On Finite Gain $\mathcal{L}_p$ Stability for Hybrid Systems ⋆

Dragan Nešić * **Andrew R. Teel** ** **Giorgio Valmorbida** *** **Luca Zaccarian** ****

* *Electrical and Electronic Engineering Dept., University of Melbourne, Parkville 3010 Vic., Australia (e-mail: dnesic@unimelb.edu.au)*
** *Dept. of Electrical and Computer Engineering, Univ. of California, Santa Barbara, CA 93106, USA (e-mail: teel@ece.ucsb.edu)*
*** *Dip. di Informatica, Sistemi e Produzione, Università di Roma, Tor Vergata, Italy (e-mail: valmorbi@disp.uniroma2.it)*
**** *CNRS, LAAS, 7 av. du Colonel Roche F-31400, Toulouse, France. and Université de Toulouse, LAAS; F-31400, Toulouse, France (e-mail: zaccarian@laas.fr)*

Abstract: We characterize finite gain $\mathcal{L}_p$ stability properties for hybrid dynamical systems. By defining a suitable concept of hybrid $\mathcal{L}_p$ norm, we provide sufficient Lyapunov conditions for $\mathcal{L}_p$ stability of hybrid dynamics, which cover the well known continuous-time and discrete-time $\mathcal{L}_p$ stability notions as special cases. We also focus on homogeneous hybrid systems and prove a result stating the equivalence between $\mathcal{L}_p$ stability, ISS, global exponential stability and local asymptotic stability of the hybrid system with no inputs. Finally, we provide some input-output results and an LMI based $\mathcal{L}_2$ gain estimate for a class of homogeneous hybrid systems.

Keywords: Input-output stability, $\mathcal{L}_p$ stability, hybrid systems, homogeneous hybrid systems.

1. INTRODUCTION

When focusing on hybrid dynamical systems, some input-output stability notions have been investigated in the context of input-to-state stability (ISS) in Cai and Teel (2009) (see also references therein for earlier solutions in similar directions) where the ISS concept introduced by Sontag in the late 1980's, and well developed in the past two decades both for the continuous-time (CT) and the discrete-time (DT) cases, is extended to the hybrid context within the general framework described in Goebel et al. (2009) (see also Goebel et al. (2012)). Despite the ISS results cited above, there seems to be a lack of results on $\mathcal{L}_p$ stability properties of hybrid systems in the literature. The goal of this paper is to provide some results in this direction. Computing the finite $\mathcal{L}_p$ gain from a control input or an exogenous disturbance can help to define the performance of interconnected systems by analyzing its components separately. In particular, since these results are obtained within the hybrid systems framework of Goebel et al. (2009), the dissipativity results that recently appeared in Sanfelice (2010); Teel (2010) can be applied by using the specific supply rates introduced here, thereby providing small gain conditions for interconnected $\mathcal{L}_p$ stable hybrid systems.

In this paper we first introduce the concept of hybrid $\mathcal{L}_p$ norms by incorporating sums and integrals in them, so that the well known continuous-time and discrete-time norms are recovered as special cases. Then, we illustrate how the use of suitable Lyapunov-like conditions can be enforced in terms of some $\mathcal{L}_p$ storage functions in such a way that finite gain input-to-state $\mathcal{L}_p$ and $\mathcal{L}_p$ to $\mathcal{L}_\infty$ stability can be established, for a given hybrid system, from its input to its state. Then we focus on homogeneous hybrid systems and establish for this special class of systems a result stating the equivalence among local asymptotic stability, global exponential stability (this is a straightforward consequence of the results in Goebel and Teel (2010)), ISS and $\mathcal{L}_p$ stability. These results are important especially in light of the recent results in Goebel and Teel (2010) about homogeneous approximations of hybrid systems. Indeed we establish here that, for these homogeneous approximations, local asymptotic stability of the system with no inputs implies ISS and finite gain $\mathcal{L}_p$ stability which can be usefully exploited for establishing properties of their interconnections. Some preliminary statements along similar directions to the ones of this paper have been reported in Nesic et al. (2011) with reference to temporally regularized homogeneous systems. Those results were instrumental to proving suitable stability properties of interconnected reset systems. In Nesic et al. (2011) the following special case was addressed: 1) systems with temporal regularization (or dwell time) and 2) external inputs only appearing in the flow map. For this special case, there was no need to introduce hybrid $\mathcal{L}_p$ norms and the results were stated in terms of classical continuous-time $\mathcal{L}_p$ norms.

The paper is organized as follows. Preliminaries are presented in Section 2. In Section 3 we show how to obtain finite gain $\mathcal{L}_p$ stability bounds from storage functions for hybrid systems. Then, in Section 4 we focus on homogeneous systems and characterize a number of equivalent properties, including ISS, LAS, GES and existence of suit-

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able Lyapunov functions. Quadratic $\mathcal{L}_2$ gain estimates for a class of homogeneous systems are given in Section 5.

Notation: $\mathcal{B}(r)$ denotes the ball of radius r. $|v|$ denotes the Euclidean norm of a vector $v \in \mathbb{R}^n$. $\mathbb{R}_{\geq 0}$ denotes the set of non-negative real numbers. Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^n$ $X + Y = \{x + y : x \in X,\ y \in Y\}$. A function $\alpha(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}_\infty$ if it is strictly increasing, $\alpha(0) = 0$ and $\lim_{s\to\infty} \alpha(s) = +\infty$. The vector $[x'\ d']'$ is denoted (x, d).

