Lazy sensors for the scheduling of measurement samples transmission in linear closed loops over networks

Fulvio Forni, Sergio Galeani, Dragan Nešić, Luca Zaccarian

Abstract— **We consider the problem of routing on a network the measurement samples coming from a linear plant stabilizer by a preassigned linear controller. Under the assumption that the linear closed-loop before the network insertion is exponentially stable, we break the communication channel between the sensors and the control law and propose a family of "lazy sensors" whose goal is to transmit the measured plant output information as little as possible while guaranteeing the closed-loop stability. We propose three transmission policies and provide conditions on the transmission parameters that guarantee global asympotic stability if the plant state is available and global practical stability if the plant state is not available. Simulation results confirm the effectiveness of the proopsed strategies.**

I. Introduction

In recent years, much attention has been devoted to the study of networked control systems. The interest in this class of control systems is motivated by the increased computational capability required by control and estimation algorithms in addition to the presence of emerging control applications wherein the systems to be controlled are spread over a wide territory or are technologically built in such a way that several subcomponents of the control system communicate over a shared and low capacity network (see, e.g., the recent surveys [17], [8] and references therein). While networked control systems denote many different situations where a network is in some sense involved in the transmission of the control signals, a case of interest is that when the network is used as a communication channel between the plant with its sensing/actuating devices and the device hosting the control algorithm. This specific context is studied, e.g., in [3], [4], [9], [10], [13], [15], [16], [18].

A typical way to represent and suitably write the dynamics of systems acting on networks is to use the hybrid systems notation, namely a state-space description wherein the state flows according to some continuous-time rules and, at some specific times, called jump times, it jumps following some discrete-time jump rule. A framework for the representation of hybrid systems that has been recently proposed in [7], [5] allows for a quite natural description of these phenomena with useful Lyapunov like results that have been proven to apply to large classes of systems described using this framework (see, e.g., [1], [2] and the survey [6]). This framework was used in connection with networked control systems in [3], [10], where Lyapunov-like tools are used to model ISS properties of network control systems and the MATI - maximum allowable transfer interval, to preserve asymptotic stability.

In this paper we consider a linear control system that consists of a controller that uses the output of a given plant and produces a suitable input to asymptotically stabilize the whole closed-loop system. Usually, the measured output y of the plant is connected to the input of the controller u, so that the signal y is continuously transmitted to the controller. Here we break this continuity by considering a not necessarily periodic sample and hold approach. In particular, we suppose that the wire from the *measured output* y to the *controller input* u is replaced by a network, that is, each measurement is sampled and routed to the controller input. Then, we define an updating policy for sending such measurements samples, based on the current state of the system and on the error between the value of the output y and the value of the samples sent to the controller. The devices performing this scheduling policy are called "lazy sensors" to resemble the fact that their goal is to avoid transmitting too often, so to keep low the network load. The structure of the considered system is represented in Figure 1.

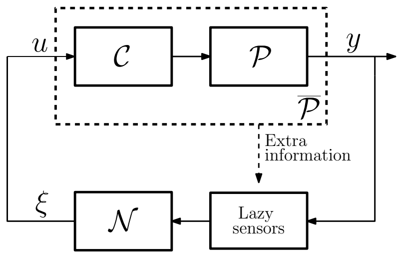

Fig. 1. A closed-loop $\mathcal{S}_{\mathcal{N}}$ over a network using lazy sensors.

Each lazy sensor is able to perform some computation on the measured plant output and, possibly, on extra input signals. Then, each sensor decides whether or not to send a sampled measurement to the input of the controller. The contribution of this paper consists in casting the above problem using the framework in [6] and proposing a number of measurement transmission (or update) policies which depend on the state of the plant and on the measurement error through a suitable Lyapunov-like function. Then, using the hybrid system tools and the framework in [6], the proposed transmission policies are shown to preserve closed-loop stability. In particular, we propose the following three solutions, suitable for different practical contexts:

- a synchronous updating policy where each sensor is aware of the conditions of the other sensors so that the samples update is a global decision. Specifically, the sensors send a new sample all together when some suitable condition occur;
- an asynchronous updating policy where each sensor knows its own measurement error and the state of the plant. Then, it decides autonomously whether or not to send a new sample to the controller;
- a synchronous updating policy based on the measurement errors and the output signal of the plant, by using an observer to reconstruct the state, assuming that it is not available for measurement.

Since the ultimate goal of the above policies is to use the network as little as possible, we call *lazy* these intelligent sensors, to resemble the fact that they are reluctant to transmit and that they do so only when it is strictly necessary, w.r.t. the satisfaction of a suitable Lyapunov-like condition, to preserve closed-loop stability. A possible implementation context could be that of a CAN network where the shared information is broadcast on the network by the controller node, which has highest priority over the other nodes. Then, the other nodes could correspond to the lazy sensors, each of them equipped with an onboard intelligence deciding whether or not to transmit over the network.

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Within the existing literature, the results in this paper can be seen as a specific application of the hybrid framework [6] to a peculiar control problem with a specific network structure. In this sense, our paper can be seen as a constructive solution along the general lines of [3], [10], where Lyapunov tools and the hybrid framework of [6] are used as well to address networked control systems. Our work can also be associated with the many interesting results in [4], [9], [13], [15] and references therein. Here, differently from [13], [15], we only take into account linear systems by proposing updating rules that do not necessarily force each sample to be updated to the current measure of the output. Moreover, asynchronous updating policies and output based updating policies studied here are not taken into account in [13], [15].

The paper is structured as follows. In Section II we introduce the notation and give some preliminaries on hybrid systems. In Section III we introduce the problem data and in the following Sections IV, V, VI we discuss the three approaches outlined above. In Section VII we give a simulation example and proofs are given in the appendix.

II. Notation and preliminaries

Given a vector v, v^T denotes the transpose vector of v. Given a set $\{a_1, \dots, a_n\}$ where $a_i \in \mathbb{R}$ for each $i = 1 \dots, n$, $\mathrm{diag}(v)$ denotes a diagonal matrix having the entries of v on the main diagonal. Both the Euclidean norm of a vector and the corresponding induced matrix norm are denoted by $|\cdot|$. For a vector $v \in \mathbb{R}^p$ and a set $\mathcal{A} \subset \mathbb{R}^n$ $|v|_{\mathcal{A}} := \inf_{y\in\mathcal{A}} |y - x|$. $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$; it is said to belong to class $\mathcal{K}_\infty$ if $a = +\infty$ and $\lim_{r\to+\infty} \alpha(r) = +\infty$. $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class $\mathcal{KL}$ if (i) for each $t{\geq}0$, $\beta(\cdot, t)$ is non decreasing and $\lim_{t\to 0} \beta(s, t) = 0$, and (ii) for each $s \geq 0$, $\beta(s, \cdot)$ is nonincreasing and $\lim_{t\to\infty} \beta(s, t) = 0$. A function is $\gamma : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class $\mathcal{KLL}$ if for each $r \geq 0$, $\gamma(\cdot, \cdot, r)$ and $\gamma(\cdot, r, \cdot)$ are $\mathcal{KL}$ functions.

We summarize next the essential notation associated with the hybrid systems framework, outlined in [5], for which several structural results have been developed in [7], [11], [12] and partially summarized in [6]. A hybrid system $\mathcal{H}$ is a tuple (C, D, F, G), where $C \subseteq \mathbb{R}^n$ and $D \subseteq \mathbb{R}^n$ are, respectively, the *flow set* and the *jump set*, while $F : C \rightrightarrows \mathbb{R}^n$ and $G : D \rightrightarrows \mathbb{R}^n$ are set-valued mappings, called the *flow map* and the *jump map*, respectively. F and G characterize the continuous and the discrete evolution of the system, that is, the motion of the state, while C and D characterize subsets of $\mathbb{R}^n$ where such evolution may occur. A hybrid system is usually represented as follows

$$\mathcal{H} = \begin{cases} \dot{x} \in F(x) & x \in C \\ x^+ \in G(x) & x \in D \end{cases} \tag{1}$$

Intuitively, the evolution of the state either continuously flows through C, by following the dynamic given by F, or it jumps from D, according to G.

As for classical dynamical systems, the evolution of the state of a hybrid system is a parameterized function of time. In particular a *solution* to the hybrid system equations is parameterized with respect to a generalized notion of time, denoted *hybrid time*, defined as follows.

