$\mathcal{L}_2$ anti-windup for linear dead-time systems

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Abstract

In this paper we address and solve the problem of anti-windup augmentation for linear systems with input and output delay. In particular, we give a formal definition of the anti-windup construction problem in the global, local, robust and nominal cases. For each of these cases we show that a specific anti-windup compensation structure (which is a generalization of the approach in [13]) is capable of solving the anti-windup problem whenever this is solvable. The effectiveness of the proposed scheme is shown on a simple example taken from the literature, in which the plant is a marginally stable linear system.

1 Introduction

The anti-windup design problem has been qualitatively stated already from the 1950's both in the analog and in the digital control framework. However, the arising solutions were at that time mainly application oriented and not applicable to large classes of control systems. It was only in the 1980's that some design techniques applicable to large classes of control systems were formalized, although the issue of performance characterization and improvement was still mostly unsolved. An interesting survey of these techniques can be found in [4]. Another phenomenon that is often found in conjunction with saturation is the presence of time delays at the input and at the output of the plant. Similar to the saturation effect, the dead-time phenomenon becomes crucial when the control task is aggressive enough so that the phase roll-off may destroy the stability (and/or performance) properties of the closed-loop system. For these saturated and retarded systems, it is of interest to address the corresponding generalization of the anti-windup problem. This can be intuitively seen as follows: assume that a *predesigned* controller (possibly including internal delays) is available for the dead-time plant without input saturation; then build an anti-windup compensator that, when interconnected to the existing control system is capable of

1. reproducing the responses induced by that predesigned controller when signals are small enough not to activate actuator saturation;
2. recovering the stability (and, partially, the performance) that would be otherwise lost due to the nonlinear effects of saturation, for all other signals.

The solution to this anti-windup problem is appealing because it makes it possible to implement control laws for dead-time systems *without* saturation also on saturated dead-time plants. Many such tools are available in the literature. See, *e.g.*, the many generalizations of the Smith predictor structure, first proposed in [10]. A very natural solution to the anti-windup design problem for dead-time systems can be obtained by suitably generalizing (in a straightforward way) the classical Internal Model Control (IMC) technique for anti-windup [16], where the stability problem is fully solved for the case where the open-loop plant is characterized by a Hurwitz matrix. However, it is known that IMC-based anti-windup designs may lead to poor performance, which leaves space for significant improvement over this straightforward construction. Despite the work of [8] (where a solution is given only for a subclass of control systems) and the IMC extension mentioned above, very little has been done so far on anti-windup design for dead-time linear control systems. Nevertheless, these ideas have been proven to be successful on several applications, including active queue management in TCP networks [9] and other experiments discussed in [2]. Also notable is the work in [11] where a static anti-windup gain is included within a saturated control design task for a dead-time plant. In [11], the anti-windup goal is not directly addressed because the controller is not considered as a design constraint but rather as a degree of freedom within the control system design. Nevertheless, that underlying technique could be easily generalized for static anti-windup design whenever the unconstrained controller is a linear system without delays.

In this paper, we address the anti-windup design problem for dead-time control systems. The proposed solution is applicable to any linear control system, including the case where the unconstrained controller contains internal time-delays (such as in the case where it arises from a Smith predictor design). Moreover, a global solution is given to the problem under the (necessary) prop-

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erty that the plant state matrix has eigenvalues with non positive real part, thus extending previous results to the case of poles on the imaginary axis. As compared to the solution in [8], the approach proposed here is stronger because it holds under weaker conditions (these conditions are actually shown to be necessary for the solvability of the problem). Moreover, whenever the approach in [8] is applicable, it can be interpreted as a special selection among a family of solutions parametrized within the framework proposed here. As compared to the potential static anti-windup solution residing in the approach of [11], our result provides a general solution to the problem, whereas the results in [11] are only applicable if certain matrix inequalities are feasible.

The paper is organized as follows. We first formalize the anti-windup design goal (which is based on a generalization of the delay-free ideas in [13]) introducing the global, local, robust and nominal problem statements in Section 2. Then in Section 3 we prove necessary and sufficient conditions for the solvability of the problem and provide a general framework for the corresponding solution (whenever it exists). In this framework, specific selections of a stabilizing signal are shown to solve the different instances of the anti-windup problem. In Section 4 we apply our design to an example where the plant state matrix is non Hurwitz (so that previous techniques are not applicable) and show the desirable performance induced by the proposed anti-windup strategy.

1.1 Notation

Given a vector $w \in \mathbb{R}^n$ and a set $\mathcal{S} \subset \mathbb{R}^n$, the distance of the vector w from the set $\mathcal{S}$ is defined as $\mathrm{dist}(w, \mathcal{S}) := \inf_{s\in\mathcal{S}} |w - s|$.