2. PRELIMINARIES

A solution to a hybrid system is defined on a hybrid time domain, which is a subset of $\mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$. A *compact hybrid time domain* is any subset of $\mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ that can be written as $\cup_{i\in\{1,\ldots,J\}}([t_i, t_{i+1}] \times \{i\})$ where $J \in \mathbb{Z}_{\geq 0}$ and $0 = t_0 \leq t_1 \leq \ldots \leq t_{J+1}$. A *hybrid time domain* is any set $E \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ such that $(T_D, J_D) \in E$ implies that $E \cap ([0, T_D] \times \{0, \ldots, J_D\})$ is a compact hybrid time domain. A *hybrid signal* is a function defined on a hybrid time domain. A *hybrid arc* is a hybrid signal x such that $t \mapsto x(t, j)$ is locally absolutely continuous for each j.

Definition 1. ($\mathcal{L}_p$ norm). For a hybrid signal z, with domain $\mathrm{dom}(z) \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, and a scalar $T \in \mathbb{R}_{\geq 0}$ the T-truncated $\mathcal{L}_p$-norm of z is given by

$$\|z_{[T]}\|_p := \left(\sum_{i=1}^{j(T)} |z(t_i, i-1)|^p + \sum_{i=0}^{j(T)} \int_{t_i}^{\sigma_i} |z(s, i)|^p ds \right)^{\frac{1}{p}} \tag{1}$$

where $t_0 = 0$, $j(T) = \max k$ such that $(t, k) \in \mathrm{dom}(z)$, $t+k \leq T$ and for all $i \in \{0, \ldots, j\}$, $\sigma_i = \min(t_{i+1}, T-i-t_i)$. Based on (1), the $\mathcal{L}_p$-norm of z is defined as

$$\|z\|_p = \lim_{T\to\infty} \|z_{[T]}\|_p. \tag{2}$$

Moreover, we say $z \in \mathcal{L}_p$ whenever the limit above exists and is finite.

Remark 1. The $\mathcal{L}_p$ norms for continuous-time and discrete time systems are particular cases of the above defined norm. Indeed, if the solution only flows, we have for any T, $j(T) = 0$ and the first sum in (1) disappears[1] so that the hybrid norm becomes the continuous-time $\mathcal{L}_p$ norm (Khalil, 2002, Chapter 5). Moreover, if the solution only jumps, then $t_k = \sigma_k = 0$ for all k, and all the integral terms in (1) disappear so that (2) corresponds to the discrete-time ℓ_p norm (Vidyasagar, 2002, Section 6.7). ⌟

Remark 2. In (1) the value of the hybrid arc just before the jump $z(t_i, i-1)$ is considered. An alternative definition can be given in terms of the values of the hybrid arc after the jump, that is, $z(t_i, i)$. Then parallel computations to the ones reported in this paper can be carried out. We choose this option here for consistency with the approach in Cai and Teel (2009) reported below (see, in particular, the definition of $\Gamma(z)$ below). ⌟

Definition 2. ($\mathcal{L}_\infty$ norm). For a hybrid signal z, with domain $\mathrm{dom}(z) \in \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, the T-truncated $\mathcal{L}_\infty$ norm is given by

$$\|z_{[T]}\|_\infty := \max \left\{ \operatorname*{ess.sup}_{(t,j)\in\mathrm{dom}(z)\setminus\Gamma(z),\ t+j\leq T} |z(t, j)|, \sup_{(t,j)\in\Gamma(z), t+j\leq T} |z(t, j)| \right\} \tag{3}$$

And the $\mathcal{L}_\infty$ norm of z is given by

$$\|z\|_\infty = \lim_{T\to\infty} \|z_{[T]}\|_\infty \tag{4}$$

where $\Gamma(z)$ denotes the set of all (t, j) such that $(t, j) \in \mathrm{dom}(z)$ and $(t, j + 1) \in \mathrm{dom}(z)$. Moreover, we say $z \in \mathcal{L}_\infty$ whenever the above limit exists and is finite.

Consider the following nonlinear hybrid system

$$\begin{cases} \dot{x} = f(x, d), & (x, d) \in \mathcal{C} \\ x^+ = g(x, d), & (x, d) \in \mathcal{D} \end{cases} \tag{5}$$

where $x \in \mathbb{R}^n$ is the state vector, $d \in \mathbb{R}^n$ is an exogenous input, $f(\cdot, \cdot)$ is the flow map, $g(\cdot, \cdot)$ is the jump map, $\mathcal{C} \subseteq \mathbb{R}^n \times \mathbb{R}^m$ is the flow set and $\mathcal{D} \subseteq \mathbb{R}^n \times \mathbb{R}^m$ is the jump set. We assume the following regularity condition for the parameters of system (5).

Assumption 1. The sets $\mathcal{C}$ and $\mathcal{D}$ are closed sets and $f(\cdot, \cdot)$ and $g(\cdot, \cdot)$ are continuous in both their arguments.

In this paper we study the finite gain $\mathcal{L}_p$ stability for system (5) which is defined as follows.

