Guaranteed stability for nonlinear systems by means of a hybrid loop

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Abstract:
We construct hybrid loops that augment continuous time control systems consisting in a continuous-time nonlinear plant in feedback with a (possibly non stabilizing) given nonlinear dynamic continuous-time controller. In particular, the arising hybrid closed-loops are guaranteed to follow the underlying continuous-time closed-loop dynamics when flowing and to jump in suitable regions of the closed-loop state space to guarantee that a positive definite function V of the closed-loop state and/or a positive definite function V_p of the plant-only state is non-increasing along the hybrid trajectories. Sufficient conditions for the construction of these hybrid loops are given for the nonlinear case and then specialized for the linear case with the use of quadratic functions. The proposed approach is illustrated on a linear and a nonlinear example.

1. INTRODUCTION

Even for nonlinear control systems which follow a purely continuous dynamics, it may be useful to consider dynamic controllers having a mixed discrete/continuous dynamics. This leads to the class of hybrid control laws, and the closed-loop system turns out to be a hybrid system. Such controllers are now instrumental in many feedback control designs, for their capability to provide asymptotic stability of the closed-loop system (see e.g. Hespanha and Morse [1999], Hespanha et al. [2004]). Such state feedback laws are also interesting for their capability to guarantee a robustness with respect to small errors in the loop, which cannot be obtained using classical (i.e. with a continuous dynamics) controllers (see e.g. Prieur [2005], Goebel and Teel [2009]).

Hybrid controllers are also instrumental to improve the performance for nonlinear systems in presence of disturbances. See Prieur and Astolfi [2003] for the non-holonomic integrator, and Sanfelice et al. [2007] for juggling systems to focus on applications only. Even for linear systems, in presence of disturbances, the hybrid systems point of view can be fruitlful. See Chen et al. [2001], Beker et al. [2004], Nesic et al. [2008], where reset controllers are used to decrease the $\mathcal{L}_2$-gain between perturbations and the output.

The aim of this paper is to design new hybrid strategies to improve the performance of the closed-loop by guaranteeing some asymptotic stability property, some decrease of a function, etc.

More precisely, the first contribution of the paper is the following. Given a control system and a dynamic controller (the closed-loop system may be unstable), and a Lyapunov-like function V, we compute a hybrid loop (i.e. with a mixed discrete/continuous dynamics), such that along the solutions of the hybrid closed-loop system, the function V is non-increasing. Moreover the closed-loop hybrid system is globally asymptotically stable. Considering again V, we may define a new function V_p which depends on the plant state only, and we may define an other hybrid loop such that, along the solutions of the hybrid closed-loop system, the function V_p is non-increasing. Moreover the state of the plant converges to zero.

The second contribution of the paper considers the case where a function V_p of the plant state is given a priori. With any continuous dynamics of the control state, we define a hybrid loop so that the sufficient conditions of the first main result hold. In other words, a hybrid stabilizer for the nonlinear plant is built.

When particularizing this study to linear plants, this hybrid controller can be seen as a generalization of reset controllers. More precisely the sufficient condition applies to the case of linear control plants, and the method can be seen as a constructive technique to compute a hybrid loop (with linear continuous and discrete dynamics) such that the hybrid closed-loop system is asymptotically stable. Even for linear systems, such nonlinear controllers may be useful since it can give better performance than linear controllers (see e.g., Chen et al. [2001], Nesic et al. [2008]).

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The paper is organized as follows. The problems under consideration in this paper and some preliminaries are given in Section 2. The first problem (namely the computation of a hybrid loop such that the closed-loop hybrid system makes decrease suitable functions) is solved in Section 3. In Section 4, the second problem is solved, i.e. some results are derived to check the sufficient condition of the first main result from the existence of a function of the plant state only. This approach is applied to linear control systems in Section 5. Section 6 contains some numerical simulations to illustrate the main results. Some concluding remarks and open questions are given in Section 7.

Notation The Euclidian norm is denoted by $|\cdot|$ and the scalar product by $\langle\cdot,\cdot\rangle$. I_n (resp. $0_{n,m}$) denotes the identity matrix (resp. the null matrix) in $\mathbb{R}^{n\times n}$ (resp. in $\mathbb{R}^{n\times m}$). The subscripts may be omitted when there is no ambiguity. Moreover, for a vector x, the diagonal matrix defined by the entries of x is noted $\operatorname{diag}(x)$, and for a matrix M, $\operatorname{He}(M) = M + M'$. For any symmetric matrix, $\star$ stands for a symmetric term. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if it is zero at zero, continuous, positive definite, and increasing, and is of class $\mathcal{K}_\infty$ if it is also unbounded.