Given numbers $\underline{a} \le a \le b \le \bar{b}$ and a function $w : [\underline{a}, \bar{b}] \to \mathbb{R}^n$, the $\mathcal{L}_2$ norm of $w(\cdot)$ restricted to the interval $[a, b]$ is defined as $\|w_{[a,b]}\|_2 := \sqrt{\int_a^b |w(\tau)|^2 d\tau}$. If $[\underline{a}, \bar{b}] = [0, \infty)$, to simplify notation we will often use $\|w\|_2$ in place of $\|w_{[0,\infty)}\|_2$. We will denote $\|w_{[0,\infty)}\|_2$ as the $\mathcal{L}_2$ norm of $w(\cdot)$ and, if $\|w_{[0,\infty)}\|_2 < \infty$, we will say that $w(\cdot) \in \mathcal{L}_2$.

Given a constant $t_d > 0$ and a function $s : [-t_d, \infty) \to \mathbb{R}^n$, then for all $t \ge 0$, the functional $s_d(\cdot)$ is defined as $s_d(t) := \{s(\tau), \tau \in [t - t_d, t]\}$. Moreover, for each $t \ge 0$, the norm $|s_d(t)|$ is defined as $|s_d(t)| := \max_{\tau\in[t-t_d,t]} |s(\tau)|$.

Let $K > 0$ and $\gamma > 0$ be given. A nonlinear functional differential equation of the form

$$\begin{array}{rcl} \dot{x} &=& f(x_d(t), w_d(t)) \\ y &=& g(x_d(t), w_d(t)) \end{array}$$

is finite gain $\mathcal{L}_2$ stable from w to y if for all functions $w(\cdot)$ and initial conditions $x_d(0)$, the following bound holds for all $t \ge 0$

$$\|y_{[0,t]}\|_2 \le K|x_d(0)| + \gamma\|w_{[-t_d,t]}\|_2.$$

2 Problem statement

Consider a linear time-invariant plant subject to input and output delays:

$$\begin{array}{l} \dot{x}(t) = Ax(t) + B\mathrm{sat}(u(t-\tau_I)) + B_e d(t) + \psi_x(t) \\ y(t) = Cx(t-\tau_O) + D\mathrm{sat}(u(t-\tau_I-\tau_O)) \\ \qquad\qquad\qquad\qquad + D_e d(t-\tau_O) + \psi_y(t) \\ z(t) = C_z x(t) + D_z \mathrm{sat}(u(t-\tau_I)) + \psi_z(t) \end{array} \tag{1}$$

where $\tau_I > 0$ is a uniform delay at the plant control input $u \in \mathbb{R}^m$, $\tau_O > 0$ is a uniform delay at the plant output measurement $y \in \mathbb{R}^p$, z represents the performance output (without loss of generality we can assume that this output is not delayed) and d represents a disturbance input. The three extra signals ψ_x, ψ_y, ψ_z can be stacked in a single vector Ψ representing the output of the following linear system (represented in the Laplace domain)

$$\Psi(s) := \begin{bmatrix} \psi_x(s) \\ \psi_y(s) \\ \psi_z(s) \end{bmatrix} := \Delta(s) \begin{bmatrix} x(s) \\ u(s) \\ d(s) \end{bmatrix} \tag{2}$$

which may be infinite dimensional (it may have internal delays) and represents unmodeled dynamics and/or parameter uncertainties in the model (1). We will need the following assumption for the perturbed plant (1), (2).

Assumption 1 *The pair (C_z, A) is detectable.* [1] *The system (2) is finite-gain $\mathcal{L}_2$ stable from $(x(\cdot), u(\cdot), d(\cdot))$ to $\Psi(\cdot)$ with $\mathcal{L}_2$ gain equal to γ_Δ.*

Assume that a linear controller (defined, in general, by linear functional differential equations) has been designed for the following linear dead-time system *without* input saturation

$$\begin{array}{l} \dot{x}(t) = Ax(t) + Bu(t-\tau_I) + B_e d(t) + \psi_x(t) \\ y(t) = Cx(t-\tau_O) + Du(t-\tau_I-\tau_O) \\ \qquad\qquad\qquad\qquad + D_e d(t-\tau_O) + \psi_y(t) \\ z(t) = C_z x(t) + D_z u(t-\tau_I) + D_{dz} d(t) + \psi_z(t) \end{array} \tag{3}$$