Definition 3. Given $p \in [1,\ +\infty)$, system (5) is finite gain $\mathcal{L}_p$ stable from d to x with gain (upper bounded by) $\gamma_p \geq 0$ if there exists a scalar $\beta \geq 0$ such that any solution to (5) satisfies

$$\|x\|_p \leq \beta|x(0, 0)| + \gamma_p\|d\|_p. \tag{6}$$

for all $d \in \mathcal{L}_p$. Moreover, it is finite gain $\mathcal{L}_{p,\infty}$ ($\mathcal{L}_p$ to $\mathcal{L}_\infty$) stable from d to x with gain $\gamma_{p,\infty} > 0$ if there exists a scalar, $\beta \geq 1$ such that any solution to (5) satisfies

$$\|x\|_\infty \leq \beta|x(0, 0)| + \gamma_{p,\infty}\|d\|_p. \tag{7}$$

for all $(t, j) \in \mathrm{dom}(x)$ and all $d \in \mathcal{L}_p$.

Remark 3. Notice that, because of the comments in Remark 1, the $\mathcal{L}_p$ (ℓ_p) stability definitions of (Khalil, 2002, Chapter 5) and (Vidyasagar, 2002, Section 6.7) respectively for continuous and discrete-time systems correspond to particular cases of inequality (6). ⌟

Definition 4. The origin of (5) with $d = 0$ is *(locally) asymptotically stable (LAS)* if there exists a ball $\mathcal{B} \subset \mathbb{R}^n$, centered at the origin and a class $\mathcal{KLL}$ function β such that for any initial condition $x(0, 0) = x_0 \in \mathcal{B}$, all solutions satisfy:

$$|x(t, j)| \leq \beta(|x_0|, t, j)\ \forall x(t, j) \in \mathrm{dom}(x).$$

Definition 5. (Exponential ISS). System (5) is exponentially finite gain input-to-state stable from d if there exist positive scalars m, ℓ and γ_∞ such that for any initial conditions $x(0, 0)$ and any $d \in \mathcal{L}_\infty$, all solutions to (5) satisfy

$$|x(t, j)| \leq \max\left\{ me^{-\ell(t+j)}|x(0, 0)|, \gamma_\infty\|d\|_\infty \right\}, \tag{8}$$

for all $(t, j) \in \mathrm{dom}(x)$. Moreover the origin of (5) is *globally exponentially stable (GES)* if (8) holds with $d = 0$.

3. STORAGE FUNCTIONS FOR $\mathcal{L}_P$ GAIN COMPUTATION

In this section we establish $\mathcal{L}_p$ stability of system (5) by using storage functions.

[1] We adopt the convention $\sum_{i=1}^{0} f(i) := 0$.

Definition 6. ($\mathcal{L}_p$ storage function). Given $p \in [1, \infty)$, a positive semidefinite continuously differentiable function $V(\cdot) : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a finite gain $\mathcal{L}_p$ storage function for system (5) if there exist positive constants c_2, k_{xf} and k_{xg} and non-negative constants k_{dg}, k_{df} such that

$$0 \leq V(x) \leq c_2|x|^p \ \forall (x, d) \in \mathcal{C} \cup \mathcal{D} \tag{9}$$

$$\langle \nabla V(x), f(x, d) \rangle \leq -k_{xf}|x|^p + k_{df}|d|^p \ \forall (x, d) \in \mathcal{C} \tag{10}$$

$$V(g(x, d)) - V(x) \leq -k_{xg}|x|^p + k_{dg}|d|^p \ \forall (x, d) \in \mathcal{D} \tag{11}$$

Proposition 1. Consider system (5) and assume that there exists a function $V(\cdot) : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfying (9)-(11). Then the system is finite-gain $\mathcal{L}_p$ stable with gain upper-bounded by $\gamma_p = \sqrt[p]{\frac{k_d}{k_x}}$ where $k_d = \max\{k_{df}, k_{dg}\}$, $k_x = \min\{k_{xf}, k_{xg}\}$. Moreover, if $\exists\, c_1 > 0$ such that

$$c_1|x|^p \leq V(x), \ \forall (x, d) \in \mathcal{C} \cup \mathcal{D}, \tag{12}$$

then the system is $\mathcal{L}_{p,\infty}$ stable with gain $\gamma_{p,\infty} \leq \sqrt[p]{\frac{k_d}{c_1}}$.

Remark 4. Notice that, since $V(x)$ is a positive-semidefinite function, finite gain $\mathcal{L}_p$ stability does not imply asymptotic stability of the origin. ⌟

Remark 5. In Teel (2010), the concept of "mixed dissipativity" was introduced to assess the stability of interconnected hybrid system. The dissipativity concepts were illustrated by considering quadratic supply rates. Here, the "mixed dissipativity" is obtained with supply rate functions of degree p and can be established by $\mathcal{L}_p$ storage functions $V(x)$ satisfying inequalities (10) and (11). ⌟

Proof of Proposition 1. Consider the function $U(t, j) = V(x(t, j))$ with $x(t, j)$ being a solution to system (5) having domain $\text{dom}(x)$. From inequalities (10) and (11) we get that if $[t_k, t_{k+1}] \times \{k\} \subset \text{dom}(x)$, then

$$\dot{U}(t, k) \leq -k_{xf}|x(t, k)|^p + k_{df}|d(t, k)|^p \quad \text{for almost all } t \in [t_k, t_{k+1}], \tag{13}$$

and

$$U(t_k, k) - U(t_k, k-1) \leq -k_{xg}|x(t_k, k-1)|^p + k_{dg}|d(t_k, k-1)|^p \text{ if } k \geq 1. \tag{14}$$