2. PROBLEM STATEMENT

Consider a nonlinear plant:

$$\dot{x}_p = \bar{f}_p(x_p, u), \quad y = \bar{h}_p(x_p), \tag{1}$$

with x_p in $\mathbb{R}^{n_p}$, in feedback interconnection with a (not necessarily stabilizing) dynamic controller:

$$\dot{x}_c = \bar{f}_c(x_c, y), \quad u = \bar{h}_c(x_c, y), \tag{2}$$

with x_c in $\mathbb{R}^{n_c}$. Then defining the closed-loop functions $f_p(x_p, x_c) := \bar{f}_p(x_p, \bar{h}_c(x_c, \bar{h}_p(x_p)))$ and $f_c(x_p, x_c) = \bar{f}_c(x_c, \bar{h}_p(x_p))$, the interconnection between (1) and (2) can be described in a compact way as:

$$\frac{d}{dt}(x_p, x_c) = (f_p(x_p, x_c), f_c(x_c, x_p)) \ , \tag{3}$$

where $f_p : \mathbb{R}^{n_p}\times\mathbb{R}^{n_c} \to \mathbb{R}^{n_p}$ and $f_c : \mathbb{R}^{n_c}\times\mathbb{R}^{n_p} \to \mathbb{R}^{n_c}$. We will assume that $\bar{f}_p(\cdot,\cdot)$, $\bar{f}_c(\cdot,\cdot)$, $\bar{h}_p(\cdot)$ and $\bar{h}_c(\cdot,\cdot)$ are such that $f_p(\cdot,\cdot)$ and $f_c(\cdot,\cdot)$ are continuous functions satisfying $f_p(0, 0) = 0$ and $f_c(0, 0) = 0$.

By exploiting the properties of Lyapunov functions, the aim of this paper is to construct a hybrid closed-loop system which follows the flow dynamics (3) when the state is in a set (called the flow set) and follows a suitable discrete dynamics when the state is in an other set (called the jump set). These flow and jump sets, together with the discrete dynamics, define a hybrid system and have to be designed to guarantee the decrease of some suitable scalar Lyapunov-like functions (more precisely positive definite functions as considered in Problems 1nd 2 below) of the closed-loop state $x := (x_p, x_c)$ and/or of the plant state x_p.

In particular, we will address the following two problems.

Problem 1. *Consider the closed-loop system (3) and a function* $V : \mathbb{R}^{n_p} \times \mathbb{R}^{n_c} \to \mathbb{R}_{\geq 0}$.

Design a hybrid system which follows the dynamics (3) when flowing and which satisfies one or both of the following two items:

1. *V is non-increasing along the solutions and the origin is globally asymptotically stable;*
2. *a suitable positive definite function $V_p : \mathbb{R}^{n_p} \to \mathbb{R}_{\geq 0}$ of the plant state is non-increasing along solutions and the plant state x_p converges to zero.*

Problem 2. *Consider the closed-loop system (3) and a function* $V_p : \mathbb{R}^{n_p} \to \mathbb{R}_{\geq 0}$.

Design a hybrid system which follows the dynamics (3) when flowing and which satisfies one or both of the following two items:

1. *V_p is non-increasing along solutions and the plant state x_p converges to zero;*
2. *a suitable positive definite function $V : \mathbb{R}^{n_p} \times \mathbb{R}^{n_c} \to \mathbb{R}_{\geq 0}$ of the closed-loop state is non-increasing along the solutions and the origin is globally asymptotically stable.*

It will be shown next that Problems 1 and 2 correspond to two faces of the same three constructions satisfying item 1, item 2 or both items of each problem. These constructions hinge upon a function $\phi : \mathbb{R}^{n_p} \to \mathbb{R}^{n_c}$ such that for all (x_p, x_c) in $\mathbb{R}^{n_p} \times \mathbb{R}^{n_c}$

$$V(x_p, \phi(x_p)) \leq V(x_p, x_c) \ , \tag{4}$$

and establishe the following relation between V and V_p:

$$V_p(x_p) = V(x_p, \phi(x_p)), \forall x_p \in \mathbb{R}^{n_p} \ . \tag{5}$$

In particular, when solving Problem 1 we will require some conditions on the function V (see Assumption 1 in Section 3) that guarantees the existence [3] of ϕ. Instead, when solving Problem 2 we will start from an asymptotic controllability assumption for the plant, corresponding to the existence of a static state feedback stabilizer as specified in Assumption 2 of Section 4. That assumption will give us the necessary functions V_p and ϕ from which it is possible to construct the function V.

