On necessary and sufficient conditions for exponential and $\mathcal{L}_2$ stability of planar reset systems

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Abstract— **In this paper we provide necessary and sufficient conditions for exponential stability and $\mathcal{L}_2$ stability of planar reset systems, i.e., systems involving a First Order Reset Element (FORE) and a linear plant having dimension one. The proof relies on Lyapunov tools developed in a recent novel representation of a class of reset systems incorporating this special planar case. Explicit Lyapunov functions are given to show both exponential and $\mathcal{L}_2$ stability. Based on this Lyapunov function, an explicit estimate of the $\mathcal{L}_2$ gain, depending on the system's parameters, is provided. Moreover, via the same tools, it is shown that the gain estimates go to zero as certain parameters (in particular, the FORE pole) become arbitrarily large, thus allowing to establish a small gain result showing stability of certain higher order SISO linear plants under the action of a FORE.**

I. INTRODUCTION

Reset controllers were proposed for the first time by Clegg in 1958 with the aim of providing more flexibility in linear controller designs and potentially removing fundamental performance limitations of linear controllers (see, e.g., [1] for one such example). Subsequently, a new reset device called the *first order reset element (FORE)* was introduced in [6] and a controller design procedure based on FOREs was proposed. The design procedure was based on linear frequency domain techniques for robust stabilization. These early results on reset control systems are summarized in a recent paper [3].

There has been a renewed interest in this class of systems in the late 1990's. First attempts to rigorously analyze stability of reset systems with Clegg integrators can be found in [7], [5]. In particular, an integral quadratic constraint was proposed in [5] to analyze stability of a class of reset systems. However, the proposed condition was conservative as it was independent of reset times. Stability analysis of reset systems consisting of a second order plant and a FORE was conducted in [4]. The proofs are based on an explicit characterization of reset times which are proved to be equidistant under mild conditions. Using this fact, the authors prove asymptotic and BIBO stability of the reset system via the discrete-time model of the system that describes the system at reset times only. However, the same approach could not be used to analyze higher order reset systems. Stability analysis of general reset systems can be found in [2] where Lyapunov based conditions for asymptotic stability of general reset systems were presented. Moreover, the authors proposed computable conditions for quadratic stability based on linear matrix inequalities (LMIs). Bounded-input bounded-state stability of general reset systems was obtained as a consequence of quadratic stability. Finally, an internal model principle was proved for reference tracking and disturbance rejection.

Recently, in [10], [11] we have presented Lyapunov like conditions for $\mathcal{L}_2$ stability and exponential stability of general reset systems. Our results apply to a more general class of models than those considered in the literature (see [2]); in particular, we allow resets to occur on more complicated sets than those considered in [2]. In this paper, we continue the investigation of exponential stability and $\mathcal{L}_2$ stability properties of reset systems.

Our first main result presents necessary and sufficient conditions under which planar reset systems are exponentially or $\mathcal{L}_2$ stable. The result provides an explicit algebraic condition on the parameters in the model that guarantee exponential or $\mathcal{L}_2$ stability. Moreover, we provide an explicit estimate of the $\mathcal{L}_2$ gain of the system. We note that this gain estimate is conservative when compared to some other estimates obtained for smaller classes of planar reset systems given in [13] and especially so when compared to numerical estimates provided in [12]. Our second main result shows that the $\mathcal{L}_2$ gain can be arbitrarily reduced when the pole of the FORE and/or the loop gain are increased to infinity. Note that increasing the pole of the FORE to infinity means that the linear system without resets is more and more unstable. While this result may be counterintuitive, our analysis shows that it holds and we comment on the underlying intuition. Finally, based on this result on the $\mathcal{L}_2$ gain trend, we establish sufficient stability results for higher dimensional SISO reset systems based on a small gain reasoning. Note that, as compared to the high gain result in [12], this result is more powerful because it doesn't require the loop gain to be sufficiently large.

The paper is organized as follows: in Section II we prove new Lyapunov tools that apply to general reset systems under strengthened conditions on the jump and flow set. In Section III we concentrate on planar FORE control loops and state our main results, while in Section IV we illustrate

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how these results can be used to establish sufficient stability conditions for higher dimensional systems with FOREs.

Notation: $(x, y) := [x^T \ y^T]^T$. Given a state variable x of a system with jumps, we denote its derivative with respect to time (which is defined almost everywhere) by $\dot{x}$ while at jump times, we denote the value of the state after the jump by x^+ and the value of the state before the jump simply by x.