and that the controller equations can be written as

$$\begin{array}{rcl} \dot{x}_c(t) &=& f(x_{c,d}(t), u_{c,d}(t), r_d(t)) \\ y_c(t) &=& g(x_{c,d}(t), u_{c,d}(t), r_d(t)), \end{array} \tag{4}$$

where $f(\cdot,\cdot,\cdot)$ and $g(\cdot,\cdot,\cdot)$ are linear functionals. [2] In particular, the controller (4) is assumed to enforce a desirable closed-loop behavior on the unconstrained plant (3) when interconnected through the following unconstrained interconnection equations

$$u(t) = y_c(t), \quad u_c(t) = y(t). \tag{5}$$

[1] This assumption is only needed to prove the necessity of the results of Theorem 1. The sufficiency statements still hold when (C_z, A) is not detectable.

[2] As defined in the notation Section 1.1, the subscripts d denote the dependence of the functional on the past history of the signal under consideration.

The corresponding closed-loop system (3), (4), (5) will be referred to as the *unconstrained closed-loop system* henceforth. Moreover, its response will be called *unconstrained response*. The following assumption will hold for the unconstrained closed-loop.

Assumption 2 *There exists a small enough gain $\gamma_\Delta > 0$ such that the unconstrained closed-loop system (3), (2), (4), (5) is well-posed (namely, solutions exist unique for all initial states and for all inputs) and finite-gain $\mathcal{L}_2$ stable from the input $\Psi = (\psi_x, \psi_y, \psi_z)$ to the closed-loop state and output, uniformly over all selections of the system (2) satisfying Assumption 1.*

We will address in this paper the anti-windup problem arising when saturation is present at the plant input, so that the unconstrained performance of the closed-loop (3), (2), (4), (5) is only feasible for small enough signals. For simplicity, we will consider decentralized saturation functions, although the results can be extended in a straightforward way to the more general class of nonlinearities characterized in [13, Assumption 2]. Similar to the approach in [13], to properly formalize the anti-windup problem, we need to introduce a subset $\mathcal{U}$ of the plant input vector space $\mathbb{R}^m$, which is a strict subset of the linear region of the saturation function, namely such that [3]

$$\exists \delta > 0, \text{ s.t. } u + \delta \frac{v}{|v|} \in \{w \in \mathbb{R}^m : \; w = \text{sat}(w)\}, \qquad (6)$$

for all $u \in \mathcal{U}, v \in \mathbb{R}^m$. Based on this set $\mathcal{U}$ and following the anti-windup approach for undelayed linear plants, the main anti-windup goal will address the design of an *anti-windup compensator* with the goal of recovering as much as possible the "response without saturation" (herein called unconstrained response) on the saturated (and compensated) closed-loop system (this will be called *anti-windup closed-loop system* henceforth). In the following, for any selection of the external signals $r(\cdot)$, $d(\cdot)$, given initial conditions for the plant (3) and for the controller (4), we will denote the unconstrained closed-loop response using overlines. Moreover, given the same initial conditions for the saturated plant (1) and the controller (4), we will denote the corresponding anti-windup closed-loop response without overlines.

Definition 1 Given a plant (1) and a controller (4) satisfying Assumption 2, and a set $\mathcal{U} \in \mathbb{R}^m$ satisfying (6), an anti-windup compensator solves the corresponding

• *robust global anti-windup problem* if there exists a continuous positive nondecreasing function $\overline{\gamma} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and a (sufficiently small) $\mathcal{L}_2$ gain γ_Δ for the unmodeled dynamics (2) such that for all initial conditions and all external inputs, [4]

$$\|z - \overline{z}\|_2 \leq \overline{\gamma}(\|\text{sat}(\overline{u}) - \overline{u}\|_2); \qquad (7)$$

• *nominal global anti-windup problem* if the bound (7) holds with $\Psi \equiv 0$ (namely, in the absence of unmodeled dynamics);

• *robust local anti-windup problem* if the bound (7) holds for small enough values of the initial conditions of the anti-windup compensator and of $\|\text{sat}(\overline{u}) - \overline{u}\|_2$. ∘

Remark 1 Note that (7) enforces a bound on the output mismatch $(z - \overline{z})(\cdot)$ based on the energy spent by the (ideal) unconstrained input response outside the saturation limits. This characterization is reasonable because that input energy cannot be instantly recovered on the saturated control system, regardless of what the anti-windup compensation is. Therefore, (7) successfully captures the intuitive performance recovery goals of the anti-windup design. Note also that (7) implicitly enforces the property that all the unconstrained trajectories that never exceed the saturation limits (so that $\|\text{sat}(\overline{u}) - \overline{u}\|_2 = 0$), will be exactly reproduced by the anti-windup closed-loop system. Indeed, in that case, (7) implies that $\|z - \overline{z}\|_2 = 0$, namely $z(\cdot) \equiv \overline{z}(\cdot)$. ∘