The interest in computing the hybrid controller solving the two problems above is that they introduce more degrees of freedom (through the additional dynamics) and may allow better performance than the underlying dynamics (3). Consider e.g. the use of reset controllers for controllable linear systems which yields lower $\mathcal{L}_2$-gain than classical linear and static controller (see Chen et al. [2001], Nesic et al. [2008]). Moreover, there might be cases where it is desirable to impose some flow condition of the closed-loop via a dynamic controller, disregarding the asympotic stability property, and then enforcing stability by way of the hybrid loops.

Problems 1 and 2 will be also addressed in the particular case where system (3) is linear. In this special case, it will be shown that Assumption 1 (respectively Assumption 2) reduces to reasonably weak properties required for the closed-loop dynamics and for the functions V (respectively V_p) in Problem 1 (respectively Problem 2).

Let us make precise the framework of hybrid systems that is considered in this paper. For an introduction, see e.g. the recent survey Goebel et al. [2009]. Such a system combines a continuous dynamics in a set F (called flow set) and

[3] Here, to keep the discussion simple, it is assumed that V is continuously differentiable and that there exists $\phi(x_p) = \operatorname*{argmin}_{x_c\in\mathbb{R}^{n_c}} V(x_p, x_c)$, which implies (4).

discrete dynamics in a set J (called jump set), and it is formally written as

$$\begin{aligned} \dot{x} &= f(x) \text{ if } x \in F \ , \\ x^+ &= g(x) \text{ if } x \in J \ , \end{aligned} \tag{6}$$

where $x = (x_p, x_c) \in \mathbb{R}^n$, $n = n_p + n_c$, $f(x) = (f_p(x), f_c(x))$ for each $x \in \mathbb{R}^n$, and $g : \mathbb{R}^n \to \mathbb{R}^n$ is a given function. We recall some basic ingredients on hybrid system theory, and solutions of (6). Due to mixed discrete/continuous dynamics, a solution of (6) will be defined on a mixed discrete/continuous time domain. More precisely, a set E is a *hybrid time domain* if for all $(T, J) \in E$, $E \cap ([0, T] \times \{0, 1, \dots J\})$ is a *compact hybrid time domain*, i.e. it can be written as

$$\bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$$

for some finite sequence of times $0 = t_0 \leq t_1 \dots \leq t_J$. A solution x to (6) consists of a hybrid time domain $\operatorname{dom} x$ and a function $x : \operatorname{dom} x \to \mathbb{R}^n$ such that $x(t, j)$ is absolutely continuous in t for a fixed j and $(t, j) \in \operatorname{dom} x$ satisfying

(S1) for all $j \in \mathbb{N}$ and almost all t such that $(t, j) \in \operatorname{dom} x$,

$$x(t, j) \in F, \quad \dot{x}(t, j) = f(x(t, j)) \ ;$$

(S2) For all $(t, j) \in \operatorname{dom} x$ such that $(t, j+1) \in \operatorname{dom} x$,

$$x(t, j) \in J, \quad x(t, j+1) = g(x(t, j)) \ .$$

Then, the state solution x is parameterized by (t, j) where t is the ordinary time and j is an independent variable that corresponds to the number of jumps of the solution. When the state $x(t, j)$ belongs to the intersection of the flow set and of the jump set, then the solution can either flow or jump. This parameterization may be omitted when there is no ambiguity.

A solution x to (6) is said to be *complete* if $\operatorname{dom} x$ is unbounded, *Zeno* if it is complete but the projection of $\operatorname{dom} x$ onto $\mathbb{R}_{\geq 0}$ is bounded, and *maximal* if there does not exist an other solution $\tilde{x}$ of (6) such that x is a truncation of $\tilde{x}$ to some proper subset of $\operatorname{dom} \tilde{x}$. Hereafter, only maximal solutions will be considered. For more details about this hybrid systems framework, we refer the reader to Goebel et al. [2009], Prieur et al. [2007].

Definition 1. *The hybrid system (6) is said to be*

- stable: *if for each $\epsilon > 0$ there exists $\delta > 0$ such that each solution x to (6) with $|x(0,0)| \leq \delta$ satisfies $|x(t, j)| \leq \epsilon$ for all $(t, j) \in \operatorname{dom} x$;*
- attractive: *if every solution x to (6) is complete and satisfies $\lim_{t+j\to\infty} |x(t, j)| = 0$;*
- globally asymptotically stable: *if is both stable and attractive.*

3. SOLUTION TO PROBLEM 1

In this section we consider the closed-loop nonlinear system (3) and a function V of the closed-loop state and give a construction to solve Problem 1. To this aim, we make the following assumption on the function V.