Consider now any $T \in \mathbb{R}_{\geq 0}$ and define $j(T) := \max k$ such that $(t, k) \in \text{dom}(x)$, $t + k \leq T$ and $\forall i \in \{0, \ldots, j(T)\}$, $\sigma_i := \min(t_{i+1}, T - i)$ and, for simplicity, denote $\sigma_{j(T)} = t$. Then integrate (13) on each interval $[t_k, \sigma_k]$, $k \leq j(T)$, to obtain

$$0 \leq -U(t_{k+1}, k) + U(t_k, k) - k_{xf} \int_{t_k}^{\sigma_k} |x(s, j)|^p ds + k_{df} \int_{t_k}^{\sigma_k} |d(s, j)|^p ds \ \forall k \in \{0, \ldots, j(T)\} \tag{15}$$

and rearrange the terms in (14) to get

$$0 \leq -U(t_k, k) + U(t_k, k-1) - k_{xg}|x(t_k, k-1)|^p + k_{dg}|d(t_k, k-1)|^p, \ \forall k \in \{1, \ldots, j(T)\}. \tag{16}$$

Summing up the $j(T)+1$ terms in (15) and the $j(T)$ terms in (16), we obtain

$$0 \leq U(0, 0) - U(\sigma_{j(T)}, j(T)) - k_{xg}\sum_{k=1}^{j(T)}|x(t_k, k-1)|^p - k_{xf}\sum_{k=0}^{j(T)}\int_{t_k}^{\sigma_k} |x(s, k)|^p ds + k_{dg}\sum_{k=1}^{j(T)}|d(t_k, k-1)|^p + k_{df}\sum_{k=0}^{j(T)}\int_{t_k}^{\sigma_k} |d(s, k)|^p ds.$$

Defining $k_d = \max\{k_{df}, k_{dg}\}$ and $k_x = \min\{k_{xf}, k_{xg}\}$, we have

$$\begin{aligned} k_x\|x_{[T]}\|_p^p &\leq -U(\sigma_{j(T)}, j(T)) + U(0, 0) + k_d\|d_{[T]}\|_p^p \\ &\leq U(0, 0) + k_d\|d\|_p^p \end{aligned} \tag{17}$$

where $\|x_{[T]}\|$ is the T-truncated $\mathcal{L}_p$ norm accordingly to Definition 1, and we used the fact that $U(t, j) \geq 0, \ \forall (t, j) \in \text{dom}(x)$.

$\mathcal{L}_p$ *gain*: Using (17) and the upper bound for $U(t, j)$ from (9), we obtain

$$\begin{aligned} \|x_{[T]}\|_p^p &\leq \frac{1}{k_x}c_2|x(0, 0)|^p + \frac{k_d}{k_x}\|d\|_p^p \\ \|x_{[T]}\|_p &\leq \sqrt[p]{\frac{1}{k_x}c_2}|x(0, 0)| + \sqrt[p]{\frac{k_d}{k_x}}\|d\|_p \end{aligned}$$

which holds for any $T > 0$, hence, taking the limit as $T \to \infty$ the bound for the finite-gain $\mathcal{L}_p$ is given by $\sqrt[p]{\frac{k_d}{k_x}}$.

$\mathcal{L}_{p,\infty}$ *gain*: Consider the first inequality in (17). Since $\|x_{[T]}\|_p^p \geq 0$, we have $U(\sigma_{j(T)}, j(T)) \leq U(0, 0) + k_d\|d\|_p^p$. Using the lower bound in (12) and the upper bound in (9), we obtain for all $(\sigma_{j(T)}, j) \in \text{dom}(x)$

$$\begin{aligned} |x(\sigma_{j(T)}, j)|^p &\leq c_1^{-1}U(0, 0) + c_1^{-1}k_d\|d\|_p^p \\ &\leq c_1^{-1}c_2|x(0, 0)|^p + c_1^{-1}k_d\|d\|_p^p \end{aligned}$$

which implies,

$$|x(\sigma_{j(T)}, j)| \leq \sqrt[p]{\frac{c_2}{c_1}}|x(0, 0)| + \sqrt[p]{\frac{k_d}{c_1}}\|d\|_p, \quad \forall (\sigma_{j(T)}, j) \in \text{dom}(x)$$

therefore according to Definition 2, we have

$$\|x\|_\infty \leq \sqrt[p]{\frac{c_2}{c_1}}|x(0, 0)| + \sqrt[p]{\frac{k_d}{c_1}}\|d\|_p,$$

namely the finite $\mathcal{L}_{p,\infty}$ gain of system (5) is upper-bounded by $\sqrt[p]{\frac{k_d}{c_1}}$. ■

4. HOMOGENEOUS SYSTEMS

In this section, using the results of Section (3), we establish finite gain $\mathcal{L}_p$ stability properties of (5) under a homogeneity assumption for the system without disturbances given by

$$\begin{cases} \dot{x} = f(x, 0), & x \in \mathcal{C}_0 \\ x^+ = g(x, 0), & x \in \mathcal{D}_0 \end{cases} \tag{18}$$

where $\mathcal{C}_0$ and $\mathcal{D}_0$ are suitable projected versions of the sets $\mathcal{C}$ and $\mathcal{D}$ on the direction of the state x, satisfying the following assumption.