II. Lyapunov approaches for the analysis of reset systems

Consider the following general model for a linear reset system

$$\begin{cases} \dot{x} = Ax + B_d d, & \text{if } x \in \mathcal{F}, \\ x^+ = A_r x, & \text{if } x \in \mathcal{J}, \end{cases} \tag{1}$$

where A denotes the flow matrix, A_r denotes the reset matrix, d is an external disturbance and the jump and flow sets $\mathcal{J}$ and $\mathcal{F}$ are closed set performing a partition of the overall state space.

When augmenting the system with temporal regularization (see [12], [11], [10] for details), the overall dynamics becomes

$$\begin{cases} \left.\begin{array}{l} \dot{\tau} = 1, \\ \dot{x} = Ax + B_d d \end{array}\right\} & \text{if } x \in \mathcal{F} \text{ or } \tau \leq \rho, \\ \left.\begin{array}{l} \tau^+ = 0, \\ x^+ = A_r x \end{array}\right\} & \text{if } x \in \mathcal{J} \text{ and } \tau \geq \rho, \end{cases} \tag{2}$$

which for any (arbitrarily small) selection of $\rho > 0$, rules out the possibility of Zeno solutions.

In the remainder of this section, we will comment upon two types of techniques that might be used to establish useful properties of system (2) based on Lyapunov functions satisfying suitable properties for the dynamics in (1).

A. *Inflating the flow set or imposing strict decrease at jumps*

The model (2) has been first introduced in [11], [10] and Lyapunov conditions have been given in those papers for establishing exponential stability and $\mathcal{L}_2$ stability from a disturbance input to a suitable output. Along the same line, in [12] and [13], numerical and analytic constructions of Lyapunov functions satisfying these conditions were given.

When using the model (1) and its extended version with temporal regularization (2), what makes the Lyapunov construction in the above referenced papers hard is that temporal regularization may cause the system's state to slightly overflow into the jump set $\mathcal{J}$ before the reset occurs. Therefore, all the Lyapunov-based results reported in [11], [10], [12], [13] require the following items to hold for a suitable Lyapunov function $V(\cdot)$ (see [11], [10] for more formal statements):

1) it satisfies some regularity conditions (e.g., quadratic upper and lower bounds) everywhere;
2) it satisfies a suitable growth condition when the state flows in the jump set $\mathcal{J}$;
3) it is a disturbance attenuation Lyapunov function in a slightly inflated version $\mathcal{F}_\varepsilon$ of the flow set $\mathcal{F}$;
4) it does not increase when jumping from the jump set $\mathcal{J}$.

While items 1 and 2 are not surprising because they simply establish suitable regularity conditions for $V(\cdot)$, what is typically hard to obtain is that the Lyapunov function satisfies together items 3 and 4. This is because one would typically want to patch two functions leading to a Lipschitz selection of $V(\cdot)$, where a first function is tailored to satisfy the jump condition in the jump set, while a second function is tailored to satisfy the flow condition in the flow set. However, this approach doesn't directly apply to the results of [11], [10], [12], [13] because of item 3 above, which requires the flow condition to hold on a *slightly inflated* version of the flow set. This requirement establishes a disturbing coupling between items 3 and 4 above because there is a small region where both the jump and the flow condition should hold. The reason why this fact is necessary is that when analyzing the system's trajectory in light of temporal regularization, it is necessary to guarantee that whenever the state overflows in the jump set before the next jump is allowed (namely until $\tau \leq \rho$), a nice bound on the trajectory still holds. This good bound comes from the bound at item 3 which is guaranteed to hold not only within the flow set but also in a slight portion of the jump set adjacent to the flow set boundary.

Motivated by the difficulty arising in the Lyapunov analysis and Lyapunov construction characterizing the techniques relying on items 1–4 above, we propose here a different model which appears to be a valuable alternative at least in some relevant cases. This model is based on relaxing the flow condition at item 3 and only requiring that it holds within the flow set (with no inflation) at the price of strengthening the jump condition at item 4 in requiring a strict decrease at jumps, namely items 3 and 4 above are replaced by

3a. $V(\cdot)$ is a disturbance attenuation Lyapunov function in the flow set $\mathcal{F}$;
4a. $V(\cdot)$ strictly decreases when jumping from the jump set $\mathcal{J}$.