Assumption 1. *The function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is such that there exists a continuous differentiable*[4] *function $\phi : \mathbb{R}^{n_p} \to \mathbb{R}^{n_c}$ such that*

$$\phi(x_p) = \operatorname{argmin}_{x_c \in \mathbb{R}^{n_c}} V(x_p, x_c) \ . \tag{7}$$

Moreover, there exists a class $\mathcal{K}$ function α such that, for all x_p in $\mathbb{R}^{n_p}$

$$\langle \nabla_p V(x_p, \phi(x_p)), f(x_p, \phi(x_p)) \rangle < -\alpha(V(x_p, \phi(x_p))), \tag{8}$$

where $\nabla_p V$ denotes the gradient of V with respect to its first argument.

Note that in Assumption 1 we do not insist that (3) is globally asymptotically stable, because (8) only requires the function V to be decreasing only on the subset of the state space defined by $(x_p, x_c) = (x_p, \phi(x_p))$. Nevertheless, if system (3) is globally asymptotically stable, then there exist a function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and a class $\mathcal{K}$ function α such that

$$\langle \nabla V(x), f(x) \rangle < -\alpha(V(x)), \ \forall x \neq 0 \tag{9}$$

which implies (8).

A natural way to stabilize the closed-loop system (3) is to flow when one (or both) of the functions V and V_p is strictly decreasing and to reset the x_c-component of the state to the value $\phi(x_p)$ (where strict decrease is guaranteed by (8)) when the function (or either of the two) is not decreasing. This leads to the following hybrid system

$$\begin{aligned} \dot{x} &= f(x) & &\text{if } x \in \hat{F} \ , \\ (x_p^+, x_c^+) &= (x_p, \phi(x_p)) & &\text{if } x \in \hat{J} \ , \end{aligned} \tag{10}$$

where $\hat{F} \subset \mathbb{R}^n$ and $\hat{J} \subset \mathbb{R}^n$ are suitable closed subsets of the whole state space such that $\hat{F} \bigcup \hat{J} = \mathbb{R}^n$. In particular, $\hat{F}$ and $\hat{J}$ are defined by suitably combining the following two pairs of sets:

$$\begin{aligned} F &= \{x \in \mathbb{R}^n, \ \langle \nabla V(x), f(x) \rangle \leq -\bar{\alpha}(V(x))\} \\ J &= \{x \in \mathbb{R}^n, \ \langle \nabla V(x), f(x) \rangle \geq -\bar{\alpha}(V(x))\} \end{aligned} \tag{11}$$

$$\begin{aligned} \bar{F} &= \{x \in \mathbb{R}^n, \ \langle \nabla V_p(x_p), f_p(x_p, x_c) \rangle \leq -\bar{\alpha}(V_p(x_p))\} \\ \bar{J} &= \{x \in \mathbb{R}^n, \ \langle \nabla V_p(x_p), f_p(x_p, x_c) \rangle \geq -\bar{\alpha}(V_p(x_p))\}, \end{aligned} \tag{12}$$

where V_p is defined in (5) and where $\bar{\alpha}$ is any class $\mathcal{K}$ function such that $\bar{\alpha}(s) \leq \alpha(s)$ for all $s \geq 0$ (this will be denoted next by the shortcut notation $\bar{\alpha} \leq \alpha$).

We are now in position to solve Problem 1 as stated in the first main result.

Theorem 1. *Consider the closed-loop system (3) and a function $V(\cdot)$. Assume that there exist functions ϕ and α satisfying Assumption 1. Then for any $\bar{\alpha} \leq \alpha$ the following holds.*

1. If V is positive definite and radially unbounded, then the hybrid system (10), (11) with $\hat{F} = F$ and $\hat{J} = J$ is globally asymptotically stable and V is non-increasing along solutions.

2. If V_p is positive definite and radially unbounded, then the hybrid system (10), (12) with $\hat{F} = \bar{F}$ and $\hat{J} = \bar{J}$ is such that the plant state x_p converges to zero, and V_p is non-increasing along solutions.

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[4] In this paper it is only needed that ϕ is locally Hölder continuous of order strictly larger than $\frac{1}{2}$, but to ease the presentation, it is assumed more regularity on ϕ in Assumption 1.

3 If V is positive definite and radially unbounded, then the hybrid system (10), (11), (12) with $\hat{F} = F \bigcap \bar{F}$ and $\hat{J} = J \bigcup \bar{J}$ is globally asymptotically stable and both V and V_p are non-increasing along solutions.