Definition 1: A set $E \subseteq \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a *hybrid time domain* if it is the union of infinitely many intervals of the form $[t_j, t_{j+1}] \times \{j\}$ where $0 = t_0 \leq t_1 \leq t_2 \leq, \dots,$ or of finitely many such intervals, with the last one possibly of the form $[t_j, t_{j+1}] \times \{j\}$, $[t_j, t_{j+1}) \times \{j\}$, or $[t_j, \infty] \times \{j\}$.

Then, a solution to a hybrid system (1) can be defined as follows.

Definition 2: A *hybrid arc* x is a map $x : \mathrm{dom}\, x \to \mathbb{R}^n$ such that (i) $\mathrm{dom}\, x$ is a hybrid time domain, and (ii) for each j, the function $t \mapsto x(t, j)$ is a locally absolutely continuous function on the interval $I_j = \{t : (t, j) \in \mathrm{dom}\, x\}$.

A hybrid arc $x : \mathrm{dom}\, x \to \mathbb{R}^n$ is a *solution to the hybrid system* $\mathcal{H}$ if $x(0, 0) \in C \cup D$ and

(i) for each $j \in \mathbb{N}$ such that I_j has a nonempty interior,

$$\begin{array}{ll} \dot{x}(t, j) \in F(x(t, j)) & \text{for almost all } t \in I_j \\ x(t, j) \in C & \text{for all } t \in [\min I_j, \sup I_j) \end{array} \tag{2}$$

(ii) for each $(t, j) \in \mathrm{dom}\, x$ such that $(t, j + 1) \in \mathrm{dom}\, x$,

$$\begin{array}{rl} x(t, j + 1) & \in G(x(t, j)) \\ x(t, j) & \in D \end{array} \tag{3}$$

Note that solutions to hybrid systems may exist for a finite time, due to the constraints on the state motion enforced by the C and D sets. We say that a solution x is *maximal* if there does not exists x' such that x is a truncation of x' to some proper subset of $\mathrm{dom}\, x'$. We say that a solution x is *complete* if $\mathrm{dom}\, x$ is unbounded.

Finally, we can define on hybrid systems usual stability properties. By following [6], for a hybrid system $\mathcal{H}$, the set $\mathcal{A}$ is (i) *stable* if for each $\epsilon > 0$ there exists $\delta > 0$ such that any solution x to $\mathcal{H}$ with $|x(0, 0)|_{\mathcal{A}} \leq \delta$ satisfies $|x(t, j)|_{\mathcal{A}} \leq \epsilon$ for all $(t, j) \in \mathrm{dom}\, x$; (ii) *pre-attractive* if there exists $\delta > 0$ such that any solution x to $\mathcal{H}$ with $|x(0, 0)|_{\mathcal{A}} \leq \delta$ is bounded and $|x(t, j)|_{\mathcal{A}} \to 0$ as $t + j \to \infty$, whenever x is complete; (iii) *pre-asymptotically stable* if it is both stable and pre-attractive; (iv) *globally pre-asymptotically stable* if it is stable and from each initial condition $x_0 \in \mathbb{R}^n$ complete solutions converge to $\mathcal{A}$, that is, $|x(t, j)|_{\mathcal{A}} \to 0$ as $t + j \to \infty$.

III. Problem statement

Consider a *nominal closed-loop system*, $\mathcal{S}$, composed by a *linear controller* $\mathcal{C}$, with input u_c and output y_c, and by a *linear plant* $\mathcal{P}$, with input u_p and output y_p. The controller drives the plant by the connection $u_p = y_c$ and the output of the plant, y_p, is connected to the input u_c of the controller (feedback signal). In what follows we denote with $\overline{\mathcal{P}}$ the *cascade* of the the controller $\mathcal{C}$ and of the plant $\mathcal{P}$, through the connection $u_p = y_c$. $\overline{\mathcal{P}}$ can be represented as follows

$$\begin{cases} \dot{x} &= Ax + Bu \\ y &= Hx \end{cases} \tag{4}$$

where we assume $u = u_c$ and $y = y_p$. Thus, the nominal closed loop system $\mathcal{S}$ is constructed by connecting (4) through

$$u = y. \tag{5}$$

Then, the closed-loop system $\mathcal{S}$ of Equations (4),(5) can be characterized as follows.

$$\begin{cases} \dot{x} &= (A + BH)\, x \\ y &= Hx \end{cases} \tag{6}$$

and we consider the following standing assumption

Assumption 1: The nominal closed-loop system $\mathcal{S}$ is exponentially stable.

Consider now to replace the direct feedback interconnection (5) with a non-continuous communication policy $u = \mathcal{N}(y)$ between the output y and the input u. $\mathcal{N}$ can be considered a sample and hold network of digital sensor devices, that brings each sensor measurement of y to the input u of the controller. The *networked closed-loop system* $\mathcal{S}_\mathcal{N}$, namely the closed-loop system of (4) through the interconnection $u = \mathcal{N}(y)$, combines together the continuous dynamics of the plant-controller cascade $\overline{\mathcal{P}}$ and the discrete behavior of the network of digital sensor devices $\mathcal{N}$. Thus, it can be conveniently characterized within the *hybrid systems framework*.

In particular, we can write a hybrid model for the networked closed-loop system $\mathcal{S}_\mathcal{N}$. It is characterized by three main components: (i) the continuous dynamics of the cascade $\overline{\mathcal{P}}$ of controllers and plant; (ii) the dynamics of the sample-and-hold device, namely the mechanism that holds a sensors sample until a new one occur; (iii) the updating policy, that decides when a new sample of the measured output y must be submitted to the controllers. Then, a possible model for $\mathcal{S}_\mathcal{N}$ is

$$\left\{\begin{array}{lcl} \dot{x} & = & Ax + B\xi \\ \dot{\xi} & = & 0 \end{array}\right. \qquad (x,\xi) \in C \tag{7a}$$

$$\left\{\begin{array}{lcl} x^+ & = & x \\ \xi^+ & = & g(x,\xi) \end{array}\right. \qquad (x,\xi) \in D \tag{7b}$$

$$y = Hx \tag{7c}$$

Consider the continuous dynamics in (7a): x takes into account the dynamics of the plant-controller cascade, while ξ is the value that is currently enforced at the input of the controller. The dynamics of ξ takes into account the sample-and-hold behavior of the network, whose derivative must be zero (it "holds"). Moreover, the dynamics of $\overline{\mathcal{P}}$ is now driven by ξ, replacing the connection $u = y$ with $u = \mathcal{N}(y)$, that is, with $u = \xi$. The set C in which the system may flow is a design parameter, that is, it will be used to define the updating policy of the measurement samples. Consider the discrete dynamics in (7b): We model the updating mechanism of a measurement sample to the controller input as a *jump*. Therefore, during a jump, the state x of $\overline{\mathcal{P}}$ is not modified, while the state ξ of the network is modified in accordance with a suitable updating policy, whose behavior depends on the function g and on sets C and D. Intuitively, a measurement is updated to a new value given by g when some suitable condition on x and ξ is satisfied, that is, when $(x,\xi) \in D$. For now, we do not make any assumption on g. The structure of g, as well as its values, depend on the particular feedback that we consider and will be defined in the next sections.

Remark 1: In this work, we consider a very simple model for the network $\mathcal{N}$. In fact, we consider $\mathcal{N}$ as a general discrete process that routes each sensor measurement to an output point. Usually, this operation introduces time-delays and quantizations of signals. Moreover, the amount of data routed by the network is limited by the physical data-rate bounds of the network. In our model we do not take into account time-delays and quantization problems, assuming that each measurement is instantaneously routed to the controller. Instead, we consider an updating policy that guarantees a low data-rate on the network. ⌟

IV. State feedback: synchronous approach

Consider the networked closed-loop system $\mathcal{S}_\mathcal{N}$ in Equation (7) and the coordinate transformation $e = \xi - y$, related to the *error* between the measured output and its samples, induced by the sample and hold mechanism of the network. The system can be written as follows.

$$\left\{\begin{array}{lcl} \dot{x} & = & \overline{A}_{11}x + \overline{A}_{12}e \\ \dot{e} & = & \overline{A}_{21}x + \overline{A}_{22}e \end{array}\right. \qquad (x,e) \in \overline{C} \tag{8a}$$

$$\left\{\begin{array}{lcl} x^+ & = & x \\ e^+ & = & \overline{g}(x,e) \end{array}\right. \qquad (x,e) \in \overline{D} \tag{8b}$$

$$y = Hx \tag{8c}$$

where $\overline{A}_{11} = (A + BH)$, $\overline{A}_{12} = B$, $\overline{A}_{21} = -H(A+BH)$ and $\overline{A}_{22} = -HB$. $\overline{g}(x,e)$, $\overline{C}$ and $\overline{D}$ characterize the updating policy and their definition is the goal of this work. They will be defined by the design method proposed below. In general, $\overline{g}$ is a function in $\mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^q$ satisfying $\overline{g}(0,0) = 0$.

