POP23 - Future Trends in Polynomial Optimization

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Pure states for polynomial nonnegativity certificates in the presence of zeros

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(joint work with Luis Felipe Vargas)
Definition 1 \[ R[x] := R[x_1, \ldots, x_n] \] polynomial ring

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\( M \) is called a **quadratic module** of \( R[x] \) if \( \forall \in M, M + M \subseteq M \) and \( R[x]^2 M \subseteq M \).
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\( M \) is called a **quadratic module** of \( R[x] \) if \( \lambda \in M, M + M \leq M \) and \( R[x]^2 M \leq M \).

A quadratic module \( M \) of \( R[x] \) is called **Archimedean** if \( M + \mathbb{Z} = R[x] \).
Proposition 2  Let $M$ be a quadratic module of $\mathbb{R}[x]$.

$M$ Archimedean $\iff \exists N \in \mathbb{N}: N - \sum_{i=1}^{n} x_i^2 \in M$
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Example 3  

\[ S^{n-1} := \{ x \in \mathbb{R}^n \mid \| x \| = 1 \} \] \quad \text{sphere}

\[ B^n := \{ x \in \mathbb{R}^n \mid \| x \| \leq 1 \} \] \quad \text{ball}
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$M_{S^{n-1}} := \sum \mathbb{R}[x]^2 + \mathbb{R}[x] \left( 1 - \sum_{i=1}^{n} x_i^2 \right)$ is an Archimedean quadratic module with $S(M_{S^{n-1}}) = S^{n-1}$. 
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$M_{B^n} := \sum \mathbb{R}[x]^2 + \sum \mathbb{R}[x]^2 (1 - \sum_{i=1}^{n} x_i^2)$ is an Archimedean quadratic module with $S(M_{B^n}) = B^n$. 
Theorem 4 (Putinar, 1993) Let $M$ be an Archimedean quadratic module and $p \in \mathbb{R}[x]$. Then

\[ p > 0 \text{ on } S(M) \implies p \in M. \]

What if $p \geq 0$ on $S(M)$?
Theorem 4 (Putinar, 1993) Let $M$ be an Archimedean quadratic module and $p \in \mathbb{R}[x]$. Then

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Example 5

$p := (x_1^2 + \ldots + x_5^2)^2 - 4(x_1^2 x_3^2 + x_1^2 x_4^2 + x_2^2 x_4^2 + x_2^2 x_5^2 + x_3^2 x_5^2) 
\in \mathbb{R}[x] = \mathbb{R}[x_1, \ldots, x_5] \quad \text{Horn form}$

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\[ p \geq 0 \text{ on } \mathbb{R}^5 \]

What if $p \geq 0$ on $S(M)$?
**Theorem 4** (Putinar, 1993) Let \( M \) be an Archimedean quadratic module and \( p \in \mathbb{R}[x] \). Then
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**Example 5** \( p := \left(x_1^2 + \ldots + x_5^2\right)^2 - 4\left(x_1^2 x_3^2 + x_1 x_4^2 + x_2^2 x_4^2 + x_2^2 x_5^2 + x_3^2 x_5^2\right) \in \mathbb{R}[x] = \mathbb{R}[x_1, \ldots, x_5] \) Horn form
\[
p \geq 0 \text{ on } \mathbb{R}^5
\]
\[
p(x_1, 0, x_3, x_4, 0) = (x_1^2 + x_2^2 + x_3^2) - 4(x_1^2 x_3^2 + x_1 x_4^2) = (x_1^2 - x_3^2 - x_4^2)^2
\]
\( p \) has infinitely many zeros on \( S^4 \).

*What if \( p \geq 0 \text{ on } S(M) \)?*
Theorem 4 (Putinar, 1993) Let $M$ be an Archimedean quadratic module and $p \in \mathbb{R}[x]$. Then $p > 0$ on $S(M) \implies p \in M$.