4. SOLUTION TO PROBLEM 2

In this section we consider the closed-loop nonlinear system (3), a function V_p of the plant state and give a construction to solve Problem 2. To this aim, we make the following assumption on the function V_p.

Assumption 2. *The function $V_p : \mathbb{R}^{n_p} \to \mathbb{R}_{\geq 0}$ is continuously differentiable and radially unbounded and there exist a continuously differentiable function $\phi : \mathbb{R}^{n_p} \to \mathbb{R}^{n_c}$, and a class $\mathcal{K}$ function α such that, for all x_p in $\mathbb{R}^{n_p}$, $x_p \neq 0$,*

$$\langle \nabla V_p(x_p), f_p(x_p, \phi(x_p)) \rangle < -\alpha(V_p(x_p)) \ . \qquad (13)$$

Note that, when the top equation of the closed-loop system (3) is affine with respect to x_c, this condition is related to the asymptotic controllability to the origin Artstein [1983]. In this case, a control law ϕ can be computed from a Control Lyapunov Function V_p and from the so-called *universal formulas* (see Freeman and Kokotović [1996], Lin and Sontag [1991]).

Assumption 2 is sufficient to construct a function V satisfying Assumption 1 so that the design strategy of the previous section can be employed. In particular, let M be any positive semidefinite matrix [5] in $\mathbb{R}^{n_c} \times \mathbb{R}^{n_c}$ and define $V : \mathbb{R}^{n_p} \times \mathbb{R}^{n_c} \to \mathbb{R}_{\geq 0}$ for all (x_p, x_c) in $\mathbb{R}^{n_p} \times \mathbb{R}^{n_c}$,

$$V(x) = V_p(x_p) + (x_c - \phi(x_p))' M (x_c - \phi(x_p)) \ . \qquad (14)$$

Note that V is continuously differentiable, and radially unbounded. Moreover, if $\phi(0) = 0$ and $M > 0$, then it is a positive definite function because, for each x_p, it is the sum of two positive definite terms, the first one strictly positive when $x_p \neq 0$ and the second one strictly positive when $x_p = 0$. The following theorem is a straightforward application of Theorem 1 in light of the V_p and ϕ given in Assumption 2 and of the V in (14).

Theorem 2. *Consider the closed-loop system (3) and a function V_p. Assume that there exist functions ϕ and α satisfying Assumption 2. Given any positive semidefinite matrix $M \in \mathbb{R}^{n_c} \times \mathbb{R}^{n_c}$ and the corresponding function V in (14), then for any $\bar{\alpha} \leq \alpha$ the three items of Theorem 1 hold.*

5. APPLICATION TO THE LINEAR CASE

When focusing on linear dynamics, the problem statement in Section 2 can be carried out as follows. Here we illustrate the fact that the strict properness assumption on the plant (1) can be removed as long as a well posedness condition holds for the closed-loop. Consider the following linear plant:

$$\dot{x}_p = \bar{A}_p x_p + \bar{B}_p u, \quad y = \bar{C}_p x_p + \bar{D}_p u, \qquad (15)$$

with $x_p \in \mathbb{R}^{n_p}$, in feedback interconnection with a (not necessarily stabilizing) linear dynamic controller:

$$\dot{x}_c = \bar{A}_c x_c + \bar{B}_c y, \quad u = \bar{C}_c x_c + \bar{D}_c y, \qquad (16)$$

with $x_c \in \mathbb{R}^{n_c}$. Assuming that $I - \bar{D}_p \bar{D}_c$ is nonsingular, the closed-loop is well posed and described by the following linear system

$$\dot{x} = Ax := \begin{bmatrix} A_p & B_p \\ B_c & A_c \end{bmatrix} x \ , \qquad (17)$$

where $x = (x_p, x_c)$ and A_p, B_p, A_c, and B_c are matrices of appropriate dimensions uniquely defined based on the matrices in (15) and (16).

In the linear case, it is reasonable to restrict V and V_p to the class of quadratic functions, ϕ to the class of linear stabilizers and α and $\bar{\alpha}$ to the class of linear gains. Based on this, the closed-loop function V can be selected as $V(x) = x'Px$ where $P = \begin{bmatrix} P_p & P_{pc} \\ \star & P_c \end{bmatrix}$ is a symmetric positive definite matrix. Then, since $\nabla_c V(x_p, x_c) = 2(P_{pc}' x_p + P_c x_c)$, from the positive definiteness of P, the function $\phi : \mathbb{R}^{n_p} \to \mathbb{R}^{n_c}$ satisfying (4) is given by