From an intuitive viewpoint, having a strict decrease at jumps allows to compensate for a possible growth of $V(x(t))$ that might have happened while $x(t)$ was overflowing in the jump set (for $\tau \leq \rho$), thus making it possible to require a less stringent flow condition and to only rely on the regularity assumption of item 2.

The advantage of this new technique is that the proof of the main results of [11], [10] becomes extremely simpler (it is given in Theorem 1 in the next section) and in some cases the Lyapunov construction might be simpler. Moreover, for situations where strict decrease at jumps is a natural condition to impose, the new Lyapunov tools of Theorem 1 might be more effective at exploiting the underlying system features to effectively design Lyapunov functions that establish exponential and $\mathcal{L}_2$ stability of the system with temporal regularization.

B. *Lyapunov conditions with strict decrease at jumps*

The qualitative requirements introduced at items 1, 2, 3a and 4a in the previous section can be formalized in the following assumption.

Assumption 1: Given system (1) and a suitable output y satisfying $|y|^2 \leq \lambda_y |x|^2$, the Lyapunov function $V(\cdot) : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is such that there exist positive real numbers λ_i, $i = 1, \dots, 7$ and $\eta \in (0,1)$ such that for all d:

$$\lambda_1 |x|^2 \leq V(x) \leq \lambda_2 |x|^2 \tag{3a}$$

$$< \nabla V(x), Ax + B_d d > \leq \lambda_3 V(x) + \lambda_4 |x||d|, \quad \forall x, \tag{3b}$$

$$\begin{array}{rcl} < \nabla V(x), Ax + B_d d > & \leq & -\lambda_5 V(x) - \lambda_6 |y|^2 \\ & & +\lambda_7 |d|^2, \quad \forall x \in \mathcal{F}, \end{array} \tag{3c}$$

$$V(A_r x) \leq \eta V(x), \quad \forall x \in \mathcal{J}. \tag{3d}$$

The following theorem establishes the sufficiency of the new Lyapunov conditions of Assumption 1 to establish the exponential and $\mathcal{L}_2$ stability properties of the reset system with temporal regularization (2). This theorem should be though of as a valuable alternative to the approaches in [11], [10], [12], [13].

Theorem 1: Consider the reset system (1) without temporal regularization and assume that there exists a function $V(\cdot)$ satisfying Assumption 1. Then the reset system with temporal regularization (2) satisfies

1) the origin of the x dynamics is exponentially stable, namely there exist positive numbers c, λ such that for all $x(0)$, $|x(t)| \leq c|x(0)| \exp(-\lambda t)$, $\forall t \geq 0$;
2) the system is finite gain $\mathcal{L}_2$ stable from d to y and for any $\epsilon > 0$, there exists ρ^* such that for all $\rho \leq \rho^*$ the $\mathcal{L}_2$ gain from d to y is upper bounded by $\sqrt{\frac{\lambda_7}{\lambda_6}} + \epsilon$.

Proof: Consider equation (3b) and complete squares to get

$$< \nabla V(x), Ax + B_d d > \leq \lambda_8 V(x) + \lambda_7 |d|^2, \quad \forall x, \tag{4}$$

where $\lambda_8 = \lambda_3 + \frac{\lambda_4^2}{4\lambda_7}$.

Define now the function $W(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ as follows:

$$W(x,\tau) := \exp(-L \min\{\tau, \rho\}) V(x) \tag{5}$$

where $L > 0$ is to be determined. If $\tau \in [0,\rho]$ then from (4) and adding the term $\exp(-L\tau)\left(-\lambda_6 |y|^2 + \frac{\lambda_6 \lambda_y}{\lambda_1} V(x)\right)$ which is positive by (3a),

$$\begin{array}{l} \frac{\partial W}{\partial \tau} + \frac{\partial W}{\partial x}(Ax + Bd) \leq -L \exp(-L\tau) V(x) \\ \quad + \exp(-L\tau)\left(\lambda_8 V(x) + \lambda_7 |d|^2\right) \\ \leq W(\tau, x)\left(-L + \lambda_8 + \frac{\lambda_6 \lambda_y}{\lambda_1}\right) - \lambda_6 \exp(-L\tau)|y|^2 + \lambda_7 |d|^2 \end{array}$$