Remark 2: Suppose that $\overline{g}(x,e)$, $\overline{C}$ and $\overline{D}$ have been constructed by a suitable design method. Then, $g(x,\xi)$, C and D of (7) can be defined from $\overline{g}(x,e)$, $\overline{C}$ and $\overline{D}$ as follows.

- Suppose $\overline{C} = \{(x,e)\,|\,r(x,e)\}$ where r is a given relation on x and e. Then, $C = \{(x,\xi)\,|\,r(x,\xi - Hx)\}$, which is equivalent to defining a set C parameterized by the current output y, namely $\{(x,\xi)\,|\,r(x,\xi - y)\}$. Analogously for D.
- $g(x,\xi) = Hx + \overline{g}(x,\xi - Hx)$. An equivalent characterization for g, based on the current output y, is $y + \overline{g}(x,\xi - y)$.

The first transformation is straightforward. To see the second one, note that $e^+ = \xi^+ - y^+ = g(x, e+y) - y^+ = g(x,e+y) - Hx^+ = -Hx + g(x,e+y)$. Then, the result follows by solving $\overline{g}(x,\xi - Hx) = -Hx + g(x,\xi)$. ⌟

Remark 3: It is worth to mention that (8a) and (8c) can now be compared to the dynamics of (6), by adding the effect of the error $e = \xi - y$ to the right-hand side of (6). Moreover, from Assumption 1, there exists a symmetric and positive definite matrix P_{11} such that the function $V_{11}(x) = \frac{1}{2}x^T P_{11} x$ decreases along the trajectories of (6), that is $\langle \nabla V_{11}(x), \overline{A}_{11}x\rangle \leq -x^T Q x$, for any given symmetric and positive definite matrix Q. ⌟

In what follows we work with the model (8), and we define a possible updating policy for the lazy sensors that decides when the measurements samples must be routed to the controller input, so that the stability of the networked closed -loop systems is preserved. Indeed, we propose a Lyapunov-like characterization of the updating policy, that is, we find a policy whose routing events are defined with respect to a suitable Lyapunov function, so that the point $(x,e) = (0,0)$ is asymptotically stable. Consider the following *Lyapunov-function candidate*

$$V(x,e) = \frac{1}{2}\begin{bmatrix} x \\ e \end{bmatrix}^T \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \tag{9}$$

where $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}$ is symmetric and positive definite.

Then, by denoting by $F(x,e)$ the right-hand side of (8a), the directional derivative $\langle \nabla V(x,e), F(x,e)\rangle$ of V is less than or equal to

$$-x^T Q x + x^T R_{11} x + x^T R_{12} e + e^T R_{22} e \tag{10}$$

where Q is a symmetric and positive definite matrix, still to be selected, and

$$\begin{aligned} R_{11} &= P_{12}\overline{A}_{21} \\ R_{12} &= P_{11}\overline{A}_{12} + P_{12}\overline{A}_{22} + \overline{A}_{11}^T P_{12} + \overline{A}_{21}^T P_{22} \\ R_{22} &= P_{12}^T \overline{A}_{12} + P_{22}\overline{A}_{22} \end{aligned} \tag{11}$$

Note that the existence of Q is guaranteed by Assumption 1 (See Remark 3). By denoting by $G(x,e)$ the right-hand side of (8b), the increment $V(G(x,e)) - V(x,e)$ of V is

$$xP_{12}(\overline{g}(x,e) - e) + \frac{1}{2}\overline{g}(x,e)^T P_{22}\overline{g}(x,e) - \frac{1}{2}e^T P_{22} e. \tag{12}$$

Define now

$$\overline{C} = \left\{(x,e)\,|\langle \nabla V(x,e), F(x,e)\rangle \leq -\varepsilon |x|^2\right\} \tag{13a}$$

$$\overline{D} = \left\{(x,e)\,|\langle \nabla V(x,e), F(x,e)\rangle \geq -\varepsilon |x|^2\right\} \tag{13b}$$

where ε and Q are chosen so that

$$Q - R_{11} - \varepsilon I > 0. \tag{14}$$

Then, the following theorems hold (the proofs are in Appendix A).

Theorem 1: Let $\overline{C}$ and $\overline{D}$ be defined as in (13). Under Assumption 1, for each continuous function $\overline{g}$ such that

(1) $V(G(x,e)) - V(x,e) \leq 0$ for all $(x,e) \in \overline{D}$,
(2) $(x, \overline{g}(x,e)) \notin \overline{D} \setminus \{(0,0)\}$ for all $(x,e) \in \overline{D}$,

the origin of the system $\mathcal{S}_\mathcal{N}$ of equations (8) is globally pre-asymptotically stable (GpAS).

Theorem 2: Let $\overline{C}$ and $\overline{D}$ be defined as in (13) and α be a class $\mathcal{K}$ function. Under Assumption 1, for each continuous function $\overline{g}$ such that

(1) $V(G(x,e)) - V(x,e) \leq -\alpha(|e|)$ for all $(x,e) \in \overline{D}$,

the origin of the system $\mathcal{S}_\mathcal{N}$ of equations (8) is globally pre-asymptotically stable.

Remark 4: Note that the existence of an updating policy for the lazy sensors is guaranteed by Assumption 1. In fact, the closed-loop system (4), (5) is exponentially stable, therefore it is robust with respect to small error signals e that vanish with x. Consider now (8). The dynamics of e is linear, therefore there exists a sufficiently small τ such that an updating policy that routes a new measurement sample ($e = 0$) with an intersample time not greater than τ would preserve the stability of the closed-loop system. ⌟

Remark 5: Note that the asymptotic stabilization of the point $(x, e) = (0, 0)$ can be relaxed to the asymptotic stabilization of the set $\mathcal{A} = \{(x, e) \,|\, x = 0, -\underline{c} \leq |e|_\infty \leq \overline{c}\}$, for some given $\underline{c}, \overline{c} \in \mathbb{R}_{\geq 0}$. In fact, if $\mathcal{A}$ is globally pre-asymptotically stable, then the state of $\overline{\mathcal{P}}$ is driven to zero as in (6). In such a case, we are relaxing the stabilization problem by requiring only a bounded error (e.g. a periodic non zero error). Note that stabilizing $\mathcal{A}$ instead of 0 would not affect the output of the system. ⌟

A. A possible construction for $\overline{C}$ and $\overline{D}$

By using the exponential stability property of the nominal closed-loop system, a solution to the stabilization problem of the networked closed-loop system can be constructed as follows. A candidate Lyapunov function V can be defined as

$$V = \frac{1}{2} \begin{bmatrix} x \\ e \end{bmatrix}^T \begin{bmatrix} P_{11} & 0 \\ 0 & P_{22} \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \tag{15}$$

where P_{11} and P_{22} are positive definite matrices and P_{11} satisfies

$$\overline{A}_{11}^T P_{11} + P_{11} \overline{A}_{11} \leq -Q \tag{16}$$

with Q symmetric positive definite matrix. Therefore, $\overline{C}$ and $\overline{D}$ can be defined as in equations (13), with $R_{11} = 0$, $R_{12} = P_{11}\overline{A}_{12} + \overline{A}_{21}^T P_{22}$ and $R_{22} = P_{22}\overline{A}_{22}$.

By resetting the error to zero whenever a jump occurs, that is, by defining $\overline{g}(x, e) = 0$ for all x and all e, we fullfill the requirements of both Theorems 1 and 2. Indeed,

$$V(G(x, e)) - V(x, e) = -\frac{1}{2} e^T P_{22} e \tag{17}$$

which satisfies condition (1) of both Theorem 1 and of Theorem 2. Moreover, by resetting the error to zero we have that

$$\begin{aligned} -x^T Q x + x^T R_{11} x + x^T R_{12} e + e^T R_{22} e = x^T (R_{11} - Q) x \\ < -\varepsilon x^T x, \end{aligned} \tag{18}$$

which, by (14), brings the state to the interior of $\overline{C}$, fullfilling condition (2) of Theorem 1.