$$\phi(x_p) = -P_c^{-1} P_{pc}' x_p =: K_p x_p \ .$$

Moreover, the function $V_p : \mathbb{R}^{n_p} \to \mathbb{R}_{\geq 0}$ defined by (5) becomes, for all $x_p \in \mathbb{R}^{n_p}$,

$$\begin{aligned} V_p(x_p) &= \begin{bmatrix} x_p \\ -P_c^{-1} P_{pc}' x_p \end{bmatrix}' \begin{bmatrix} P_p & P_{pc} \\ \star & P_c \end{bmatrix} \begin{bmatrix} x_p \\ -P_c^{-1} P_{pc}' x_p \end{bmatrix} \\ &= x_p' (P_p - P_{pc} P_c^{-1} P_{pc}') x_p =: x_p' \bar{P}_p x_p \ . \end{aligned}$$

Let us now focus on Problem 1 and specialize it to the linear case. Based on the above identities, after some calculations, it follows that Assumption 1 is guaranteed if given V, there exists $\tilde{\alpha} > 0$ such that

$$\mathrm{He}(\bar{P}_p (A_p + B_p K_p)) < -\tilde{\alpha} \bar{P}_p, \qquad (18)$$

which, by the definition of $\bar{P}_p$ above, establishes an interesting nonlinear constraint on the function V.

Consider now the sets in (11) and (12). Given any $0 < \bar{\tilde{\alpha}} \leq \tilde{\alpha}$ and with the definitions above, after some calculations they become

$$\begin{aligned} F &= \{x \in \mathbb{R}^n, \ x'Nx \leq -\bar{\tilde{\alpha}} x'Px\} \ , \\ J &= \{x \in \mathbb{R}^n, \ x'Nx \geq -\bar{\tilde{\alpha}} x'Px\} \ , \end{aligned} \qquad (19a)$$

$$\begin{aligned} \bar{F} &= \{x \in \mathbb{R}^n, \ x'N_p x \leq -\bar{\tilde{\alpha}} x_p' \bar{P}_p x_p\} \ , \\ \bar{J} &= \{x \in \mathbb{R}^n, \ x'N_p x \geq -\bar{\tilde{\alpha}} x_p' \bar{P}_p x_p\} \ , \end{aligned} \qquad (19b)$$

where

$$N := \mathrm{He}\left(\begin{bmatrix} P_p A_p + P_{pc} B_c & P_{pc} A_c + P_p B_p \\ P_{pc}' A_p + P_c B_c & P_c A_c + P_{pc}' B_p \end{bmatrix} \right) \qquad (19c)$$

$$N_p := \mathrm{He}\left(\begin{bmatrix} \bar{P}_p A_p & \bar{P}_p B_p \\ 0 & 0 \end{bmatrix} \right) \ . \qquad (19d)$$

With the above definitions, the following proposition particularizes the results of Theorem 1 to the linear case.

Proposition 1. *Consider the closed-loop system (17) and a function $V(x) = x'Px = x' \begin{bmatrix} P_p & P_{pc} \\ \star & P_c \end{bmatrix} x$ such that $\bar{P}_p := P_p - P_{pc} P_c^{-1} P_{pc}'$ satisfies (18) for some $\tilde{\alpha} > 0$ and for $K_p = -P_c^{-1} P_{pc}'$. Then the hybrid system*

$$\begin{aligned} \dot{x} &= Ax && \text{if } x \in \hat{F} \ , \\ (x_p, x_c)^+ &= (x_p, K_p x_p) && \text{if } x \in \hat{J} \ , \end{aligned} \qquad (20)$$

satisfies all the items of Theorem 1 with $V_p(x_p) = x_p' \bar{P}_p x_p$ and using the sets in (19) with any $\bar{\tilde{\alpha}} \leq \tilde{\alpha}$.

[5] Note that the matrix M may be a function of x. This extra degree of freedom could be used to perform convenient selections of V.

Let us focus now on Problem 2 for the linear case. With (17), Assumption 2 is satisfied whenever the pair (A_p, B_p) is stabilizable and $V_p(x_p) = x_p' \bar{P}_p x_p$, with $\bar{P}_p > 0$, is a control Lyapunov function for (A_p, B_p). As a matter of fact in that case there exist a static state feedback matrix K_p and a constant $\tilde{\alpha} > 0$ such that equation (18) holds, and then equation (13) will hold with $\phi(x_p) = K_p x_p$. In particular, given $\bar{P}_p$, K_p and $\tilde{\alpha}$ can be computed using a Linear Matrix Inequality (LMI) solver. Alternatively, under a stabilizability assumption, one can always solve a generalized eigenvalue problem (gevp) and find an optimal pair $(\bar{P}_p, K_p)$ maximizing $\tilde{\alpha}$.