If $\tau \notin [0,\rho]$ but $x \in \mathcal{F}$ then using (3c) we get

$$\begin{array}{l} \frac{\partial W}{\partial \tau} + \frac{\partial W}{\partial x}(Ax + Bd) \leq \\ \quad \leq \exp(-L\rho)(-\lambda_5 V(x) - \lambda_6 |y|^2 + \lambda_7 |d|^2) \\ \quad \leq -\lambda_5 W(\tau, x) - \lambda_6 \exp(-L\rho)|y|^2 + \lambda_7 |d|^2 \ . \end{array} \tag{6}$$

In general, selecting $L \geq \lambda_5 + +\lambda_8 + \frac{\lambda_6 \lambda_y}{\lambda_1}$ we have

$$\begin{array}{l} \frac{\partial W}{\partial \tau} + \frac{\partial W}{\partial x}(Ax + d) \leq -\lambda_5 W(\tau, x) - \\ - \exp(-L\rho)\lambda_6 |y|^2 + \lambda_7 |d|^2, \quad \forall x \in \mathcal{F} \text{ and } \tau \leq \rho, \end{array} \tag{7}$$

which corresponds to the flow set condition in (2).

Consider the change in W due to jumps. We have

$$\begin{array}{rcl} W(0, A_r x) & = & V(A_r x) \\ & \leq & \eta V(x) \\ & = & \eta \exp(L\rho) W(\tau, x) \ . \end{array} \tag{8}$$

Therefore, selecting $\rho \leq \rho^* = \dfrac{\exp(1/\eta)}{L}$, we have

$$W(0, A_r x) \leq W(\tau, x), \quad \forall x \in \mathcal{J} \text{ or } \tau \geq \rho, \tag{9}$$

which corresponds to the jump set condition in (2).

The proof is completed integrating equations (7) and (9) along the trajectories of the system to derive an exponential bound on $|x|$ and the $\mathcal{L}_2$ bound from $\|d\|_2$ to $\|y\|_2$. ■

III. PLANAR FORE CONTROL LOOPS

In this section, we discuss how the general models (1), (2) specialize to the case of scalar reset control systems involving a first order reset element (FORE). In particular, as compared to the parallel studies carried out in [12], [13], we will slightly modify here the reset rule of the FORE so that the strict decrease condition at jumps required to apply Theorem 1 will be satisfied. The simple idea adopted here is to slightly enlarge the flow set and shrink the jump set by tilting one of the two boundaries of the corresponding sector. Then any state in the jump set will be mapped into the interior of the flow set (except for the origin) and it will be possible to construct Lyapunov functions guaranteeing the strict decrease condition (3d) for some $\eta < 1$.

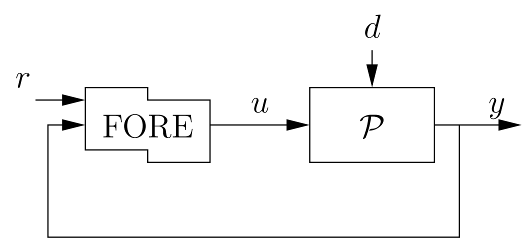

Fig. 1. A linear first-order plant controlled by a FORE.

In the most elementary setting, the closed-loop is a planar system where the plant is scalar and is not subject to resets. The underlying dynamics can be represented as

$$\begin{array}{rcl} \dot{x}_p & = & a_p x_p + b_p u + d, \\ y & = & x_p \end{array} \tag{10}$$

where u is the control input, d is a disturbance input and x_p is the plant state. For the plant (10), assume that a control system is designed, according to Figure 1, where the FORE element is described by the following dynamics:

$$\text{FORE} \left\{ \begin{array}{ll} \dot{x}_r = a_c x_r + b_c e, & \text{if } \varepsilon e^2 + 2 e x_r \geq 0 \\ x_r^+ = 0, & \text{if } \varepsilon e^2 + 2 e x_r \leq 0, \end{array} \right. \tag{11}$$

$$\text{Interc.} \left\{ \begin{array}{l} u = x_r, \\ e = -y \end{array} \right. \tag{12}$$

where $a_c \in \mathbb{R}$ denotes the time constant of the FORE. Note that a_c can be any number (including positive ones). For example, choosing $b_c = 1$ and $a_c = 0$ corresponds to implementing in the FORE the Clegg integrator.