It is important to note that this possible construction can be an effective model of the updating policy only if the state of the plant $\overline{\mathcal{P}}$ is known. In fact, a data is updated only if the state of the plant $\overline{\mathcal{P}}$ and the error $e = \xi - y$ characterize a configuration that do not belongs to $\overline{C}$. From a constructive point of view, we need sensors that evaluate the inequality in (13) and, based on such an evaluation, decide whether or not to update the data. Such a configuration is illustrated in Figure 2. Note that resetting e to zero is equivalent to reseting ξ to y.

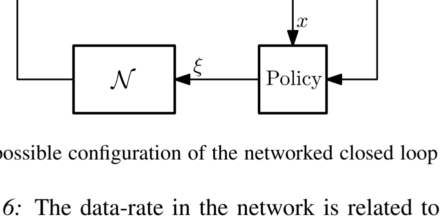


Fig. 2. A possible configuration of the networked closed loop system $\mathcal{S}_{\mathcal{N}}$.

Remark 6: The data-rate in the network is related to the definition of $\overline{C}$ and $\overline{D}$. In fact, longer flow intervals for $\mathcal{S}_{\mathcal{N}}$ guarantee lower data-rate on the network. For example, by choosing P_{22} so that $\overline{\sigma}(P_{22})$ is small, we are giving less consideration to the error e. This *naive* selection of P_{22} increases the length of the flow interval, therefore the jump rate decreases. ⌟

V. State feedback, asynchronous approach

The characterization of $\overline{C}$, $\overline{D}$ and $\overline{g}(x, e)$ of the previous section is based on the knowledge of the full state vector of the plant $\overline{\mathcal{P}}$ and of the complete error $e = \xi - y$. Such architecture needs that the sensors take into account the state x and the error e and decide whether or not to update the whole vector of (measured) output to the input vector of $\overline{\mathcal{P}}$.

In this section we propose an asynchronous updating policy in which each sensor decides autonomously its own update time. For instance, the knowledge of each sensor is limited to the state x of $\overline{\mathcal{P}}$ and to its own error, say e_i, given by $e_i = \xi_i - y_i$, where ξ_i and y_i are the ith components of ξ and y, respectively. Each sensor i decides to update ξ_i by taking into account the state vector x and the error e_i only. No shared knowledge of the state of others sensors, say $j \neq i$, is allowed.

Consider the hybrid system $\mathcal{S}_{\mathcal{N}}$ in (8). The *asynchronous behavior* of each sensor and the effect of such a behavior on the dynamics of the whole system can be modeled by the following definition of $\overline{C}$, $\overline{D}$ and $\overline{g}(x, e)$:

- $\overline{C}$ and $\overline{D}$ as the intersection and union of sets $\overline{C}_i$ and $\overline{D}_i$. For any given $i \in \{1, \ldots, q\}$, $\overline{C}_i$ or $\overline{D}_i$ are subsets of $\mathbb{R}^n \times \mathbb{R}$ whose elements are the pairs (x, e_i), where e_i is the ith component of e;
- define $\overline{g} \in \mathbb{R}^n \times \mathbb{R}^q \rightarrow \mathbb{R}^q$ as the vector $[\,\overline{g}_1(x, e_1), \ldots, \overline{g}_q(x, e_q)\,]^T$ where each $\overline{g}_i$, $i \in \{1, \ldots, q\}$, is a function in $\mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$.

Let us consider a candidate Lyapunov function V as in (9). Equations (10) and (12) characterize the directional derivative and the increment of V. For each $i \in \{1, \ldots, q\}$, define now

$$\begin{aligned} \overline{C_i} = \{(x, e_i) \,|\, & -\alpha_i x^T Q x + \alpha_i x^T R_{11} x \\ & + K_1 |x||e_i| + K_2 e_i^2 \leq -\alpha_i \varepsilon |x|^2 \,\} \\ \overline{D_i} = \{(x, e_i) \,|\, & -\alpha_i x^T Q x + \alpha_i x^T R_{11} x \\ & + K_1 |x||e_i| + K_2 e_i^2 \geq -\alpha_i \varepsilon |x|^2 \,\} \end{aligned} \tag{19}$$

where

- for each $i \in \{1, \ldots, q\}$, $\alpha_i \in \mathbb{R}_{>0}$ and $\sum_{i=1}^q \alpha_i = 1$,
- $K_1 = \max_{|x|=1, |e|=1} |x^T R_{12} e|$,
- $K_2 = \max_{|e|=1} |e^T R_{22} e|$,
- Q and ε satisfy (14).

Then, we can define $\overline{C}$ and $\overline{D}$ as follows.

$$\overline{C} = \{(x, e) \,|\, \text{for each } 1 \leq i \leq q,\ (x, e_i) \in \overline{C_i}\} \tag{20a}$$

$$\overline{D} = \{(x, e) \,|\, \text{there exists } 1 \leq i \leq q,\ (x, e_i) \in \overline{D_i}\} \tag{20b}$$

Remark 7: Note that $\overline{D}$ in (20b) is the closed complement of $\overline{C}$ in (20a). This fact and the definition of $\overline{D}$ imply that a jump occurs when at least one combination of e_i and x, $i \in \{1, \ldots, q\}$, satisfies the condition in $\overline{D_i}$. ⌟

The asynchronous behavior of the sensors is then guaranteed by assuming that each function $\overline{g}_i$, $i \in \{1, \ldots, q\}$, coincides with the identity function that maps e_i to e_i, for $(x, e_i) \notin \overline{D}_i$. In fact, suppose that a jump is enabled by the ith sensor only, that is, $(x, e_i) \in \overline{D}_i$. Then the ith sensor sends a new sample based on the value given by $\overline{g}_i(x, e_i)$, while the behavior of all the other sensors, say $j \neq i$, is given by $\overline{g}_j(x, e_j) = e_j$, that is, their value is not modified.

To state the main result of this section, in Theorem 3, we need the following technical definition.

Definition 3: For each $i \in \{1, \ldots, q\}$, $\overline{g}_i \in \mathbb{R}^n \times \mathbb{R}^q \rightarrow \mathbb{R}^q$ is a function such that

- $\overline{g}_i(x, e_i) = e_i$ if $(x, e_i) \notin \overline{D}_i$;
- the restriction of g_i on $\overline{D}_i$ is a continuous function.

Then, we say that $\overline{g} \in \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^q$ defined by $\overline{g}(x,e) = [\overline{g}_1(x,e_1), \dots, \overline{g}_q(x,e_q)]^T$ is *asynchronous*.

Theorem 3: Let $\overline{C}$ and $\overline{D}$ be defined as in (20) and α a $\mathcal{K}$ function. Under Assumption 1, for each *asynchronous* function $\overline{g}$, if for each $(x,e) \in \overline{D}$

(1) $V(G(x,e)) - V(x,e) < 0 \quad$ if $e \neq 0$,

then the origin of system $\mathcal{S}_\mathcal{N}$ (8) is globally pre-asymptotically stable.

Proof: See Appendix B. ■

A. A possible construction for $\overline{C}$ and $\overline{D}$

A solution to the stabilization problem of the networked closed loop system can be constructed as follows. Consider a candidate Lyapunov function V defined as

$$V = \frac{1}{2} \begin{bmatrix} x \\ e \end{bmatrix}^T \begin{bmatrix} P_{11} & 0 \\ 0 & P_{22} \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \tag{21}$$

where P_{11} and P_{22} are positive definite matrices, P_{11} satisfies

$$\overline{A}_{11}^T P_{11} + P_{11}\overline{A}_{11} \leq -x^T Q x \tag{22}$$

for some given positive definite and symmetric Q, and

$$P_{22} = \mathrm{diag}\{ P_{22}^{(1)}, \dots, P_{22}^{(q)} \}. \tag{23}$$

The sets $\overline{C}$ and $\overline{D}$ can be defined as in equation (20) with

$$\begin{aligned} \overline{C_i} &= \Big\{ (x,e_i) \mid -\alpha_i x^T Q x + K_1|x||e_i| + K_2 e_i^2 \leq -\alpha_i \varepsilon |x|^2 \Big\} \\ \overline{D_i} &= \Big\{ (x,e_i) \mid -\alpha_i x^T Q x + K_1|x||e_i| + K_2 e_i^2 \geq -\alpha_i \varepsilon |x|^2 \Big\} \end{aligned} \tag{24}$$

where K_1 and K_2 satisfy

$$\begin{aligned} K_1 &= \max_{|x|=1,|e|=1} |x^T(P_{11}\overline{A}_{12} + \overline{A}_{21}^T P_{22})e| \\ K_2 &= \max_{|e|=1} |e^T P_{22}\overline{A}_{22} e| \end{aligned} \tag{25}$$