Assumption 2. (Flow and jump sets). The sets $\mathcal{C}_0$ and $\mathcal{D}_0$ are closed and there exist scalars L_C and L_D such that for all $(x, d) \in \mathbb{R}^{n+m}$

$$(x, d) \in \mathcal{C} \Rightarrow x \in \mathcal{C}_0 + L_C\mathcal{B}(|d|) \tag{19a}$$

$$(x, d) \in \mathcal{D} \Rightarrow x \in \mathcal{D}_0 + L_D\mathcal{B}(|d|) \tag{19b}$$

where $\mathcal{C}_0 \subset \mathbb{R}^n$ and $\mathcal{D}_0 \subset \mathbb{R}^n$ are closed sets satisfying $\mathcal{C}_0 \times \{0\} \supset (\mathbb{R}^n \times \{0\}) \cap \mathcal{C}$ and $\mathcal{D}_0 \times \{0\} \supset (\mathbb{R}^n \times \{0\}) \cap \mathcal{D}$.

An homogeneous system is defined as follows:

Definition 7. System (18) is homogeneous of degree zero if given any scalar $\lambda > 0$, we have

$$\begin{aligned} f(\lambda x, 0) = \lambda f(x, 0)\ \forall x \in \mathcal{C}_0 \\ g(\lambda x, 0) = \lambda g(x, 0)\ \forall x \in \mathcal{D}_0 \end{aligned} \tag{20}$$

$$\begin{aligned} x \in \mathcal{C}_0 \Rightarrow \lambda x \in \mathcal{C}_0 \\ x \in \mathcal{D}_0 \Rightarrow \lambda x \in \mathcal{D}_0 \end{aligned} \tag{21}$$

Remark 6. From (19a) and (19b) we have that $\mathcal{C}_0$ and $\mathcal{D}_0$ must respectively contain the flow and jump sets projected on the space of x, that is, the set of x such that $(x, 0) \in \mathcal{C}$ and x such that $(x, 0) \in \mathcal{D}$. Conic flow sets and jump sets in the form

$$\left\{ (x, d) : \left[\, x' \ \ d' \,\right] M \begin{bmatrix} x \\ d \end{bmatrix} \geq 0 \right\} \tag{22}$$

are homogeneous of degree zero but, in general, (19) do not hold for these sets. The reason why is that the quadratic dependence on d in (22) can, in some cases, prevent the existence of a linear bound as in (19). For example, consider the case $M = diag\,(M_x, -I)$ and note that the condition in inequality (22) becomes $x' M_x x \geq |d|^2$ for which (19) can not possibly hold.

Despite the above limitation, if the conic flow and jump sets are defined as

$$\mathcal{C} = \{(x, d) : (x + S_C d)' M_C (x + S_C d) \geq 0\} \tag{23a}$$

$$\mathcal{D} = \{(x, d) : (x + S_D d)' M_D (x + S_D d) \geq 0\} \tag{23b}$$

with $M_C,\ M_D \in \mathbb{R}^{n \times n}$, $S_C,\ S_D \in \mathbb{R}^{n \times m}$, then (19) holds with $L_C = |S_C|$ and $L_D = |S_D|$. ⌟

The following assumption states some properties for the jump and flow maps in (5).

Assumption 3. (Flow and jump maps). Consider systems (5) and (18) satisfying Assumption 2. There exist two positive constants L_{df} and L_{dg} such that for all $d \in \mathbb{R}^m$,

$$|f(z+v, d) - f(z, 0)| \leq L_{df}|d|,\ \forall z \in \mathcal{C}_0, |v| \leq L_C|d| \tag{24a}$$

$$|g(z+v, d) - g(z, 0)| \leq L_{dg}|d|,\ \forall z \in \mathcal{D}_0, |v| \leq L_D|d| \tag{24b}$$

Remark 7. If $f(\cdot,\cdot)$ and $g(\cdot,\cdot)$ are globally Lipschitz in both arguments, then (24) hold. However (24) correspond to weaker assumptions as they are required to hold only in some particular subsets of the state space and with $|v|$ upper-bounded by some function of $|d|$. ⌟

The following theorem states equivalent properties for homogeneous systems which satisfy the above assumptions. Its proof is omitted due to space constraints.

Theorem 1. If Assumptions 1, 2 and 3 hold, and system (5) is homogeneous of degree zero in the sense of Definition 7, then the following statements are equivalent:

(1) The origin of (5) with $d = 0$, (namely (18)) is (locally) asymptotically stable;
(2) The origin of (5) with $d = 0$, (namely (18)) is globally exponentially stable;
(3) System (5) satisfies the following
   (a) for each $p \in [1, +\infty)$, $\exists V(\cdot)$ satisfying (9)-(12) and system (5) is finite gain $\mathcal{L}_p$ stable and finite gain $\mathcal{L}_{p,\infty}$ stable from d to x;
   (b) it is finite gain exponentially ISS from d to x;
(4) for each $p \in [1, +\infty)$, there exists a function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ that is smooth in $\mathbb{R}^n \setminus \{0\}$ and positive constants c_1, c_2, L_1, L_2, μ and $\nu \in [0, 1)$ such that

$$c_1|x|^p \leq V(x) \leq c_2|x|^p,\ \forall x \in \mathbb{R}^n \tag{25a}$$

$$\langle \nabla V(x), f(x, 0)\rangle \leq -\mu V(x), \forall x \in \mathcal{C}_0 \setminus \{0\}\,, \tag{25b}$$

$$V(g(x, 0)) \leq \nu V(x), \quad \forall x \in \mathcal{D}_0, \tag{25c}$$

$$|\nabla V(x)| \leq L_1|x|^{p-1}, \quad \forall x \in \mathbb{R}^n \setminus \{0\}, \tag{25d}$$

$$|V(x + v) - V(x)| \leq 2L_1\left(|x|^{p-1}|v| + |v|^p\right), \tag{25e}$$

$$|\nabla V(x + v) - \nabla V(x)| \leq L_2\left(|x|^{p-2}|v| + |v|^{p-1}\right) \tag{25f}$$

$$\forall x, v \text{ such that } x \neq 0 \text{ and } x + v \neq 0.$$

$$\begin{tikzcd} & \text{GES} \arrow[dr, Rightarrow] & \\ \text{LAS} \arrow[ur, Rightarrow] & & \exists V \Rightarrow \mathcal{L}_p \text{ and } \mathcal{L}_{p,\infty} \arrow[dl, Rightarrow] \\ & \text{ISS} \arrow[ul, Rightarrow] & \end{tikzcd}$$