3 Main result

In this section we will address the anti-windup problem of Definition 1 and give necessary and sufficient conditions for its solvability, together with a constructive solution whenever the problem is solvable. For the solution of this problem, we will select the anti-windup compensator as the following dynamical system:

$$\begin{aligned} \dot{x}_{aw}(t) =& \; Ax_{aw}(t) + B\Big(\text{sat}(u(t)) - y_c(t)\Big) \\ v_1(t) =& \; f_{aw}(x_{aw}(t), \text{sat}(u(t)) - y_c(t)) \\ v_2(t) =& \; Cx_{aw}(t - \tau_I - \tau_O) \\ & + D\Big(\text{sat}(u(t - \tau_I - \tau_O)) - y_c(t - \tau_I - \tau_O)\Big), \end{aligned} \qquad (8)$$

where the selection of the function $f_{aw}(\cdot, \cdot)$ will be specified later. This filter will modify the interconnection between the plant (1) and the controller (4) through the following equations

$$u(t) = y_c(t) + v_1(t), \quad u_c(t) = y(t) - v_2(t). \qquad (9)$$

The corresponding closed-loop system is represented in Figure 1.

The effectiveness of the structure (8), (9) of the anti-windup compensator in solving the anti-windup problem of Definition 1 is based on the fact that (at least in the case where $\Psi \equiv 0$) the arising closed-loop system

[3] A typical example of this set is $\mathcal{U} := [u_{m1} + \delta, u_{M1} - \delta] \times \cdots \times [u_{Mm} + \delta, u_{Mm} - \delta]$, where u_{mi}, u_{Mi}, $i = 1, \ldots, m$ denote the saturation limits.

[4] For simplicity of notation, in equation (7) and throughout the proof of the paper, the $\mathcal{L}_2$ bounds are all given omitting the initial conditions.

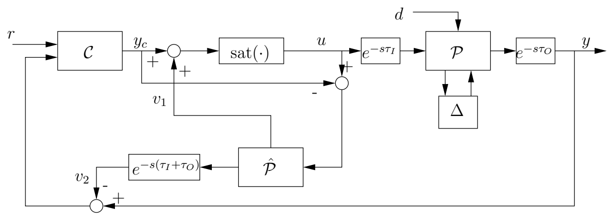

Figure 1: The proposed anti-windup scheme.

can be transformed into the cascade interconnection between the functional differential equation corresponding to the unconstrained closed-loop (3), (4), (5) and an extra subsystem consisting of the filter (8) which needs to be stabilized by suitably designing the compensation signal v_1 in (9). This scheme arises from a generalization of the scheme adopted in [13] for anti-windup design for linear undelayed systems. Indeed, the same tools introduced in [13] for the design of the signal v_1 can be also used in the framework (8), (9). To this aim, we report in the following the result in [13, Lemma 1], which will be useful next. As reported in [13], this lemma can be proven by combining the results in [6] and [12].

Lemma 1 *For the control system*

$$\dot{x}_{aw} = Ax_{aw} + B\left(\mathrm{sat}(v + \varphi(t)) - \varphi(t)\right),$$

where (A, B) is stabilizable, and $\mathcal{U}$ satisfies (6),

1. *there always exists a globally Lipschitz feedback $v = k(x_{aw})$ such that if $\|\mathrm{dist}(\varphi,\mathcal{U})\|_2$ and $|x_{aw}(0)|$ are sufficiently small, then $x_{aw}(\cdot) \in \mathcal{L}_2$ and the $\mathcal{L}_2$ gain from $\mathrm{dist}(\varphi(\cdot),\mathcal{U})$ to $x_{aw}(\cdot)$ is finite;*

2. *if all the eigenvalues of A have non-positive real part, then there exists a globally Lipschitz feedback $v = k(x_{aw})$ such that if $\mathrm{dist}(\varphi(\cdot),\mathcal{U}) \in \mathcal{L}_2$, then $x_{aw}(\cdot) \in \mathcal{L}_2$; moreover, if A is critically stable, i.e., if there exists $P = P^T > 0$ such that $A^TP + PA \leq 0$, then the function $k(\cdot)$ can be taken to be the linear function $x_{aw} \mapsto k(x_{aw}) := -B^TPx_{aw}$. Finally, when A is Hurwitz, the $\mathcal{L}_2$ gain from $\mathrm{dist}(\varphi(\cdot),\mathcal{U})$ to $x_{aw}(\cdot)$ is finite.*

We are now ready to state our main result whose proof is only sketched due to space constraints.