Example 5 $p := (x_1^2 + \ldots + x_5^2)^2 - 4(x_1 x_3^2 + x_2 x_4^2 + x_2 x_4^2 + x_2^2 x_5^2 + x_3 x_5^2) 
\in \mathbb{R}[x] = \mathbb{R}[x_1, \ldots, x_5]$ Horn form

$p \geq 0$ on $\mathbb{R}^5$
$p(x_1, 0, x_3, x_4, 0) = (x_1^2 + x_2^2 + x_3^2) - 4(x_1 x_3^2 + x_2 x_4^2) = (x_1^2 - x_3^2 - x_4^2)^2$

$p$ has infinitely many zeros on $S^4$.

$p \in M_{S^4}$, $p \notin M_{B^5}$

What if $p \geq 0$ on $S(M)$?
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Many partial answers but mainly in the case where $p$ has only finitely many zeros on $S(M)$.


Marshall 2006 \& Annales de la Faculté des sciences de Toulouse

Nie 2014
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Many partial answers but mainly in the case where $p$ has only finitely many zeros on $S(M)$.

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Here we will deal with the case of infinitely many zeros!

I don’t tell you what is a pure state since we have introduced test states.
Definition 6

Let $I$ be an ideal and $M$ be a quadratic module of $R[x]$. Let $v \in I$ and $a \in R^n$. We call $\psi : I \to R$ a test state on $I$ for $M$ at $a$ with respect to $v$ if

- $\psi(v) = 1$,
- $\psi(I \cap M) \subseteq R_{\geq 0}$,
- $\forall p, q \in I : \psi(p+q) = \psi(p) + \psi(q)$ and
- $\forall p \in R[x] : \forall q \in I : \psi(pq) = p(a) \psi(q)$.

https://arxiv.org/abs/2310.12853
**Definition 6** Let $I$ be an ideal and $M$ be a quadratic module of $\mathbb{R}[x]$. Let $\nu \in I$ and $\alpha \in \mathbb{R}^n$. We call $\psi : I \to \mathbb{R}$ a test state on $I$ for $M$ at $\nu$ wrt. $\alpha$ if
- $\psi(\nu) = 1$,
- $\psi(I \cap M) \subseteq \mathbb{R}_{\geq 0}$,
- $\forall p, q \in I : \psi(p+q) = \psi(p) + \psi(q)$ and
- $\forall p \in R[x] : \forall q \in I : \psi(pq) = p(\alpha) \psi(q)$.

**Example 7**

$n = 1$, $\mathbb{R}[x] = \mathbb{R}[x_1]$
$I = \mathbb{R}[x]$
$M = M_{_{\beta}}$
$\alpha = 0$
$\nu = 1$

If $\psi : I \to \mathbb{R}$ is a test state for $M$ at $\nu$, then $\psi(p) = \psi(p\nu) = p(0) \psi(\nu) = p(0)$ for all $p \in R[x]$. Conversely, this defines a test state. Evaluation at zero is the only test state!
Definition 6 Let $I$ be an ideal and $M$ be a quadratic module of $\mathbb{R}[x]$. Let $u \in I$ and $a \in \mathbb{R}^n$. We call $\gamma : I \to \mathbb{R}$ a test state on $I$ for $M$ at $u$ if:
- $\gamma(u) = 1$,
- $\gamma(I \cap M) \subseteq \mathbb{R}_{\geq 0}$,
- $\forall p, q \in I : \gamma(p+q) = \gamma(p) + \gamma(q)$ and
- $\forall p \in \mathbb{R}[x] : \forall q \in I : \gamma(pq) = p(a) \gamma(q)$.

Example 8

$n = 1$, $\mathbb{R}[x] = \mathbb{R}[x_1]$

$I = \mathbb{R}[x] \times\{x\} \times\{a\}$

$M = M_{\mathbb{R}^n}$

$a = 0$

$u = x$

If $\gamma : I \to \mathbb{R}$ is a test state for $M$ at $u$, then $\gamma(x) = \gamma(px) = p(0) \gamma(u) = p(0)$ for all $p \in \mathbb{R}[x]$. Conversely, this defines a test state, derivative at zero is the only test state!
**Definition 6** Let $I$ be an ideal and $M$ be a quadratic module of $\mathbb{R}[x]$. Let $v \in I$ and $a \in \mathbb{R}^n$. We call $\Psi : I \rightarrow \mathbb{R}$ a \textit{test state} on $I$ for $M$ at $a$ wrt. $v$ if

- $\Psi(v) = 1$,
- $\Psi(I \cap M) \subseteq \mathbb{R}_{\geq 0}$,
- $\forall p, q \in I : \Psi(p + q) = \Psi(p) + \Psi(q)$ and
- $\forall p \in \mathbb{R}[x] : \forall q \in I : \Psi(pq) = \Psi(p) \Psi(q)$.