By defining $\overline{g}(x,e)$ as follows

$$\overline{g}(x,e) = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_q \end{bmatrix} \text{ where } \begin{cases} v_i = 0 & \text{if } (x,e_i) \in \overline{D}_i \\ v_i = e_i & \text{otherwise} \end{cases} \tag{26}$$

we fullfill the requirements of Theorem 3. Indeed, $\overline{g}$ has an asynchronous structure because the reset of v_i to zero depends on x and e_i only, for each $i = 1, \dots, q$. Moreover,

$$\begin{aligned} V(G(x,e)) - V(x,e) &= \frac{1}{2}(\overline{g}(x,e)^T P_{22}\overline{g}(x,e) - e^T P_{22} e) \\ &= \frac{1}{2}\sum_{i=1}^{q} P_{22}^{(i)}(v_i^2 - e_i^2). \end{aligned} \tag{27}$$

Since $\overline{g}$ is applied only if the state (x,e) is in $\overline{D}$, it follows that there exists at least one $j \in \{1 \dots, q\}$ such that $v_j = 0$. Therefore,

$$V(G(x,e)) - V(x,e) \leq -\frac{1}{2}P_{22}^{(j)} e_j^2 \tag{28}$$

for some $j \in \{1, \dots, q\}$. This satisfies condition (1) of Theorem 3.

From a constructive point of view we need q sensors. Each sensor, say i, evaluates the inequality in (19), that depends only on the measured output y_i and on the state x. Based on such an evaluation, the sensor decides whether or not to update the sample ξ_i, namely whether or not to transmit its measurement. Such a configuration is illustrated in Figure 3. Note that resetting e_i to zero is equivalent to reset ξ_i to y_i.

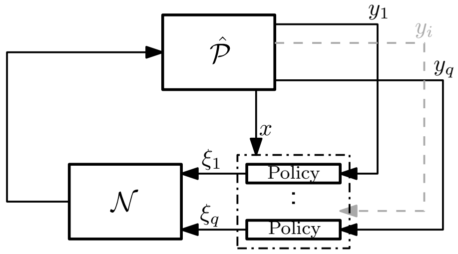


Fig. 3. A possible asynchronous configuration of the networked closed loop system $\mathcal{S}_\mathcal{N}$

Remark 8: Note that α_i can be used to increase the update-rate of a sensor with respect to the others. Indeed, a greater α_i allows for a larger error bound, therefore the update-rate decreases. Note also that each α_i can be modified at runtime. As long as $\sum_{i=0}^{q} \alpha_i = 1$, the stability is preserved. ⌟

Remark 9: In general, if $\overline{g}(x,e)$ does not depend on the state, that is, its definition does not use x to define the value of $\overline{g}(x,e)$, then we can reduce the quantity of information sent to the sensors. In fact, for each $i \in \{1, \dots, q\}$, both $\overline{C}_i$ and $\overline{D}_i$ can be redefined by using only e_i (i.e. ξ_i) and the following two signals $s_1 = x^T(-Q + R_{11})x$ and $s_2 = |x|$. ⌟

VI. Output feedback approach

Consider the nominal closed-loop system of equations (4) and (5) and assume now that the state of controller x of the $\mathcal{C}$ and of the plant $\mathcal{P}$ can only be reconstructed from the output measurements. Despite the lack of information on the state, the approach of Section IV can still be used by considering a suitable estimate of the state. We need the following assumption.

Assumption 2: The pair (A, H) in (4) is detectable.

The introduction of a classical continuous-time observer of the state in the networked closed-loop system $\mathcal{S}_\mathcal{N}$ leads to the following model

$$\begin{cases} \dot{x} = Ax + B\xi \\ \dot{\xi} = 0 \\ \dot{\hat{x}} = A\hat{x} + B\xi + L(y - H\hat{x}) \end{cases} \quad (\hat{x},\xi) \in C \text{ or } \left\| \begin{bmatrix} \hat{x} \\ \xi - H\hat{x} \end{bmatrix} \right\| \leq \rho \tag{29a}$$

$$\begin{cases} x^+ = x \\ \xi^+ = g(\hat{x},\xi) \\ \hat{x}^+ = \hat{x} \end{cases} \quad (\hat{x},\xi) \in D \text{ and } \left\| \begin{bmatrix} \hat{x} \\ \xi - H\hat{x} \end{bmatrix} \right\| \geq \rho \tag{29b}$$

$$y = Hx \tag{29c}$$

where L is the observer matrix in $\mathbb{R}^{n \times q}$ and g is a function in $\mathbb{R}^q \times \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^q$ with q dimension of the output y of $\overline{\mathcal{P}}$. C and D are subsets of $\mathbb{R}^n$ and $\rho \in \mathbb{R}_{\geq}0$. Note that the flow and jump sets of (29) can be considered as the combination of the flow and jump sets of (7) with a new condition $\left\| \begin{bmatrix} \hat{x} \\ \xi - H\hat{x} \end{bmatrix} \right\| \geq \rho$. This condition guarantees that if the estimate $\hat{x}$ and the sampling error $\xi - H\hat{x}$ are small enough (than ρ), then the system continues to flow without updating the value of the samples.

We can use the coordinate transformation $\hat{x}$, $e = \xi - H\hat{x}$ and $\eta = x - \hat{x}$ to rewrite (29) as follows:

$$\begin{cases} \dot{\hat{x}} = \overline{A}_{11}\hat{x} + \overline{A}_{12}e + LH\eta \\ \dot{e} = \overline{A}_{21}\hat{x} + \overline{A}_{22}e - HLH\eta \\ \dot{\eta} = (A - LH)\eta \end{cases} \quad (\hat{x},e) \in \overline{C} \text{ or } \left\| \begin{bmatrix} \hat{x} \\ e \end{bmatrix} \right\| \leq \rho \tag{30a}$$

$$\begin{cases} \hat{x}^+ = \hat{x} \\ e^+ = \overline{g}(\hat{x},e) \\ \eta^+ = \eta \end{cases} \quad (\hat{x},e) \in \overline{D} \text{ and } \left\| \begin{bmatrix} \hat{x} \\ e \end{bmatrix} \right\| \geq \rho \tag{30b}$$

$$y = H\hat{x} + H\eta \tag{30c}$$

where $\overline{A}_{11}$, $\overline{A}_{12}$, $\overline{A}_{21}$, $\overline{A}_{22}$ are defined as in Section IV. To extend the results of Section IV, $\overline{C}$ and $\overline{D}$ are defined as in (13), and $\overline{g}(\hat{x}, e) = M[\,\hat{x}^T \; e^T\,]^T$, where M is a matrix of dimensions $q \times (n+q)$. Then, the following theorem holds.

Theorem 4 (Global practical asymptotic stability): Suppose that the conditions of Theorem 1 or of Theorem 2 are satisfied with the state x replaced by the estimation $\hat{x}$ and with $\overline{g}(\hat{x}, e) = M[\,\hat{x}^T \; e^T\,]^T$, where M is a matrix of dimension $q \times (n+q)$. Suppose that the gain-matrix L of the observer guarantees that $\mathrm{eig}(A - LH)$ is hurwitz.

Then, there exists a $\overline{\gamma} \in \mathbb{R}_{>0}$ such that for any given ρ in (30), there exists a set $\mathcal{A} \subseteq \overline{\gamma}\rho\mathbb{B} \subset \mathbb{R}^{n+q}$, such that $\mathcal{A} \times \{0\} \subset \mathbb{R}^{n+q} \times \mathbb{R}^n$ is globally pre-asymptotically stable.

Proof: See Appendix C. ■

Corollary 1: If the conditions of Theorem 4 are satisfied, then each solution $(x, \xi, \hat{x})$ to (29) is such that $\hat{x}$ converges to x and $(x, \xi - y)$ converges to a ball of radius $\overline{\gamma}\rho\mathbb{B}$.

Proof: Note that the union of the flow set and of the jump set of (30) coincides with the whole state-space $\mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}^n$. Thus, from any given initial condition, each maximal solution is a complete solution. The coordinate transformation $(x, \xi, \hat{x}) \to (\hat{x}, e, \eta)$ is invertible, therefore the convergence of η of (30) to 0 implies that $\hat{x}$ converges to x. Thus, the convergence of $(\hat{x}, e)$ to $\overline{\gamma}\rho\mathbb{B}$ implies $(x, \xi - Hx)$ converges to $\overline{\gamma}\rho\mathbb{B}$. ■

Remark 10: g, C and D of (29) can be constructed from $\overline{g}$, $\overline{C}$ and $\overline{D}$ as follows.