Theorem 1 *Suppose Assumptions 1 and 2 hold for the plant (1), (2) and for the controller (4). Then, according to Definition 1, the following holds*

1. *the local robust anti-windup problem is always solvable;*

2. *the global nominal anti-windup problem is solvable if and only if A has poles in the closed left half plane;*

3. *the global robust anti-windup problem is solvable if and only if A is Hurwitz;*

4. *whenever any of the anti-windup problems is solvable, the anti-windup filter (8) with the interconnection equations (9) and with the selection $f_{aw}(x_{aw}, \mathrm{sat}(u) - y_c) = k(x_{aw})$ according to the constructive result of Lemma 1, is well-posed and is a solution to the problem.*

Proof: *Necessity.* The necessity in items 2 and 3 can be shown following similar steps to [13, Theorem 1].

Sufficiency. The proof of the sufficiency is constructive and is based on the structure (8), (9) with $f_{aw}(\cdot,\cdot)$ selected as $f_{aw}(x_{aw}, \mathrm{sat}(u) - y_c) = k(x_{aw})$ according to Lemma 1. Due to space constraints, we will only address here the case where the unmodeled dynamics (2) are absent (namely, $\Psi \equiv 0$). The proof extends to the robust case by small gain arguments similar to those carried out in [13].

Consider the anti-windup closed-loop (1), (4), (8), (9), write the closed-loop dynamics in the coordinates $(e(t), x_c(t), x_{aw}(t)) = (x(t) - x_{aw}(t - \tau_I), x_c(t), x_{aw}(t))$ as follows (here, $y_e(t) = y(t) - v_2(t)$; moreover, z_e and z_{aw} are new outputs of the closed-loop system):

$$\left\{\begin{array}{rcl}\dot{e}(t) &=& Ae(t) + By_c(t-\tau_I) + B_ed(t)\\ y_e(t) &=& Ce(t-\tau_O) + Dy_c(t-\tau_I-\tau_O)\\ && \qquad +D_ed(t-\tau_O)\\ z_e(t) &=& C_ze(t) + D_zy_c(t-\tau_I) + D_{dz}d(t)\\ \dot{x}_c(t) &=& f(x_{c,d}(t), y_{e,d}(t), r_d(t))\\ y_c(t) &=& g(x_{c,d}(t), y_{e,d}(t), r_d(t))\end{array}\right. \tag{10a}$$

$$\left\{\begin{array}{rcl}\dot{x}_{aw}(t) &=& Ax_{aw}(t) + B\Big(\mathrm{sat}(u(t)) - y_c(t)\Big)\\ v_1(t) &=& Kx_{aw}(t) + L\Big(\mathrm{sat}(u(t)) - y_c(t)\Big)\\ z_{aw}(t) &=& C_zx_{aw}(t) + D_z\Big(\mathrm{sat}(u(t)) - y_c(t)\Big)\\ u(t) &=& y_c(t) + v_1(t).\end{array}\right. \tag{10b}$$

It is evident that the closed-loop (10) is a cascade interconnection of two subsystems, where (10a) coincides exactly with the unconstrained closed-loop system (3), (4), (5). Therefore, if the initial condition of (8) is $x_{aw,d}(0) = 0$, then [5]

$$y_c(t) = \overline{u}(t), \forall t \geq 0. \tag{11}$$

Moreover, the additional output z_e of (10a) satisfies $z_e(t) = \overline{z}(t)$ for all times. Consider now the additional output z_{aw} of the second subsystem (10b) and notice that, since by definition $e(t) = x(t) - x_{aw}(t - \tau_I)$, then $z_e(t) = z(t) - z_{aw}(t - \tau_I)$ for all times. Therefore, $z(t) - \overline{z}(t) = z_{aw}(t - \tau_I)$, $\forall t \geq \tau_I$, and consequently, [6]

$$\|(z - \overline{z})\|_2 = \|z_{aw}\|_2. \tag{12}$$

[5]In the case when $x_{aw,d}(0) \neq 0$ there will be an additional term depending on the initial condition in the $\mathcal{L}_2$ bound (7). To keep the discussion simple, we are omitting the initial conditions in the $\mathcal{L}_2$ bounds of this paper (the corresponding relations are a straightforward generalization of the initial condition-free ones).

[6]Note that due to the presence of the input delay, since the plant initial conditions are assumed to be the same in the unconstrained and in the anti-windup case, then $z(t) - \overline{z}(t) = 0$ for all $t \in [0, \tau_I)$.