Based on V_p and K_p, consider any symmetric positive definite matrix P_c in $\mathbb{R}^{n_c \times n_c}$. The function $V : \mathbb{R}^{n_p} \times \mathbb{R}^{n_c} \to \mathbb{R}_{\geq 0}$ in (14) is defined, for all (x_p, x_c) in $\mathbb{R}^{n_p} \times \mathbb{R}^{n_c}$, as

$$\begin{aligned} V(x) &= V_p(x_p) + (x_c - K_p x_p)' P_c (x_c - K_p x_p) \\ &= x' \begin{bmatrix} \bar{P}_p + K_p' P_c K_p & -K_p' P_c \\ \star & P_c \end{bmatrix} x =: x' P x. \end{aligned} \qquad (21)$$

Since $P_c > 0$, this function is continuously differentiable, radially unbounded and positive definite.

We are then in the position to state the following proposition which particularizes the results of Theorem 2 to the linear case.

Proposition 2. *Assume that the pair (A_p, B_p) is stabilizable and that $V_p(x) = x_p' \bar{P}_p x_p$ is a control Lyapunov function for this pair. Then there exist $\tilde{\alpha} > 0$ and K_p satisfying (18). Moreover, given any symmetric positive definite matrix P_c in $\mathbb{R}^{n_c \times n_c}$ and the corresponding function V defined in (21), for any selection of $0 < \bar{\tilde{\alpha}} \leq \tilde{\alpha}$, the reset system (20) with the sets in (19) satisfies all the items of Theorem 1.*

6. ILLUSTRATION ON NUMERICAL SIMULATIONS

In this section some numerical simulations are performed to illustrate the main results, first on nonlinear and then on linear systems. To simulate the hybrid systems we use the simulator presented in [Goebel et al., 2009, pages 78-81] [6].

6.1 Nonlinear example

Let us illustrate the second item of Theorem 2 by performing some numerical simulations on the following system in $\mathbb{R}^2$

$$\begin{cases} \dot{x}_1 = x_2 + x_1^2 \\ \dot{x}_2 = u + x_1^2 \end{cases} \qquad (22)$$

where (x_1, x_2) is the plant state and u stands for the control variable in $\mathbb{R}$. By using a backstepping method (see e.g. **?**), the following controller $\phi : \mathbb{R}^2 \to \mathbb{R}$ defined by, for all (x_1, x_2) in $\mathbb{R}^2$,

$$\phi(x_1, x_2) = -2x_1 - 2x_2 - 3x_1^2 - 2x_1(x_2 + x_1^2) \qquad (23)$$

and the positive definite function $V_p : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ defined by, for all (x_1, x_2) in $\mathbb{R}^2$,

$$V_p(x_1, x_2) = \frac{1}{2} x_1^2 + \frac{1}{2}(x_2 + x_1 + x_1^2)^2$$

[6] The simulation code can be downloaded from `homepages.laas.fr/cprieur/Papers/nolcos2010.zip`

may be computed. Assumption 2 holds with $\alpha(s) = \frac{s}{2}$, for all $s \geq 0$. To check the attractivity property, let us consider the initial condition $(x_1(0,0), x_2(0,0)) = (10, 10)$ and let us numerically compute the solution of system (22) in closed-loop with the controller (23). The time evolution of the x_1 and x_2 variables, and of the controller ϕ are depicted on Figure 1, in solid line.

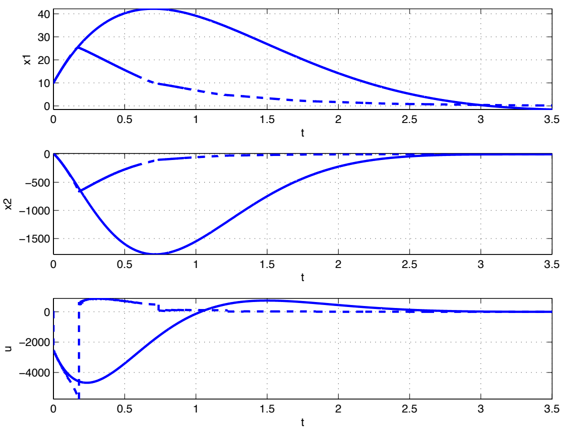

Fig. 1. Time evolution of x_1 (top), of x_2 (middle) for system (22) with the control values u (down). The control values are given by the controller (23) (in solid line), or by the third component of system (10), (12) with $\hat{F} = \bar{F}$ and $\hat{J} = \bar{J}$ (in dashed line)

Now let us consider the system (22) in closed-loop with the controller $\dot{x}_3 = x_3 - x_1^3$. This nonlinear closed-loop system is unstable (the initial condition $(x_1(0,0), x_2(0,0), x_3(0,0)) = (10, 10, 10)$ gives a diverging solution). Let us apply Theorem 2 with $\bar{\alpha} = 10^{-3}\alpha$. Consider the hybrid system (10), (12) with $\hat{F} = \bar{F}$ and $\hat{J} = \bar{J}$.

