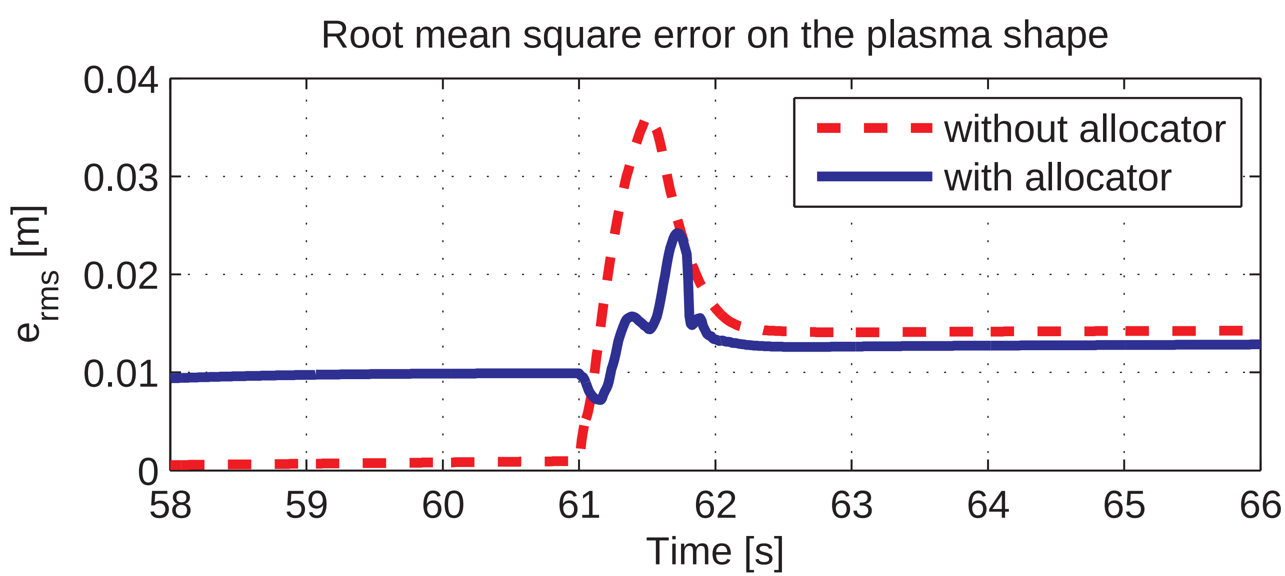
So current in the D1 circuit is permanently saturated at 0 (upper figure, red).

With the allocator, current in the D1 circuit saturates only during the transient (upper figure, blue).

Closed loop simulations

Without the allocator before t_1 the RMS shape error (red) is small, because the current in D1 is not saturated so much, but after t_2 the steady state error increases.

With the allocator (blue) before t_1 the current in D1 is moved away from the saturation at the price of an increased shape error, but after t_2 the reallocation results in a smaller error.



Conclusions

- Dynamic allocation scheme proposed for the linear case
- Input redundancy can be fake, then trade-off minimizing a **nonlinear** cost
- No need to compute explicitly the minimum (hard for nonlinear): allocator converges to it with speed ρ
- Applications in plasma control: allocator parameters penalize physically relevant quantities

Extensions and future work

- Apply to nonlinear plants: some results with satellite control, compliant robotics
- Include actuators dynamics: preliminary results obtained with control of hybrid cars
- Extend set point regulation to reference tracking: results under investigation with application to HyperSonic Vehicles