Since $u(t) = y_c(t) + k(x_{aw}(t))$, by the global Lipschitz property of the saturation function, we have for all $t \geq 0$, $|\mathrm{sat}(u(t)) - y_c(t)| \leq |k(x_{aw}(t))| + |\mathrm{sat}(y_c(t)) - y_c(t)|$. Therefore, since by Lemma 1 $k(\cdot)$ is globally Lipschitz, substituting the previous bound in the third equation of (10b), it follows that there exists $\gamma > 0$ such that for all $t \geq 0$

$$|\mathrm{sat}(u(t)) - y_c(t)| \leq \gamma(|x_{aw}(t)| + |\mathrm{sat}(y_c(t)) - y_c(t)|). \quad (13)$$

Finally, the proof is completed by applying Lemma 1 with $\varphi(\cdot) \equiv y_c(\cdot)$ and combining the resulting $\mathcal{L}_2$ bound with equations (11), (12) and (13). ∎

Remark 2 The selection for v_1 proposed in Lemma 1 is sufficient to solve the anti-windup problems of Definition 1. However, from a performance perspective, alternative selections may be more desirable because they improve the unconstrained response recovery transient. To this aim, a useful property arising from the structure (8), (9) is that as shown in the proof of Theorem 1, regardless of the selection of v_1, the mismatch between the unconstrained and the actual performance response is given by the extra output z_{aw} in (10b). Therefore, v_1 can be selected by only focusing on the stabilization of the (undelayed!) subsystem (10b) and on the performance seen at this particular output z_{aw}.

In the past years, several techniques have been proposed in the context of the undelayed anti-windup problem to improve the corresponding transient responses. Most of these results correspond to linear matrix inequality (LMI) formulations of convex optimization problems. Fortunately, the same approaches can be applied also to the dead-time problem addressed here because of the cascade structure (10) induced by the filter (8). Among these techniques, nonlinear scheduled ones were proposed in [14] and sampled-data ones were proposed in [1]. Moreover, in [15] the selection of the function $f_{aw}(\cdot,\cdot)$ is optimized among the linear functions:

$$f_{aw}(x_{aw}, \mathrm{sat}(u) - y_c) = Kx_{aw} + L(\mathrm{sat}(u) - y_c), \quad (14)$$

where the gains K and L arise from suitable LMIs easily solvable by convex optimization. The difference between (14) and the selection proposed in Lemma 1 (at least for the Hurwitz case) is in the presence of the feedthrough term L, which evidently enforces an algebraic loop around the saturation. It is commonly acknowledged (see, *e.g.*, [7]) that this algebraic loop may significantly improve the transient performance of the control system, especially in the MIMO case. Although the LMI-based selection of K and L proposed in [15] guarantees that the interconnection is well-posed, the corresponding optimal solution might often lead to values of L that are very close to a non well-posed interconnection. In those cases it is useful to augment the LMIs of [15] with an extra matrix inequality of the form

$$\begin{bmatrix} 2(\rho-1)W - \rho X_2 - \rho X_2^T & \rho X_2^T \\ \rho X_2 & \dfrac{\eta-\rho}{2}W \end{bmatrix} > 0, \quad (15)$$

where $\rho \in (0,1)$ and $\eta > \rho$. Based on the results of [3], this bound will ensure that the explicit solution to the implicit equation imposed by the algebraic loop in (8) is Lipschitz of level $\eta\sqrt{\frac{\sigma_M(W)}{\sigma_m(W)}}$ (where $\sigma_M(\cdot)$, $\sigma_m(\cdot)$ denote the maximum and minimum singular value of the matrix at argument, respectively), thereby not allowing the arising anti-windup solution to be ill-posed. ∘

Remark 3 The approach given here can be easily seen as a generalization of the construction in [8]. This generalization allows to remove the technical Assumption (A3) in [8] (thus solving the anti-windup problem also when this Assumption (A3) doesn't hold), it allows to establish GAS of the arising closed-loop whenever the plant is non exponentially unstable and it allows, in general, to guarantee improved performance by way of the degrees of freedom in the selection of v_1.

Indeed, by suitable loop transformations and some tedious calculations, it can be shown that the anti-windup solution proposed in [8] for dead-time plants is equivalent to using the filter (8) with the selection

$$v_1(s) = -D_c(s)v_2(s), \quad (16)$$

where the matrix $D_c(s)$ corresponds to the input-output link of the unconstrained controller. In particular, in [8, §3], the unconstrained controller is a delay-free LTI system and $D_c(s) = L$ is its (constant) input-output matrix; in [8, §4], the unconstrained controller has a specific structure with an internal time delay τ_3 and $D_c(s) = L_3L_1(I - e^{-s\tau_3}L_2L_1)^{-1}$ is the corresponding term [7] related to the generalized input-output link (see [8] for details).