- Suppose $\overline{C} = \{(\hat{x}, e) \,|\, r(\hat{x}, e)\}$ where r is a given relation on $\hat{x}$ and e. Then, $C = \{(\hat{x}, \xi) \,|\, r(\hat{x}, \xi - H\hat{x})\}$. For D is the same.
- $g(\hat{x}, \xi) = H\hat{x} + \overline{g}(\hat{x}, \xi - H\hat{x})$. In fact, $e^+ = \xi^+ - H\hat{x}^+ = g(\hat{x}, \xi) - H\hat{x} = \overline{g}(\hat{x}, \xi - H\hat{x})$, where the last equality follows from (30b). ┘

Remark 11: The result of Theorem 4 extends to the asynchronous case in Section V but it requires that the output $\hat{x}$ of the observer is shared among the sensors, thus breaking the decentralized structure of that approach. ┘

VII. Simulation Example

We consider the following exponentially unstable linear plant $\mathcal{P}$ defined as follows

$$\mathcal{P} = \begin{cases} \dot{x}_p &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_p + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u_p \\ y_p &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_p. \end{cases} \tag{31}$$

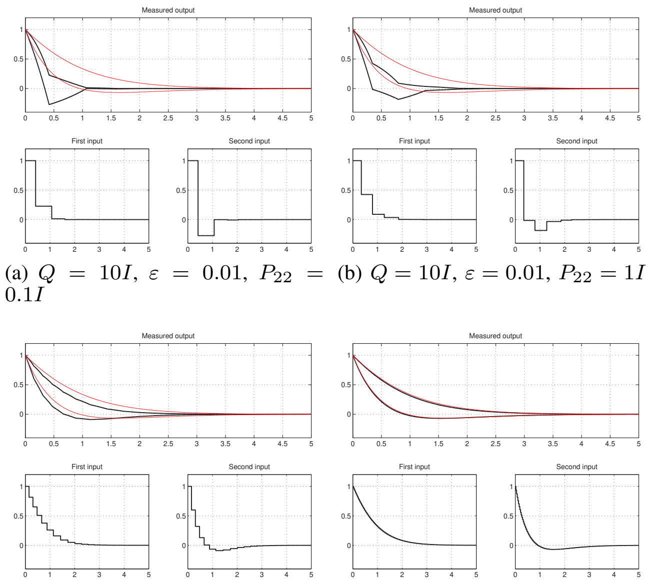
The nominal closed-loop system is constructed by connecting the plant $\mathcal{P}$ to the following LQR static controller $\mathcal{C}$.

$$y_c = \begin{bmatrix} -2.1961 & -0.7545 \\ -0.7545 & -2.7146 \end{bmatrix} u_c. \tag{32}$$

through the interconnection $u_p = y_c$ and $u_c = y_p$. With this controller the nominal closed-loop system is exponentially stable.

In the networked closed-loop system, the interconnection $u_c = y_p$ is replaced by $u_c = \xi$ where ξ is the vector of samples of the measured output that the controller is currently using. The vector of samples ξ is updated by following the policy defined in section IV-A. Four simulations tests with different values of the parameters are reported in Figure 4.

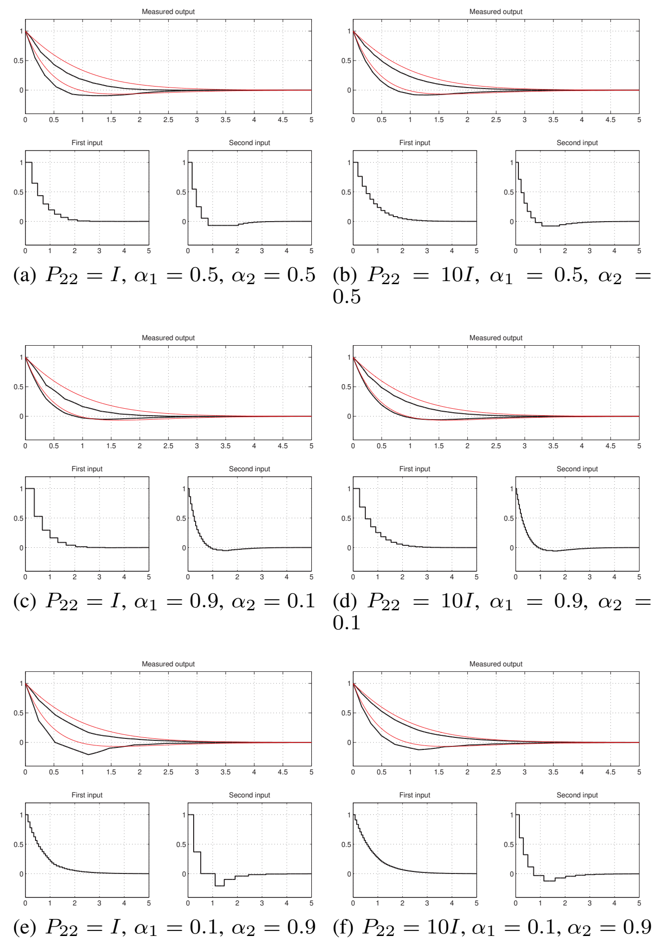
Consider now the closed-loop system given by equation (31) and (32). In this example we use the asynchronous policy of section V-A, The results are illustrated in Figure 5, where six diferent parameters values are used. Note that each sensor i updates the measured output sample ξ_i without any kind of synchronization with the other sensors. Moreover, by choosing different α_1 and α_2, we force one sensor to allow for a larger error bound on $e_i = \xi_i - y_i$ before forcing an update. Therefore, one sensor will reset its state ξ_i more frequently than the other.



(a) $Q = 10I$, $\varepsilon = 0.01$, $P_{22} = 0.1I$ (b) $Q = 10I$, $\varepsilon = 0.01$, $P_{22} = 1I$

(c) $Q = 10I$, $\varepsilon = 0.01$, $P_{22} = 10I$ (d) $Q = 10I$, $\varepsilon = 0.01$, $P_{22} = 100I$

Fig. 4. Input and output of $\mathcal{S}_{\mathcal{N}}$ in the synchronous case, for different choices of P_{22}. The thin line in each figure is the output of the nominal closed loop.



(a) $P_{22} = I$, $\alpha_1 = 0.5$, $\alpha_2 = 0.5$ (b) $P_{22} = 10I$, $\alpha_1 = 0.5$, $\alpha_2 = 0.5$

(c) $P_{22} = I$, $\alpha_1 = 0.9$, $\alpha_2 = 0.1$ (d) $P_{22} = 10I$, $\alpha_1 = 0.9$, $\alpha_2 = 0.1$

(e) $P_{22} = I$, $\alpha_1 = 0.1$, $\alpha_2 = 0.9$ (f) $P_{22} = 10I$, $\alpha_1 = 0.1$, $\alpha_2 = 0.9$

Fig. 5. Input and output of $\mathcal{S}_{\mathcal{N}}$ in the asynchronous case, for different choices of P_{22}. The thin line in each figure is the output of the nominal closed-loop system. Extra paramters common to all cases are parameters are $Q = 10I$ and $\varepsilon = 0.01$.

APPENDIX

A. Proofs of Theorems 1 and 2.

System of equations (8), with $\overline{C}$ and $\overline{D}$ defined as in Section IV or in Section V, and the function g continuous or asynchronous (Definition 3) satisfies the *basic assumptions* of [6]. Therefore, the following result from [6] can be used to prove Theorems 1, 2 and 3.

We denote with $F(x,e)$ the right-hand side of (8a) and we denote with $G(x,e)$ the right-hand side of (8b).

Proposition 1: [6, Theorem 23]
For a system of equations (8), with C and D closed sets and g continuous or asynchronous (Definition 3), if

$$\begin{aligned} \langle \nabla V(x,e), F(x,e)\rangle \leq 0 \quad & \text{for all } x \in C \setminus \{(0,0)\} \\ V(G(x,e)) - V(x,e) \leq 0 \quad & \text{for all } x \in D \setminus \{(0,0)\} \end{aligned} \tag{33}$$

then $(x,e) = (0,0)$ is stable. Moreover, if there exists a compact neighborhood K of $(x,e) = (0,0)$ such that, for each $\mu > 0$, no complete solutions to $\mathcal{S}_\mathcal{N}$ remain in the set $\{(x,e) \,|\, V(x,e) = \mu\} \cap K$, then $(x,e) = (0,0)$ is pre-asymptotically stable. Finally, $(x,e) = (0,0)$ is globally pre-asymptotically stable if K can be arbitrarily large and the set $\{(x,e) \,|\, V(x,e) \leq \mu\}$ is compact.