**Example 9**

$n$ arbitrary

$I = \sum_{i,j=1}^{n} \mathbb{R}[x] x_i x_j$

$M = M_{\mathbb{R}}^n$

$a = 0$

$U = x_1^2 + \ldots + x_n^2$

If $\Psi : I \rightarrow \mathbb{R}$ is a test state, then

$$A := (\Psi(x_i x_j))_{1 \leq i, j \leq n}$$

is psd and

$$\Psi(p) = \text{tr} \left( (\nabla^2 f(0)) A \right)$$

for all $p \in \mathbb{R}[x]$. Up to a positive constant, the test states are exactly the non-zero conic combinations of second directional derivatives at 0.
Definition 6 Let $I$ be an ideal and $M$ be a quadratic module of $\mathbb{R}[x]$. Let $v \in I$ and $a \in \mathbb{R}^n$. We call $\mathcal{Y} : I \rightarrow \mathbb{R}$ a test state on $I$ for $M$ at $a$ wrt. $v$ if

- $\mathcal{Y}(v) = 1$,
- $\mathcal{Y}(I \cap M) \subseteq \mathbb{R}_{\geq 0}$,
- $\forall p, q \in I : \mathcal{Y}(p+q) = \mathcal{Y}(p) + \mathcal{Y}(q)$ and
- $\forall p \in \mathbb{R}[x] : \forall q \in I : \mathcal{Y}(pq) = p(a) \mathcal{Y}(q)$.

Warning 10 We do not think that test states always have a nice geometric interpretation. Although they are associated with a point $a \in \mathbb{R}^n$, we think that they are of algebraic nature in general.
Theorem 11

Let \( F \subseteq R[x] \) generate the ideal \( I \).

Let \( M \) be an Archimedean quadratic module of \( R[x] \) and \( f, v \in I \). Suppose that

In particular, \( f \in M \).
Theorem 11

Let $F \subseteq \mathbb{R}[x]$ generate the ideal $I$. Let $M$ be an Archimedean quadratic module of $\mathbb{R}[x]$ and $f, u \in I$. Suppose that

(a) $f \geq 0$ on $S(M)$

In particular, $f \in M$. 
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(a) $f \geq 0$ on $S(M)$

(b) $\forall a \in S(M) : (f(a) = 0 \Rightarrow u(a) = 0)$

(c) $\cup M \subseteq M$

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Theorem 11. Let $F \subseteq \mathbb{R}[x]$ generate the ideal $I$. Let $M$ be an Archimedean quadratic module of $\mathbb{R}[x]$ and $f, u \in I$. Suppose that

(a) $f \geq 0$ on $S(M)$
(b) $\forall a \in S(M) : (f(a) = 0 \Rightarrow u(a) = 0)$
(c) $uM \subseteq M$
(d) $u$ is $F$-stably contained in $M$, i.e., $\forall f \in F : \exists \varepsilon > 0 : u \pm \varepsilon f \in M$

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(b) \( \forall a \in S(M) : (f(a) = 0 \Rightarrow v(a) = 0) \)

(c) \( uM \subseteq M \)

(d) \( u \) is \( F \)-stably contained in \( M \),
i.e., \( \forall f \in F : \exists \epsilon > 0 : u + \epsilon f \in M \)

(e) \( \forall(f) > 0 \) for all zeros \( a \) of \( f \) on \( S(M) \)
and all test states \( \forall \) on \( I \) for \( M \) at \( a \) wrt. \( v \).