This re-interpretation of the scheme of [8] actually clarifies the technical Assumption (A3) therein reported, which corresponds to requiring that the matrix $S = A - e^{-s(\tau_I+\tau_O)}B(I + e^{-s(\tau_I+\tau_O)}D_c(s)D)^{-1}D_c(s)C$ has stable eigenvalues. Indeed, since the unconstrained closed-loop is stable (this is assumed in our Assumption 2 and in Assumption (A2) of [8]), the stability of the equivalent cascaded structure (10) reduces to the stability of the second subsystem (10b). When constraining v_1 to be selected as in (16), a necessary (but not sufficient because of the presence of saturation) condition for (10b) to be stable is that the system is stable for small signals, where the saturation is not active. The corresponding equations with (16), written in the Laplace domain, are given by

[7] Actually, in [8, §4], $D_c(s)$ is defined as $L_u := L_3[I + e^{-s\tau_3}L_1(I - e^{-s\tau_3}L_2L_1)^{-1}L_2]L_1$. However, it can be shown that this last expression coincides with the more intuitive one reported above.

$$sx_{aw}(s) - x_{aw}(0) = Ax_{aw}(s) + e^{-s\tau_I} B v_1(s)$$
$$v_1(s) = -e^{-s\tau_O} D_c(s) C x_{aw}(s) - e^{-s(\tau_I+\tau_O)} D_c(s) D v_1(s),$$

whose state matrix corresponds to the matrix S defined above. The anti-windup solution in [8] was motivated by the goal of minimizing the mismatch between the controller states in the unconstrained and anti-windup responses. When reinterpreted within the cascaded structure (10), it becomes clear that this solution is a specific selection among a family of solutions parametrized by the signal v_1. Indeed, as shown in the proof of the theorem, regardless of the selection of v_1 (and by way of the compensation signal v_2), the controller states behave exactly as in the unconstrained response and the above mentioned mismatch is always zero. One other relevant selection among these corresponds to the IMC-based anti-windup (see, *e.g.*, [16]) which can be easily extended to the case of dead-time linear plants and corresponds to selecting $v_1 \equiv 0$. Note, however, that both the IMC selection and the selection (16) of [8] are only applicable when the matrix A in (1) is Hurwitz. The approach proposed here, instead, always solves the anti-windup problem as long as A has poles in the closed left-hand plane.

Moreover, when A is Hurwitz, different from the selection (16), which might lead to an unstable closed-loop in some cases, IMC-based anti-windup is always asymptotically stabilizing because the dynamics in (10b) correspond to the asymptotically stable plant dynamics. On the other hand, it is well known that IMC anti-windup solutions often lead to poor closed-loop performance, especially when the plant contains slow modes. Such a performance may be improved by the approach in [8], where an extra (sometimes stabilizing) action is enforced through the nonzero selection (16) for v_1.

An advantage of our approach is that the anti-windup problem is reduced to a stabilization problem where the performance output z_{aw} needs to be minimized in some sense. [8] The arising solution, in addition to being stabilizing (thus recovering the stability merits of the IMC approach), also benefits from the degrees of freedom residing in the selection of compensation signal v_1. ◦

4 Simulation example

In this section we apply the proposed anti-windup technique to a simple scalar simulation example taken from [5, Example 4]. This example is a control system where the plant is represented by an integrator with a 5 seconds output delay and the controller has a modified Smith predictor structure. Notice that since the state matrix of the plant is not Hurwitz, then the anti-windup approach of [8] cannot be applied here.

For this particular example, since the plant is an integrator, then $A = 0$ and following the construction at item 2 of Lemma 1, v_1 can be selected as $v_1 = -\rho x_{aw}$, where ρ is an arbitrary positive number. In particular, the larger ρ is, the stronger the anti-windup action will be for the recovery of the unconstrained response. In these simulations we have chosen $\rho = 10$, but different values for ρ could also be selected to suitably impose the speed of convergence of the anti-windup response to the unconstrained one. In [5], it is shown that in the absence of saturation (input saturation is not addressed in [5]), the unconstrained controller induces an improved response as compared to previous results. In particular, two simulations are reported therein, referred to the nominal case, and to the robust case, where a 10% error is introduced in the delay present at the plant output. In both cases, the reference input r and the disturbance input d are selected as follows:

$$r(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0, \end{cases} \qquad d(t) = \begin{cases} 0, & t < 15 \\ -0.1, & t \geq 15, \end{cases} \tag{17}$$