Proof: Theorem 1.
From the definition of $\overline{C}$ in (13a) and by condition (1) of the theorem,

$$\begin{aligned} \langle \nabla V(x,e), F(x,e)\rangle \leq 0 \quad & \text{for all } (x,e) \in \overline{C} \\ V(G(x,e)) - V(x,e) \leq 0 \quad & \text{for all } (x,e) \in \overline{D}. \end{aligned} \tag{34}$$

Moreover, (13) and condition (2) of the theorem guarantee that for each $(x,e) \in \overline{D}$, $(x, \overline{g}(x,e))$ belongs to the interior of $\overline{C}$ or $(x, \overline{g}(x,e)) = (0,0)$. For the first case, it follows that there exists a compact interval, say $[0,\bar{t}]$, with $\bar{t} \in \mathbb{R}_{>0}$, in which the system can only flow.

Consider now a state (x,e) in $\overline{C}$ and such that $V(x,e) = \mu$, for some $\mu \in \mathbb{R}_{>0}$. By (13a), $x \neq 0$, then $\langle \nabla V(x,e), F(x,e)\rangle \leq -\varepsilon|x|^2$. It follows that no complete solutions to $\mathcal{S}_\mathcal{N}$, from $x \neq 0$ remains in the set $\{(x,e) \,|\, V(x,e) = \mu\}$, for any given $\mu > 0$.

Suppose now (x,e) belongs to $\overline{C}$ but $x = 0$ and $e \neq 0$. By (8a), the continuous dynamics of x is driven by $\dot{x} = \overline{A}_{12}e$. It follows that $x(t) = 0$ cannot be a solution to $\mathcal{S}_\mathcal{N}$ in the interval $t \in [0,\bar{t}]$. Therefore,

- if (x,e) is an interior point of $\overline{C}$ then there exist a time $t \in [0,\bar{t}]$ such that $x(t) \neq 0$, from which $\langle \nabla V(x(t),e(t)), F(x(t),e(t))\rangle \leq -\varepsilon|x(t)|^2$,
- if (x,e) is on the border of $\overline{C}$ two cases are possible: a jump occurs, that forces the state of the system in the interior of $\overline{C}$, or there exists a compact interval $[0,\bar{t}]$ in which the system can flow. In such case, there exist a time $t \in [0,\bar{t}]$ such that $\langle \nabla V(x(t),e(t)), F(x(t),e(t))\rangle \leq -\varepsilon|x(t)|^2$.

It follows that, no complete solutions to $\mathcal{S}_\mathcal{N}$, from $x = 0$ and $e \neq 0$ remains in the set $\{(x,e) \,|\, V(x,e) = \mu\}$, for any given $\mu > 0$.

By Proposition 1, $(x,e) = (0,0)$ is global pre-asymptotically stable ∎

Proof: Theorem 2.
From (13a) and (1) of the theorem, (34) hold also for Theorem 2.

Suppose that (x,e) belongs to $\overline{D}$ and $V(x,e) = \mu$, for some given $\mu > 0$. By (i), if $e \neq 0$ then $V(G(x,e)) - V(x,e) \leq -\alpha(|e|)$ therefore no complete solutions to $\mathcal{S}_\mathcal{N}$, from $e \neq 0$ remains in the set $\{(x,e) \,|\, V(x,e) = \mu\}$. Moreover, inequality (14) guarantees that if $x \neq 0$ and $e = 0$ then (x,e) cannot belong to $\overline{D}$. For instance, let $x \neq 0$ and $e = 0$ then

$$\begin{aligned} -x^TQx + x^TR_{11}x + x^TR_{12}e + e^TR_{22}e = \\ = -x^TQx + x^TR_{11}x < -\varepsilon|x|^2 \end{aligned} \tag{35}$$

therefore (x,e) belongs to the interior of $\overline{C}$ and the system flows only.

The analysis of the continuous dynamics of $\mathcal{S}_\mathcal{N}$ follows the line of the proof of Theorem 1. It follows that no complete solutions to $\mathcal{S}_\mathcal{N}$ from $(x,e) \in \overline{C}$ remains in the set $\{(x,e) \,|\, V(x,e) = \mu\}$, for any given $\mu > 0$. By Proposition 1, it follows that $(x,e) = (0,0)$ is global pre-asymptotically stable. ∎

B. Proof of Theorem 3.

By (1) of Theorem 3, for all $(x,e) \in \overline{D}$

$$V(G(x,e)) - V(x,e) < 0 \quad \text{if } e \neq 0. \tag{36}$$

From the definitions of $\overline{C_i}$ and $\overline{C}$ in (19) and (20a), we have that $\langle \nabla V(x,e), F(x,e)\rangle$ is equal to

$$= -x^TQx + x^TR_{11}x + x^TR_{12}e + e^TR_{22}e \tag{37a}$$

$$\leq -x^TQx + x^TR_{11}x + K_1|x||e| + K_2e^Te \tag{37b}$$

$$\leq -x^TQx + x^TR_{11}x + K_1|x|\sum_{i=1}^{q}|e_i| + K_2\sum_{i=1}^{q}e_i^2 \tag{37c}$$

$$\leq \sum_{i=1}^{q}\Big(-\alpha_ix^TQx + \alpha_ix^TR_{11}x + K_1|x||e_i| + K_2e_i^2\Big) \tag{37d}$$

$$\leq \sum_{i=1}^{q} -\alpha_i\varepsilon|x|^2 \leq -\varepsilon|x|^2 \quad \text{for all } (x,e) \in \overline{C}. \tag{37e}$$

The inequality between (37a) and (37b) follows from the definition of K_1 and K_2. The inequality between (37b) and (37c) follows from the fact that $|e| \leq \sum_{i=1}^{q}|e_i|$, where e_i is the ith component of e, for each $i \in \{1,\ldots,q\}$. (37d) follows from (37c) by $\sum_{i=1}^{q}\alpha_i = 1$. Finally, from $\overline{C_i}$ and $\overline{C}$, the argument of the sum in (37d) can be written as (37e). It follows that (33) holds.

Suppose now (x,e) belongs to $\overline{D}$ and $V(x,e) = \mu$, for some given $\mu > 0$. If $e \neq 0$ then (36) holds and no complete solutions to $\mathcal{S}_\mathcal{N}$ from $e \neq 0$ remains in the set $\{(x,e) \,|\, V(x,e) = \mu\}$. If $x \neq 0$ and $e = 0$ then for each $i \in \{1,\ldots,q\}$

$$\begin{aligned} -\alpha_ix^TQx + \alpha_ix^TR_{11}x + K_1|x||e_i| + K_2e_i^2 = \\ = -\alpha_ix^TQx + \alpha_ix^TR_{11}x < -\alpha_i\varepsilon|x|^2 \end{aligned} \tag{38}$$

where the last inequality follows from (14). It follows that (x,e) cannot belong to $\overline{D}$, it belongs to the interior of $\overline{C}$ and the system flows only.

The analysis of the continuous dynamics of $\mathcal{S}_\mathcal{N}$ follows the line of the proof of Theorem 1. It follows that no complete solutions to $\mathcal{S}_\mathcal{N}$ from $(x,e) \in \overline{C}$ remains in the set $\{(x,e) \,|\, V(x,e) = \mu\}$, for any given $\mu > 0$. By Proposition 1, it follows that $(x,e) = (0,0)$ is global pre-asymptotically stable. ∎

C. Proof of Theorem 4.

We need the following definition.

Definition 4: For each $p \in \mathbb{N}$, a function $\sigma : \mathbb{R}^p \to \mathbb{R}$ is said to be *homogeneous with degree* $\delta \in \mathbb{R}$ if, for all $z \in \mathbb{R}^p$ and $\lambda > 0$, $\sigma(\lambda z) = \lambda^\delta\sigma(z)$.