In particular, \( f \in M \).
Let $F \subseteq R[x]$ generate the ideal $I$. Let $M$ be an Archimedean quadratic module of $R[x]$ and $f, u \in I$. Suppose that

(a) $f \geq 0$ on $S(M)$

(b) $\forall a \in S(M) : (f(a) = 0 \implies u(a) = 0)$

(c) $uM \subseteq M$

(d) $u$ is $F$-stably contained in $M$, i.e., $\forall f \in F : \exists \varepsilon > 0 : u \pm \varepsilon f \in M$

(e) $\mathcal{V}(f) > 0$ for all zeros $a$ of $f$ on $S(M)$ and all test states $\mathcal{V}$ on $I$ for $M$ at $a$ wrt. $u$.

Then, there is $\varepsilon > 0$ such that $f - \varepsilon u \in M$. In particular, $f \in M$. 

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Let $F \subseteq R[x]$ generate the ideal $I$. Let $M$ be an Archimedean quadratic module of $R[x]$ and $f, u \in I$. Suppose that

(a) $f \geq 0$ on $S(M)$

(b) $\forall a \in S(M) : (f(a) = 0 \Rightarrow u(a) = 0)$

(c) $uM \subseteq M$

(d) $u$ is $F$-stably contained in $M$, i.e., $\forall f \in F : \exists \epsilon > 0 : u \pm \epsilon f \in M$

(e) $\forall (f) > 0$ for all zeros $a$ of $f$ on $S(M)$ and all test states $\Psi$ on $I$ for $M$ at a wrt. $u$

Then, there is $\epsilon > 0$ such that $f - \epsilon u \in M$. In particular, $f \in M$. 

$F := \{1\}$

$u := 1$
Theorem 11. Let $F \subseteq R[x]$ generate the ideal $I$.

Let $M$ be an Archimedean quadratic module of $R[x]$ and $f, u \in I$. Suppose that

(a) $f > 0$ on $S(M)$

(b) $\forall a \in S(M) : (f(a) = 0 \Rightarrow u(a) = 0)$

(c) $uM \subseteq M$

(d) $u$ is $F$-stably contained in $M$, i.e., $\forall f \in F : \exists \varepsilon > 0 : u \pm \varepsilon f \in M$

(e) $\forall (f) > 0$ for all zeros $a$ of $f$ on $S(M)$ and all test states $\varphi$ on $I$ for $M$ at a wrt. $u$.

Then, there is $\varepsilon > 0$ such that $f - \varepsilon u \in M$. In particular, $f \in M$. 

$F := \{1\}$

$u := 1$
Let $F \subseteq R[x]$ generate the ideal $I$. Let $M$ be an Archimedean quadratic module of $R[x]$ and $f, u \in I$. Suppose that:

- $f \gg 0$ on $S(M)$
- $\forall a \in S(M): (f(a) = 0 \Rightarrow u(a) = 0)$
- $uM \subseteq M$
- $u$ is $F$-stably contained in $M$, i.e., $\forall f \in F: \exists \varepsilon > 0: u \neq f \neq \varepsilon f \in M$
- $Q(f) > 0$ for all zeros $a$ of $f$ on $S(M)$ and all test states $Q$ on $I$ for $M$ at $a$ wrt. $u$.

Then, there is $\varepsilon > 0$ such that $f - \varepsilon u \in M$. In particular, $f \in M$. 

$F := \{1\}$ \hspace{1cm} $u := 1$
Theorem 11

Let \( F \subseteq R[x] \) generate the ideal \( I \).

Let \( M \) be an Archimedean quadratic module of \( R[x] \) and \( f, u \in I \). Suppose that

(a) \( f \geq 0 \) on \( S(M) \)
(b) \( \forall a \in S(M) : (f(a) = 0 \Rightarrow u(a) = 0) \)
(c) \( uM \subseteq M \)
(d) \( u \) is \( F \)-stably contained in \( M \), i.e., \( \forall f \in F : \exists \varepsilon > 0 : u \pm \varepsilon f \in M \)
(e) \( \forall f \) \( \forall \varepsilon > 0 \) for all zeros \( a \) of \( f \) on \( S(M) \) and all test states \( \varepsilon \) on \( I \) for \( M \) at \( a \) wrt. \( u \).