We consider similar reference and disturbance inputs here and we insert a saturation with limits ± 0.5 at the plant input. The unconstrained, saturated and anti-windup responses in the nominal case are reported in Figure 2 using bold solid, dashed and thin solid curves, respectively. As in [5], the parameters of the plant and unconstrained controller for this simulation are selected as $\theta = 5$, $\theta_m = 5$, $k_p = 0.1$, $T_i = 0.01$, $k_f = 4.131$, $k_d = 0.105$. In Figure 2, the upper plot compares the plant outputs in the three cases, while the lower plot represents the plant input responses.

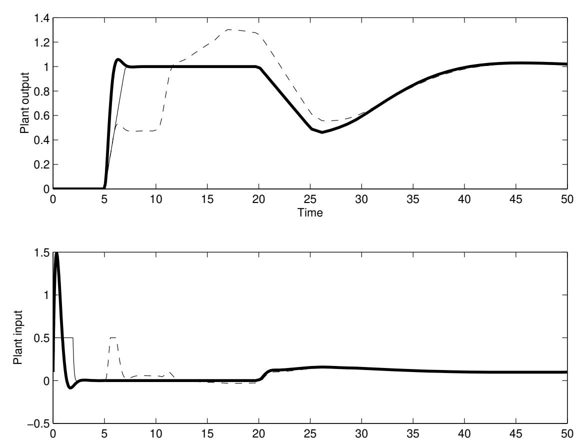

Figure 2: Nominal responses of the unconstrained closed-loop (bold solid), the saturated closed-loop (dashed) and the anti-windup closed-loop (thin solid) to the reference and disturbance selection (17).

Note that the anti-windup action is capable of rapidly recovering the performance lost due to input saturation. As a matter of fact, it is seen from the lower plot of Figure 2 that the input is kept into saturation until the output reaches the unconstrained output response (recall that there is a 5 seconds time shift between the

[8] This minimization, at least in the case when A is Hurwitz, can be carried out as suggested in Remark 2.

input profile and its effect on the output, due to the output delay characterizing the plant). The saturated response, on the other hand, exhibits undesired overshoots and slowly converges back to the unconstrained response. Note that, since the disturbance does not cause input saturation for the unconstrained trajectory, the anti-windup response perfectly reproduces the unconstrained response in the second part of the plot, as formally proven in Theorem 1.

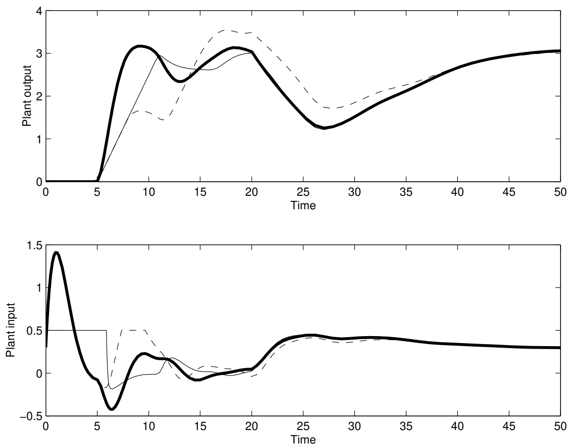

Figure 3: Perturbed responses of the unconstrained closed-loop (bold solid), the saturated closed-loop (dashed) and the anti-windup closed-loop (thin solid) to reference and disturbance inputs three times larger than (17).

Figure 3 represents the response of the perturbed system, as described in [5, Example 4]. To make the effect of saturation noticeable, in this case we have enlarged three times the reference and disturbance inputs (17) (note that by linearity, this doesn't change the nature of the unconstrained response, but just affects its size). The controller parameters are selected once again according to [5] and correspond to $\theta = 5,\ \theta_m = 5.5,\ k_p = 0.1,\ T_i = 0.1,\ k_f = 1.247,\ k_d = 0.095$. The anti-windup response shows once again good unconstrained response recovery properties, as compared to the saturated responses, thus confirming the robustness properties guaranteed in Theorem 1.

5 Conclusions

In this paper we have formalized the $\mathcal{L}_2$ anti-windup problem for linear systems with input and output delays. By suitably generalizing the approach in [13] we have given a solution to the robust, nominal, global and local problems whenever one of these is solvable. The corresponding construction is based on the augmentation of the original control scheme with a dynamic filter. Connections with existing results on anti-windup for dead-time systems have been established and the performance of the proposed scheme has been successfully tested on an example taken from the literature.

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