Let us numerically compute the solution starting from the initial condition $(x_1(0,0), x_2(0,0), x_3(0,0)) = (10, 10, 10)$. We check on Figure 1 (see the dashed line) that the plant state x_p variable is globally asymptotically stable. Moreover we note that the x_3-variable converges also to 0 and has some jumps (when it is reset to the value of the controller (23)). The number of jumps depends on the size of J and thus on the choice of the function $\bar{\alpha}$. By comparing the solid and the dashed lines on Figure 1, we note that the speed of convergence is improved using the hybrid controller (10).

6.2 Linear example

In this section, we apply Proposition 2 to a linear controllable system and we illustrate the convergence of system (20), (19a) with $\hat{F} = F$ and $\hat{J} = J$.

To do that, let us consider the system (15) with $\bar{A}_p = \begin{bmatrix} 0 & 0 \\ 1 & -0.2 \end{bmatrix}$, $\bar{B}_p = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\bar{C}_p = I_2$, $\bar{D}_p = 0$ (see **??**). For the linear dynamic controller (16), we select any matrices such that (17) is unstable. In this example we fix $\bar{A}_c = 1$, $\bar{B}_c = [0\ 1]$, $\bar{C}_c = 1$, and $\bar{D}_c = [0\ 0]$. Recalling the notations in (17), it reads $A_p = \bar{A}_p$ and $B_p = \bar{B}_p$, and thus the pair (A_p, B_p) is stabilizable. For the matrix K_p, let us choose a pole placement controller $K_p = [\,-1 \ -0.9\,]$, and let $\bar{P}_p = \begin{bmatrix} 0.6 & -0.2 \\ \star & 0.5 \end{bmatrix}$, and $\tilde{\alpha} = 1$, so that condition (18) holds. The symmetric positive definite matrix P_c and the positive value $\bar{\tilde{\alpha}} \leq \tilde{\alpha}$ can be also arbitrarily chosen. Here we let $P_c = 1$, and $\bar{\tilde{\alpha}} = \frac{1}{2}\tilde{\alpha}$.

To illustrate the global asymptotic stability of the system (20), (19a) with $\hat{F} = F$ and $\hat{J} = J$, let us consider the initial condition $x(0,0) = (10, 10, 10)'$. Figure 2 depicts the time evolution of the projection of the solution x onto the flow time axis t. The jumps of the last component x_3 may be observed.

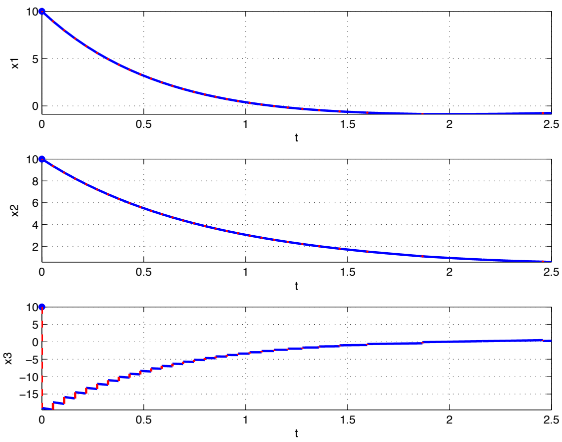

Fig. 2. Time evolution of x_1 (top), of x_2 (middle) and of x_3 for flow time between 0 and 2.5 of the system (20), (19a) with $\hat{F} = F$ and $\hat{J} = J$

7. CONCLUSION

The design problem of a stabilizing hybrid loop is considered. This class of system mixes discrete and continuous dynamics depending on the value of a nonlinear function. This allows to guarantee the stability and/or a decreasing property of some positive definite function.

This work lets many questions open. In particular the optimization problem and the best choice of the continuous dynamics for the controller state (and of the flow and jump sets) are under actual investigation. The performance of the closed-loop system in terms of the speed of convergence, or of the rejection of perturbation may be also considered. It could also be interesting to consider applications on physical systems such as the non-holonomic integrator, or the juggling systems and to compare with existing hybrid strategies (see Prieur and Astolfi [2003] and Sanfelice et al. [2007] respectively).

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