The overall closed-loop system before temporal regularization can then be described by the dynamic equations in (1), where, based on the values in (10), (11) and (12),

$$\left[\begin{array}{c|c} A & B_d \\ \hline A_r & M_\varepsilon \end{array}\right] = \left[\begin{array}{cc|cc} a_p & b_p & 1 & \\ -b_c & a_c & 0 & \\ \hline 1 & 0 & -\varepsilon & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]. \quad (13)$$

and where the flow and jump sets are defined as follows based on the matrix M_ε:

$$\begin{array}{rcl} \mathcal{F} := \mathcal{C}_\varepsilon & := & \{x : x^T M_\varepsilon x \leq 0\} \\ \mathcal{J} := \mathcal{D}_\varepsilon & := & \{x : x^T M_\varepsilon x \geq 0\} \end{array} \quad (14)$$

In particular, as commented above, the upper left term in M_ε corresponding to $-\varepsilon$ for some small constant $\varepsilon > 0$, allows to ensure that the whenever the state jumps from the jump set, it will be mapped in the interior of the flow set so that a strict Lyapunov function decrease will be achievable.

The next theorem establishes necessary and sufficient conditions for the exponential stability and finite $\mathcal{L}_2$ gain (from d to x) of the planar FORE control system (1), (13). The proof relies on the degrees of freedom available from the novel model introduced in Section II and is based on the Lyapunov results of Theorem 1. It is omitted due to space constraints.

Theorem 2: Consider the planar FORE control system (2), (13) (i.e., the closed-loop system (10), (11), (12) with temporal regularization) and suppose that the loop gain $b_p b_c$ is positive. Then the following statements are equivalent.

1) There exists $\varepsilon^* > 0$ such that for any fixed $\varepsilon \in (0, \varepsilon^*]$ there exists $\rho^* > 0$ such that for any $\rho \in (0, \rho^*]$,
   a) there exists a locally Lipschitz Lyapunov function $V(\cdot) : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfying conditions (3) for suitable values of λ_i, $i = 1, \ldots, 7$;
   b) the system is exponentially stable when $d(t) \equiv 0$;
   c) the system is finite gain $\mathcal{L}_2$ stable from d to (x_p, x_r).
2) At least one of the following two conditions holds:
   a) the matrix A in (13) is Hurwitz;
   b) the following condition is satisfied:

$$2\sqrt{b_p b_c} + a_c - a_p > 0. \quad (15)$$

While Theorem 2 establishes useful necessary and sufficient conditions for exponential stability and $\mathcal{L}_2$ stability of planar FORE systems, another interesting aspect to study is to understand how the $\mathcal{L}_2$ gain compares to the $\mathcal{L}_2$ gain of the closed-loop without resets (whenever it exists) and also the trend of the gain as certain parameters get large. In particular, it is commonly acknowledged by practical experience that introducing resets improves the performance of a linear planar control system, even though a formal proof of this fact wasn't available. Such a proof is given next. Moreover, it has been already noticed by studying certain gain estimates in [12] that the $\mathcal{L}_2$ gain seems to become smaller and smaller as the loop gain and/or the pole of the FORE (namely a_c) becomes larger and larger. This intuition arises from the fact that the step responses generated by the closed loop look increasingly aggressive, by corresponding to the patching of an exponentially diverging branch (having larger and larger growth rate) followed by a flat-top at the desired steady state (see also the simulations in [11, Figure 3]). However, a formal proof of these $\mathcal{L}_2$ gain trends has not been established yet. It is now given in the following Theorem 3, whose proof is omitted due to space constraints. For the correct statement of the theorem we need to clarify a suitable concept of gain estimate and of gain convergence, clarified in the next definition.

Definition 1: Consider the FORE control system with temporal regularization (2), (13). Assume that γ is an input/output gain. Then we say that $\bar{\gamma}$ is an asymptotic estimate of the gain γ conditionally to hierarchically small (ε, ρ) or alternatively, that

$$\gamma \overset{\varepsilon,\rho}{\leq} \bar{\gamma},$$

if for each $\Delta_\gamma > 0$ there exists ε^* such that for each $\varepsilon \in (0, \varepsilon^*]$ there exists ρ^* such that for all $\rho \in (0, \rho^*]$, $\gamma \leq \bar{\gamma} + \Delta_\gamma$.