Fig. 1. The structure of the proof of Theorem 1.

Remark 8. The proof of Theorem 1 is carried out by showing (1) $\Rightarrow$ (2) $\Rightarrow$ (4) $\Rightarrow$ (3b) $\Rightarrow$ (1), and that (4) $\Rightarrow$ (3a). Its structure is also graphically shown in Figure 1 which shows that we rely on ISS to establish the equivalence among item (3) and the remaining items. Conversely, the $\mathcal{L}_p$ stability property of item (3a) does not necessarily imply LAS, as already observed in Remark 4. ⌟

5. STORAGE FUNCTIONS FOR INPUT-OUTPUT $\mathcal{L}_P$ GAIN ESTIMATION

In this section we consider system (5) with output $y = h(x, d)$ and establish input-output $\mathcal{L}_p$ stability by using storage functions. In particular, the following definition and proposition generalize Definition 6 and Proposition 1 to the input-output case. The proof of the proposition is the same as that of Proposition 1, replacing x by y, wherever relevant.

Definition 8. ($\mathcal{L}_p$ storage function). Given $p \in [1, \infty)$, a positive semidefinite continuously differentiable function $V(\cdot) : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a finite gain input-output $\mathcal{L}_p$ storage function for system (5) with output $y = h(x, d)$ if there exist positive constants c_2, k_{yf} and k_{yg} and non-negative constants k_{dg}, k_{df} such that

$$0 \leq V(x) \leq c_2|x|^p\ \forall (x, d) \in \mathcal{C} \cup \mathcal{D} \tag{26}$$

$$\langle \nabla V(x), f(x, d)\rangle \leq -k_{yf}|y|^p + k_{df}|d|^p\ \forall (x, d) \in \mathcal{C} \tag{27}$$

$$V(g(x, d)) - V(x) \leq -k_{yg}|y|^p + k_{dg}|d|^p\ \forall (x, d) \in \mathcal{D}. \tag{28}$$

Proposition 2. Consider system (5) with $x(0, 0) = 0$ and assume that there exists a function $V(\cdot) : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfying (26)-(28). Then the system is input-output finite-gain $\mathcal{L}_p$ stable with input-output gain upper bounded by $\gamma_p = \sqrt[p]{\dfrac{k_d}{k_y}}$ where $k_d = \max\{k_{df}, k_{dg}\}$, $k_y = \min\{k_{yf}, k_{yg}\}$.

Remark 9. A parallel result to the $\mathcal{L}_{p,\infty}$ statement of Proposition 1, can also be stated here, by requiring for some non-negative scalars c_{yV}, c_{yd}

$$|y|^p \leq c_{yV} V(x) + c_{yd}|d|^p,\ \forall (x, d) \in \mathcal{C} \cup \mathcal{D}, \tag{29}$$

which generalizes (12) to the output case. Then the proof of finite gain $\mathcal{L}_{p,\infty}$ stability from d to y with gain

$\sqrt[p]{c_{yV}k_d + c_{yd}}$ follows the same steps as the proof of Proposition 1. ⌟

Consider the linear hybrid system

$$\begin{cases} \dot{x} = A_f x + B_{fd} d, & (x,d) \in \mathcal{C} \\ x^+ = A_g x + B_{gd} d, & (x,d) \in \mathcal{D} \\ y = Hx + Ld \end{cases} \tag{30a}$$

$$\begin{aligned} \mathcal{C} &= \left\{ (x,d) : \begin{bmatrix} x \\ d \end{bmatrix}' M_C \begin{bmatrix} x \\ d \end{bmatrix} \geq 0 \right\} \\ \mathcal{D} &= \left\{ (x,d) : \begin{bmatrix} x \\ d \end{bmatrix}' M_D \begin{bmatrix} x \\ d \end{bmatrix} \geq 0 \right\} \end{aligned} \tag{30b}$$

with $A_f, A_g, M_C, M_D \in \mathbb{R}^{n\times n}$, $B_{fd}, B_{gd}, \in \mathbb{R}^{n\times m}$, $H \in \mathbb{R}^{p\times n}$ and $L \in \mathbb{R}^{p\times m}$, $M_C = M_C'$ and $M_D = M_D'$.

The following corollary of Proposition (2) allows us to compute an upper-bound of the $\mathcal{L}_2$ gain of system (30) using a quadratic storage function $V(\cdot)$.