From Theorem 1, 2 or 3 we know that

$$\begin{cases} \dot{\hat{x}} = \overline{A}_{11}\hat{x} + \overline{A}_{12}e \\ \dot{e} = \overline{A}_{21}\hat{x} + \overline{A}_{22}e \end{cases} \quad (\hat{x},e) \in \overline{C} \tag{39a}$$

$$\begin{cases} \hat{x}^+ = \hat{x} \\ e^+ = \overline{g}(\hat{x},e) \end{cases} \quad (\hat{x},e) \in \overline{D} \tag{39b}$$

$$y = H\hat{x} \tag{39c}$$

is GpAS. Then, define $z = \begin{bmatrix}\hat{x} \\ e\end{bmatrix}$, $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{bmatrix}$ and $G(z) = \begin{bmatrix}\hat{x} \\ \overline{g}(\hat{x},e)\end{bmatrix}$. From the *homogeneity* of (39) (e.g. [14], continuous and discrete dynamics are defined by linear vector field and $\overline{C}$ and $\overline{D}$ are cones) and from [2, Theorem 7.9] and [14, Theorem 2], there exists a function $\overline{V} : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}_{\geq 0}$ that is smooth on $\mathbb{R}^n \times \mathbb{R}^q \setminus \{0\}$ and homogeneous with degree $\delta \in \mathbb{R}$ such that,

$$\alpha_1(|z|) \leq \overline{V}(z) \leq \alpha_2(|z|) \qquad \forall z \in \mathbb{R}^n \times \mathbb{R}^p \tag{40a}$$
$$\langle \nabla \overline{V}(z), Az \rangle \leq -\mu \overline{V}(z) \qquad \forall z \in \overline{C} \tag{40b}$$
$$\overline{V}(G(z)) \leq \nu \overline{V}(z) \qquad \forall z \in \overline{D} \tag{40c}$$

where $\mu > 0$, $\nu \in (0,1)$ and α_1, $\alpha_2 \in \mathcal{K}_\infty$.

Consider $\delta = 2$, then for each $z, w \in \mathbb{R}^n \times \mathbb{R}^p$,

$$\begin{aligned} \langle \nabla \overline{V}(z), w \rangle &= \lim_{h \to 0} \frac{\overline{V}(z + hw) - \overline{V}(z)}{h} \\ &= \lim_{h \to 0} \frac{\overline{V}(|z|\frac{z}{|z|} + |z|\frac{h}{|z|}w) - \overline{V}(|z|\frac{z}{|z|})}{|z|\frac{h}{|z|}} \\ &= \lim_{h \to 0} \frac{|z|^2}{|z|} \frac{\overline{V}(\frac{z}{|z|} + \frac{h}{|z|}w) - \overline{V}(\frac{z}{|z|})}{\frac{h}{|z|}} \end{aligned} \tag{41}$$

where the last equality is the result of the homogeneity of V. Since w is arbitrary, for any $z \neq 0$, $\nabla \overline{V}(z) = |z| \nabla \overline{V}(\frac{z}{|z|})$. Since V is smooth, $|\nabla \overline{V}(z)| \leq \lambda |z|$, where $\lambda = \max_{|z|=1} \overline{V}(z)$. Note that $\alpha_1(1)|z|^2 \leq |z|^2 \overline{V}(\frac{z}{|z|}) \leq \alpha_2(1)|z|^2$.

With these tools we can now prove the global *practical* asymptotic stability of (30). Consider $\gamma \in \mathbb{R}_{>0}$, $\gamma \ll 1$ and take $\ell = \alpha_2(\rho + \gamma\rho)$, that implies, $\{z \,|\, |z| \leq \rho\} \subseteq \{z \,|\, \overline{V}(z) < \ell\}$ and consider the compact set $\mathcal{A} = \{z \,|\, \overline{V}(z) \leq \ell\} \times \{0\}$. We prove that $\mathcal{A}$ is globally pre-asymptotically stable for (30). Define the candidate Lypunov function $V(z, \eta)$ as follows.

$$V(z,\eta) = \begin{cases} \overline{V}(z) - \ell + \frac{1}{2}\eta^T \overline{P} \eta & \overline{V}(z) \geq \ell \\ \frac{1}{2}\eta^T \overline{P} \eta & \text{otherwise} \end{cases} \tag{42}$$

where $\overline{P}$ is a positive definite symmetric matrix of dimension $n \times n$. Note that V is continuous in $\mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}^n$, 0 in $\mathcal{A}$ and smooth for points $(z, \eta) \in \mathbb{R}^{n+q} \times \mathbb{R}^n$ such that $\overline{V}(z) \neq \ell$. It is locally Lipschitz for points $(z, \eta) \in \mathbb{R}^{n+q} \times \mathbb{R}^n$ such that $\overline{V}(z) = \ell$. For such points, say $(\overline{z}, \overline{\eta})$, we consider the generalized gradient (in the sense of Clarke) of V, that coincides with the convex hull of all limits of sequences $\nabla V(z_i, \eta_i)$ where (z_i, η_i), $i \in \mathbb{N}$, is any sequence converging to $(\overline{z}, \overline{\eta})$ while avoiding an arbitrary set of measurement zero containing all the points at which V is not differentiable [11].

Define $B = \begin{bmatrix} LH \\ -HLH \end{bmatrix}$ and consider (30a). The directional derivative of V is less then or equal to

$$\begin{cases} v_1 = \langle \nabla \overline{V}(z), Az + B\eta \rangle + \eta^T \overline{P}(A - LH)\eta & \overline{V}(z) > \ell \\ v_2 = \eta^T \overline{P}(A - LH)\eta & \overline{V}(z) < \ell \\ v_3 \in \overline{\text{co}}\{v_1, v_2\} & \text{otherwise} \end{cases}$$

By Assumption 2, $\eta^T \overline{P}(A - LH)\eta \leq \eta^T \overline{Q} \eta$, where $\overline{Q}$ is a negative definite symmetric matrix of dimension $n \times n$. $\overline{Q}$ will be defined below to guarantee negativity of the derivative of V. **(i)** Consider the case $z \in \overline{C}$.

$$\begin{aligned} v_1 &\leq -\mu \overline{V}(z) + \lambda |B||z||\eta| - \eta^T \overline{Q} \eta \\ &\leq \left(-\mu\alpha_1(1) + \varepsilon^2\right)|z|^2 + \left(\frac{\lambda |B|}{\varepsilon^2} - \lambda_{min}(\overline{Q})\right)|\eta|^2 \\ v_2 &\leq -\eta^T \overline{Q} \eta \end{aligned}$$

Therefore v_1 is strictly negative in $\{(z,\eta) \,|\, z \in \overline{C} \text{ or } |z| \leq \rho\} \setminus \mathcal{A}$ for $\varepsilon^2 < \mu\alpha_1$ and $\lambda_{min}(\overline{Q}) > \frac{\lambda|B|}{\varepsilon^2}$. v_2 is strictly negative in $\{(z,\eta) \,|\, z \in \overline{C} \text{ or } |z| \leq \rho\} \setminus \mathcal{A}$ by the fact that, when $\eta = 0$, $z \in \overline{C}$ and $\overline{V}(z) \leq \ell$ imply $z \in \mathcal{A}$. **(ii)** Consider the case $z \notin \overline{C}$. Thus, $|z| \leq \rho$. In this case, $V(z) < \ell$ therefore the directional derivative of V is less then or equal to v_2, that is, it is negative in $\{(z,\eta) \,|\, z \in \overline{C} \text{ or } |z| \leq \rho\} \setminus \mathcal{A}$.

Consider now (30b). Then,

$$V(z^+, \eta^+) - V(z, \eta) \leq (\nu - 1)\overline{V}(z) \leq -(1-\nu)\alpha_1(1)|z|^2$$

that is negative in $\{(z,\eta) \,|\, z \in \overline{D} \text{ and } |z| \geq \rho\} \setminus \mathcal{A}$ by the fact that $|z| \geq \rho$. Then, by [11, Theorem 7.6] and [11, Corollary 7.7] the set $\mathcal{A} \times \{0\}$ is globally pre-asymptotically stable.

Note that $\mathcal{A} \times \{0\} \subseteq \alpha_1^{-1}(\alpha_2(\rho + \gamma\rho))\mathbb{B} \times \{0\}$. By the fact that $\alpha_i(s) = |s|^2\alpha_i(1)$, for $i \in \{1, 2\}$, it folllows that $\alpha_1^{-1}(s) = \left(\frac{s}{\alpha_1(1)}\right)^{\frac{1}{2}}$. Then, $\alpha_1^{-1}(\alpha_2(\rho + \gamma\rho)) = \left(\frac{\alpha_2(1)}{\alpha_1(1)}\right)^{\frac{1}{2}}(\rho + \gamma\rho)$, that is, $\overline{\gamma} = (1+\gamma)\left(\frac{\alpha_2(1)}{\alpha_1(1)}\right)^{\frac{1}{2}}$. ■

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