Assume that p is a suitable parameter of the closed-loop system and that $\gamma(p)$ is an input/output gain depending on p. Then we say that $\gamma(p)$ converges to zero conditionally to hierarchically small (ε, ρ) as p tends to $+\infty$, or alternatively, that

$$p \to \infty \;\Rightarrow \gamma(p) \overset{\varepsilon,\rho}{\longrightarrow} 0,$$

if for each $\Delta_\gamma > 0$ there exists[1] $p^* > 0$ such that for each $p \geq p^*$ there exists ε^* such that for each $\varepsilon \in (0, \varepsilon^*]$ there exists ρ^* such that for all $\rho \in (0, \rho^*]$, $\gamma(p) \leq \Delta_\gamma$. ∘

Remark 1: The goal of Definition 1 is to clarify what we mean by gain estimate and convergence to a value in terms of the small parameters of the system. In particular, the gain estimates and trends established in the next theorem require that first the parameter ε modifying the FORE resetting rule is sufficiently small and then that the temporal regularization constant ρ is once again sufficiently small. With reference to the second part of Definition 1, we note that in Theorem 3 we consider various situations when $p = a_c$ or $p = k := b_c b_p$ or $p = (a_c, k)$. In a design context, one should first fix the desired gain Δ_γ, then choose p sufficiently large and then impose first ε sufficiently small and subsequently ρ sufficiently small. ∘

Theorem 3: Consider the planar FORE control system (2), (13) (i.e., (10), (11), (12) with temporal regularization) where the loop gain $k := b_p b_c$ is positive.

1) ($\mathcal{L}_2$ *gain estimates*) Whenever the closed-loop is exponentially, stable (so that, by Theorem 2 at least one of the two conditions at item 2 of Theorem 2 holds), the following asymptotic estimates conditionally to hierarchically small (ε, ρ) (in the sense of Definition 1) hold for the $\mathcal{L}_2$ gain γ of the closed-loop from d to y:
   a) if item 2a holds, then

$$\gamma \overset{\varepsilon,\rho}{\leq} \gamma_L,$$

[1]The parameter p is allowed to be a vector and in this case $p > 0$ means that each entry of p is strictly larger than zero.

where γ_L is the (finite, because A is Hurwitz) gain from d to y of the linear closed-loop without resets.

b) if item 2b holds, then

$$\gamma \overset{\varepsilon,\rho}{\leq} \frac{2(2+\kappa)\exp\left(\kappa\frac{\pi}{2}\right)}{\kappa(2\sqrt{b_c b_p} - \max\{a_p - a_c, 0\}) - 4\max\{|a_c|, |a_p|\}}, \tag{16}$$

where κ is any constant satisfying $\kappa > \overline{\kappa} := \frac{4\max\{|a_c|,|a_p|\}}{2\sqrt{b_c b_p} - \max\{a_p - a_c, 0\}}$.

2) ($\mathcal{L}_2$ *gain trends*) Let a_p be fixed. Denote by $\gamma(a_c, k)$ the $\mathcal{L}_2$ gain of the closed-loop from d to y as a function of the FORE pole a_c and of the loop gain $k := b_p b_c$. Then the following trends hierarchically conditioned by (ε, ρ) in the sense of Definition 1 hold for the closed-loop system:

a) $k \to +\infty \ \Rightarrow \gamma(a_c, k) \xrightarrow{\varepsilon,\rho} 0$,

b) $a_c \to +\infty \ \Rightarrow \gamma(a_c, k) \xrightarrow{\varepsilon,\rho} 0$,

c) $k \to +\infty$ and $a_c \to +\infty \ \Rightarrow \gamma(a_c, k) \xrightarrow{\varepsilon,\rho} 0$,

namely, the $\mathcal{L}_2$ gain of the closed-loop decreases to zero (conditionally to hierarchical selections of (ε, ρ)) as the loop gain and/or the FORE pole are increased.

Remark 2: It is of interest to wonder whether for fixed values of the parameters there's an optimal selection of κ within (16) which gives the tightest estimate for the $\mathcal{L}_2$ gain. Indeed, by deriving equation (16) with respect to κ and imposing that the derivative is zero, one gets two solutions (of a second order equation), one of them always being smaller than $\overline{\kappa}$ (thus not being usable) and one of them always being larger than $\overline{\kappa}$. In particular, the optimal κ is determined as

$$\kappa^* := \frac{\overline{\kappa}}{2} - 1 + \sqrt{\left(\frac{\overline{\kappa}}{2}+1\right)\left(\frac{\overline{\kappa}}{2}+1+\frac{4}{\pi}\right)},$$

and, when substituted into the gain bound equation (16) it gives the following bound, which only depends on the system parameters:

$$\gamma^* = \frac{1+\kappa_0+\sqrt{\kappa_0(\kappa_0+2)}\exp\left(\kappa_1+\sqrt{\kappa_0(\kappa_0+2)}\right)}{2\sqrt{b_c b_p} - \max\{a_p - a_c, 0\}} \tag{17}$$

where $\kappa_0 = \frac{\pi}{4}(\overline{\kappa}+2)$ and $\kappa_1 = \frac{\pi}{4}(\overline{\kappa}-2)$.