Corollary 5.1. If there exist a positive-definite symmetric matrix $P \in \mathbb{R}^{n\times n}$, non-negative scalars γ, τ_C and τ_D satisfying the next linear matrix inequalities

$$\left[\begin{array}{cc|c} A_f'P + PA_f & PB_{fd} & H' \\ B_{fd}'P & -\gamma I & L' \\ \hline H & L & -\gamma I \end{array}\right] + \left[\begin{array}{c|c} \tau_C M_C & 0 \\ \hline 0 & 0 \end{array}\right] \leq 0, \tag{31a}$$

$$\left[\begin{array}{cc|c} A_g'PA_g - P & A_g'PB_{gd} & H' \\ B_{gd}'PA_g & -\gamma I + B_{gd}'PB_{gd} & L' \\ \hline H & L & -\gamma I \end{array}\right] + \left[\begin{array}{c|c} \tau_D M_D & 0 \\ \hline 0 & 0 \end{array}\right] \leq 0\,. \tag{31b}$$

then (26)-(28) are satisfied with $p = 2$ and the finite input-output $\mathcal{L}_2$ gain of (30) is bounded by γ. Moreover, if the inequalities in (31) are strict, then (30) is GES.

Proof. Applying a Schur complement to (31a) we arrive at

$$\begin{bmatrix} A_f'P + PA_f & PB_{fd} \\ B_{fd}'P & -\gamma I \end{bmatrix} - \frac{1}{\gamma}\begin{bmatrix} H' \\ L' \end{bmatrix}[\,H\ \ L\,] + \tau_C M_C \leq 0, \tag{32}$$

recalling that $y = [\,H\ \ L\,]\begin{bmatrix} x \\ d \end{bmatrix}$, and from the $\mathcal{S}$-procedure, we obtain the following quadratic inequality, which is implied by (32)

$$\langle 2Px, A_f x + B_{fd} d\rangle + \frac{1}{\gamma}|y|^2 - \gamma|d|^2 \leq 0 \ \forall (x,d) \in \mathcal{C},$$

where $\mathcal{C}$ is given in (30b). The above inequality is equivalent to (27) with the choices $p = 2$, $V(x) = x'Px$, $k_{yf} = \frac{1}{\gamma}$, $k_{df} = \gamma$. Similarly, we have that (31b) implies (28) with $k_{yg} = \frac{1}{\gamma}$, $k_{dg} = \gamma$. Since we have a quadratic $V(\cdot)$, then (26) holds and, by Proposition 2, the finite input-output gain is upper bounded by γ.

The proof of GES, when inequalities (31) hold strictly, arises from the fact that considering $d = 0$ one gets from the upper left blocks of (31) that the function $V(x) = x'Px$ is a strict Lyapunov function for the hybrid system. ■

Remark 10. The above result combines the bounded real lemma for continuous- and for discrete-time systems to obtain a bound to the $\mathcal{L}_2$ gain for hybrid systems when using hybrid norms. In (Zaccarian et al., 2011, Theorem 2) and (Beker et al., 2004, Theorem 13), similar results were stated for the special case of systems with dwell time (which ensures unbounded hybrid domains in the ordinary time direction). In that case, since only continuous-time norms are used, it is necessary that d do not affect the jump and flow sets and the jump map, and (31b) reduces to $A_g'PA_g - P \leq 0$. ⌟

Remark 11. Corollary 5.1 corresponds to the selection of a quadratic $V(\cdot)$, $k_{yf} = k_{yg} = \gamma^{-1}$, and $k_{df} = k_{dg} = \gamma$ in (27)-(28), and noticing the convexity of the arising conditions. It is emphasized that the selection above for the scalars is not conservative for the gain estimation, due to the following.

- Assume that (27)-(28) are satisfied by different values of k_{df} and k_{dg} or k_{yf} and k_{yg} providing the gain estimate $\gamma_p = \sqrt[p]{\frac{k_d}{k_y}}$ with $k_d = \max\{k_{df}, k_{dg}\}$ and $k_y = \min\{k_{yf}, k_{yg}\}$. Then, (27)-(28) are clearly also satisfied with k_d replacing both k_{df} and k_{dg} and with k_y replacing both k_{yf} and k_{yg}. Moreover the same gain estimate is obtained by these new inequalities.
- We showed above that it is not conservative to use k_d, k_y. For the quadratic case, it is also not conservative to have $k_d = \frac{1}{k_y}$. Indeed, assume that (31) are satisfied with $\gamma_y = \frac{1}{k_y}$ on the $(2,2)$ elements of the two leftmost matrices and with $\gamma_d = k_d$ on the $(3,3)$ elements. Then the same estimate can be established by setting $k_d = \frac{1}{k_y} = \gamma = \sqrt{\gamma_y\gamma_d}$ on all the $(2,2)$ and $(3,3)$ diagonal terms and selecting $P^\star = \sqrt{\frac{\gamma_y}{\gamma_d}}P$, $\tau_C^\star = \sqrt{\frac{\gamma_y}{\gamma_d}}\tau_C$ and $\tau_D^\star = \sqrt{\frac{\gamma_y}{\gamma_d}}\tau_D$, which satisfy both inequalities.