An example of the gain curve given by the function (17) is shown in Figure 2, when selecting $a_p = 0$ and $b_p b_c = 1$ and having a_c take values in $[-0.5, 0.5]$. This curve is compared to the gain estimates obtained when using the numerical and analytic tools given in [12] and [13], respectively. The latter estimates turn out to be tighter for this special case, but the advantage of this construction is that it provides an estimate of the gain for a larger class of systems (the constructions in [12] and [13] are limited to the case $a_p = 0$ and $b_p b_c = 1$). ◦

IV. SISO RESET SYSTEMS WITH FOREs

In this section we derive a result on higher dimensional control systems involving FOREs. In particular, we use the results derived in Section III to conclude stability of the closed-loop consisting of SISO linear plants of arbitrary dimension interconnected to FOREs in terms of sufficient conditions that establish exponential stability of a FORE interconnected to a minimum phase plant.

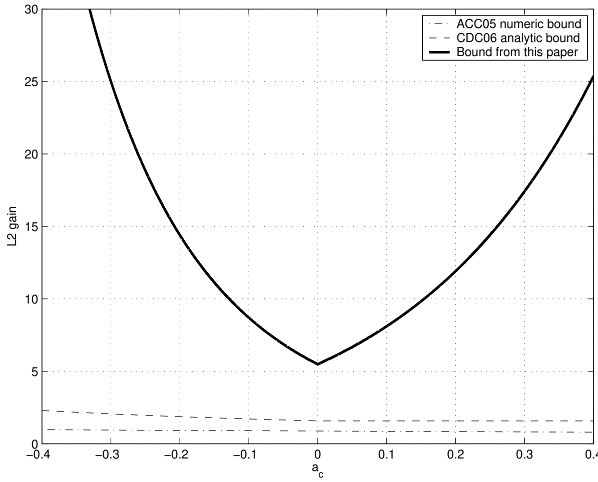

Fig. 2. Comparison of the gain estimates obtained by using equation (17) from Theorem 3 (bold), using the analytic tools from [13] (dashed) and the numerical tools from [12] (dash-dotted).

A. Models for FORE control loops

Consider a strictly proper SISO linear plant whose dynamics is described by

$$\mathcal{P}\left\{\begin{array}{rcl}\dot{x}_p &=& A_p x_p + B_{pu} u + B_{pd} d,\\ y &=& C_p x_p,\end{array}\right. \tag{18}$$

where u is the control input, d is a disturbance input and y is the measured plant output (A_p, B_{pu}, B_{pd} and C_p are matrices of appropriate dimensions). The plant (18) is connected with a FORE described by (11) and the interconnection is described by (12). The overall closed-loop system that is augmented with the temporal regularization can be described by equations (2) where, different from the planar case, now the matrices in (2) correspond to the following selections:

$$\begin{array}{l} A = \begin{bmatrix} A_p & B_{pu} \\ -b_c C_p & a_c \end{bmatrix}, B_d = \begin{bmatrix} B_{pd} \\ 0 \end{bmatrix}, \\ A_r = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \ M = \begin{bmatrix} -\varepsilon & C_p^T \\ C_p & 0 \end{bmatrix}. \end{array} \tag{19}$$

B. Stability results on minimum phase SISO plants with FORE

In this section we use results of Theorems 2 and 3 to establish sufficient conditions for stability of a class of reset systems described by (2), (19). In other words, the closed loop consists of a SISO linear plant and a FORE. Moreover, we assume in this section that the SISO plant is minimum phase and relative degree one. The underlying idea in the $\mathcal{L}_2$ stability proof is to use a small gain theorem. Then using the results in [9], we show that we have exponential stability in the absence of disturbances.