⌟

Remark 12. While the reasoning of Remark 11 shows no conservativeness in the selection of $k_{df} = k_{dg} = k_{yf}^{-1} = k_{yg}^{-1} = \gamma$, there is however conservativeness of the $\mathcal{L}_2$ gain estimate due to the fact that we restrict $V(\cdot)$ to be quadratic. Indeed, unlike the linear continuous- and discrete-time cases where the bounded real lemma is known to give the exact gain, for the hybrid case this is not true and nonconvex Lyapunov functions are needed in general (Blanchini and Savorgnan (2008)), even to assess exponential stability (see also Forni and Teel (2010) where sum-of-squares Lyapunov functions are used with homogeneous hybrid systems). ⌟

Remark 13. In this section we have considered sets in the form (30b) while in Section 4 we restricted the sets to take the special form (23) in order to obtain the bounds (19) (see also Remark 6). Therefore, for a system satisfying the conditions of Corollary 5.1 to have the properties implied by Theorem 1, the flow and jump sets must be in the form of (23) and satisfy the strict version of inequalities (31), so that GES holds. Notice, indeed, that according to the implications of Figure 1, it is not sufficient to assess $\mathcal{L}_2$ stability to assess GES, ISS and the additional properties whose equivalence is established in Theorem 1. ⌟

In the example below we compute the input-output $\mathcal{L}_2$-gain for a linear hybrid system of the form (30).

Example 1 This example is taken from (Prieur et al., 2011, Example 1) where we have introduced disturbances acting both at the input of the plant d_u and at the measured output d_y. Following Prieur et al. (2011), the hybrid closed-loop of (Prieur et al., 2011, Example 1) can be written as

$$\begin{cases} \dot{x}_p &= a_p x_c + b_p(x_c + d_u) \\ \dot{x}_c &= a_c x_c - y \end{cases} \quad (x,d) \in \mathcal{C}$$
$$\begin{cases} x_p^+ &= x_p \\ x_c^+ &= -\kappa_M y \end{cases} \quad (x,d) \in \mathcal{D} \tag{33a}$$
$$y = x_p + d_y$$

$$\mathcal{C} = \left\{ (x,d) : \begin{bmatrix} y \\ x_c \end{bmatrix}' \begin{bmatrix} a_p + b_p\kappa_M & b_p \\ b_p & 0 \end{bmatrix} \begin{bmatrix} y \\ x_c \end{bmatrix} \geq 0 \right\}$$
$$\mathcal{D} = \overline{\mathbb{R}^2 \setminus \mathcal{C}}. \tag{33b}$$

Notice that the disturbance d_u affects only the flow map while disturbance d_y affects the jump map and the flow and jump sets. Therefore the results in Zaccarian et al. (2011) and Beker et al. (2004) can not be used to estimate the gain from the disturbance d_y to the output y. Thanks to the structure of the flow and jump sets, which can be written as in (23), Theorem 1 applies to this system. System (33) can be written in the form (30) using the following values:

$$A_f = \begin{bmatrix} a_p & b_p \\ a_c & -1 \end{bmatrix}; A_g = \begin{bmatrix} 1 & 0 \\ -\kappa_M & 0 \end{bmatrix};$$
$$B_{fd} = \begin{bmatrix} b_p & 0 \\ 0 & 0 \end{bmatrix}; B_{gd} = \begin{bmatrix} 0 & 0 \\ 0 & -\kappa_M \end{bmatrix};$$
$$M_C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_p + b_p\kappa_M & b_p \\ b_p & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}; M_D = -M_C$$

In Prieur et al. (2011), it is shown that, with $a_c = 0$, $\kappa_M = 1$ and $b_p = 1$, the system is GES for all $a_p \in (-\infty, 1]$. Therefore, according to Theorem 1, the system is also finite gain $\mathcal{L}_2$ stable for these values. However, the conservativeness introduced by the quadratic Lyapunov function (see Remark 12) does not allow to compute a quadratic estimates of the $\mathcal{L}_2$ gain for all the values of a_p in the GES range. Figure 2 shows the quadratic gain estimates from d_u to y (red, dashed) and from d_y to y (blue solid) as a function of a_p. Notice that the estimate γ_{d_y} from d_y to y is constant and the estimate γ_{d_u} from d_u to y increases as a_p increases, up to the value of $a_p = -3$ which is a bound for the feasibility of (31). The shaded area in Figure 2 corresponds to the set where the system is not finite-gain $\mathcal{L}_2$ stable and the vertical dashed line corresponds to the value of a_p in one of the simulations reported in Prieur et al. (2011).

6. CONCLUSIONS

We presented results for the characterization of finite-gain $\mathcal{L}_p$ stability of hybrid systems. By deriving a mixed-dissipativity inequality we generalized to the hybrid case the $\mathcal{L}_p$ stability concepts of both continuous-time and discrete-time systems. When focusing on homogeneous hybrid system, under suitable regularity assumptions on the jump and flow maps and sets, we stated the equivalence between LAS, GES, ISS and the existence of a storage function. Each one of these properties implies finite-gain $\mathcal{L}_p$ stability of the homogeneous system.

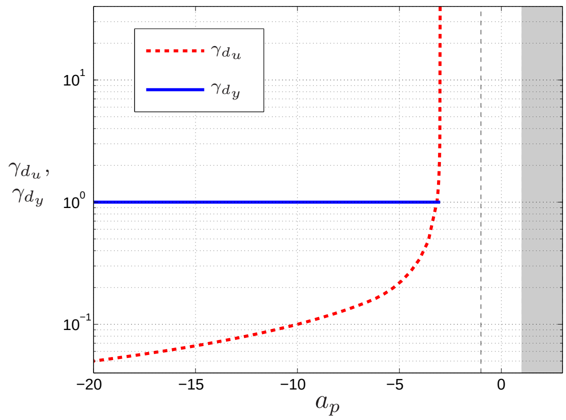

Fig. 2. Quadratic $\mathcal{L}_2$ gain estimates from d_u and d_y to the output y as a function of the parameter a_p.

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