To suitably represent the plant under consideration, first note that since the plant is minimum phase and relative degree one, there exists a nonsingular change of coordinates so that we can write its dynamics as follows [8, Remark 4.3.1]:

$$\dot{z} = A_z z + B_{zy} y + B_{zd} d \quad (20a)$$
$$\dot{y} = a_p y + b_p u + \underbrace{C_z z + E_d d}_{\tilde{d}}, \quad (20b)$$

where $y \in \mathbb{R}, z \in \mathbb{R}^{n-1}$ and $u \in \mathbb{R}$ are respectively the plant output, part of state corresponding to zero dynamics and input, A_z is Hurwitz (since the plant is minimum phase) and we assume without loss of generality that $b_p > 0$.

To state the next result we introduce the following definition:

Definition 2: Consider the FORE control system with temporal regularization (2), (13). Assume that p is a suitable parameter of the closed-loop system. Then we say that the system is exponentially stable (or finite gain $\mathcal{L}_2$ stable) conditionally to large p and hierarchically small (ε, ρ) if there exists $p^* > 0$ such that for each $p \geq p^*$ there exists ε^* such that for each $\varepsilon \in (0, \varepsilon^*]$ there exists ρ^* such that for all $\rho \in (0, \rho^*]$ we have that the system (2), (13) is exponentially stable (finite gain $\mathcal{L}_2$ stable). ○

It is understood in the above definition that the only parameters that we can change are p, ε, ρ, whereas all other constants in the model are fixed. Then we can state the following result:

Theorem 4: Consider the closed loop system consisting of the plant (20) and the FORE (11), (12), where $r \equiv 0$. Let A_z be Hurwitz and $b_p > 0$ in (20). Then, the following statements are true:

1) The system is finite gain $\mathcal{L}_2$ stable from d to x_p conditionally to large a_c and hierarchically small (ε, ρ). Moreover, when $d(t) \equiv 0$ the system is also exponentially stable conditionally to large a_c and hierarchically small (ε, ρ);
2) The system is finite gain $\mathcal{L}_2$ stable from d to x_p conditionally to large b_c and hierarchically small (ε, ρ). Moreover, when $d(t) \equiv 0$ the system is also exponentially stable conditionally to large b_c and hierarchically small (ε, ρ);
3) The system is finite gain $\mathcal{L}_2$ stable from d to x_p conditionally to large (a_c, b_c) and hierarchically small (ε, ρ). Moreover, when $d(t) \equiv 0$ the system is also exponentially stable conditionally to large (a_c, b_c) and hierarchically small (ε, ρ).

Proof: We only prove the first case and the proof of other two cases follow almost identical steps by using conditions of Theorem 3. We consider the overall system as a feedback interconnection of the linear system (20a) that has inputs (y, d) and the output z and the second order reset system consisting of (20b), (11), (12) that has inputs (z, d) and the output y. From the item 1 of Theorem 3 we have that the gain of the reset system from $\tilde{d} := C_z z + E_d d$ to y can be reduced arbitrarily by adjusting a_c, ε and ρ, that is:

$$a_c \to +\infty \;\Rightarrow \gamma(a_c) \xrightarrow{\varepsilon,\rho} 0$$

Hence, the gain from (z, d) to y can be reduced arbitrarily for the reset system. Then, since A_z in the linear system (20a) is Hurwitz, then the system is finite gain $\mathcal{L}_2$ stable from (d, y) to z with some gain γ_z. Hence, there exist sufficiently large a_c and sufficiently small ε and ρ such that the small gain condition

$$\gamma(a_c) \cdot \gamma_z < 1$$

holds, which implies that the closed loop system (20a), (20b), (11), (12) is finite gain $\mathcal{L}_2$ stable from d to (z, y), which completes the proof of $\mathcal{L}_2$ stability.

To prove exponential stability note first from the item 2 of Theorem 2 we have that we can adjust a_c, ε, ρ so that we also have a finite gain for the reset system from $\tilde{d}$ to (y, x_c) from which we can conclude $\mathcal{L}_2$ stability from d to (z, y, x_c) for the closed loop system. Now we can use[2] [9, Theorem 3] to conclude exponential stability when $d(t) \equiv 0$. Indeed, we have that all conditions of [9, Proposition 1] hold in our case and, hence, the closed loop system is uniformly globally fixed time interval stable (UGFTIS) with linear gain (see [9, Definition 6]). This implies that all conditions of [9, Theorem 3] hold and, hence, we can conclude that the system is UGES. ■

[2]Results in [9] were stated for a special class of hybrid systems that does not include reset systems. However, all results in [9] that we use can be restated for general hybrid models that include the class of reset systems considered in this paper.

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