Then, there is \( \varepsilon > 0 \) such that \( f - \varepsilon u \in M \). In particular, \( f \in M \).
Then, there is $\varepsilon > 0$ such that $f - \varepsilon \in \mathcal{W}$. In particular, $f \in \mathcal{W}$. And all test states $y$ on $I$ for $M$ are w.r.t. $u$ for all zeros $a$ of $f$ on $\text{S}(\mathcal{W})$.

Let $M$ be an Archimedean quadratic module of $\mathbb{R}[x]$. Suppose that $f \in \mathcal{W}$.

Theorem 4 (Artin, 1953): Let $M$ be an Archimedean quadratic module of $\mathbb{R}[x]$.
Theorem 11

Let $F \subseteq R[x]$ generate the ideal $I$. Let $M$ be an Archimedean quadratic module of $R[x]$ and $f, v \in I$. Suppose that

(a) $f \geq 0$ on $S(M)$
(b) $\forall a \in S(M) : (f(a) = 0 \iff v(a) = 0)$
(c) $uM \subseteq M$
(d) $u$ is $F$-stably contained in $M$, i.e., $\forall f \in F : \exists \varepsilon > 0 : u \pm \varepsilon f \in W$
(e) $\Psi(f) > 0$ for all zeros $a$ of $f$ on $S(M)$ and all test states $\Psi$ on $I$ for $M$ at a wrt. $v$.

Then, there is $\varepsilon > 0$ such that $f - \varepsilon v \in M$. In particular, $f \in M$. Under these hypotheses, one can show that actually, even more:

$\exists \varepsilon > 0 : u - \varepsilon f, f - \varepsilon v \in M$ and hence $\geq 0$ on $S(M)$. 
\[ \exists \varepsilon > 0: (f - \varepsilon u \geq 0 \text{ on } S(M) \quad \text{and} \quad u - \varepsilon f \geq 0 \text{ on } S(M)) \]

means that not only that \( f \) and \( u \) have the same zeros on \( S(M) \) but also that they behave similarly near these zeros...
Theorem 11 Let $F \subseteq \mathbb{R}[x]$ generate the ideal $I$.

Let $M$ be an Archimedean quadratic module of $\mathbb{R}[x]$ and $f, u \in I$. Suppose that

(a) $f \geq 0$ on $S(M)$
(b) $\forall a \in S(M) : (f(a)=0 \Rightarrow u(a)=0)$
(c) $uM \subseteq M$
(d) $u$ is $F$-stably contained in $M$, i.e., $\forall f \in F : \exists \varepsilon > 0 : u \pm \varepsilon f \in M$
(e) $\Phi(f) > 0$ for all zeros $a$ of $f$ on $S(M)$ and all test states $\Phi$ on $I$ for $M$ at a wrt. $u$.

Then, there is $\varepsilon > 0$ such that $f-\varepsilon u \in M$. In particular, $f \in M$. 

Step-by-step strategy
Theorem 11. Let $F \subseteq \mathbb{R}[x]$ generate the ideal $I$.

Let $M$ be an Archimedean quadratic module of $\mathbb{R}[x]$ and $f, u \in I$. Suppose that

(a) $f \geq 0$ on $S(M)$
(b) $\forall a \in S(M): (f(a) = 0 \Rightarrow u(a) = 0)$
(c) $uM \subseteq M$
(d) $u$ is $F$-stably contained in $M$, i.e., $\forall f \in F: \exists \varepsilon > 0: u + \varepsilon f \in M$
(e) $\varphi(f) > 0$ for all zeros $a$ of $f$ on $S(M)$ and all test states $\varphi$ on $I$ for $M$ at a wrt. $u$.

Then, there is $\varepsilon > 0$ such that $f - \varepsilon u \in M$. In particular, $f \in M$. 

Let $M$ be an Archimedean quadratic module of $\mathbb{R}[x]$ and $f \in \mathbb{R}[x]$ with $f \geq 0$ on $S(M)$. Want to prove $f \in M$. 

**Step-by-step strategy**
Let $F \subseteq \mathbb{R}[x]$ generate the ideal $I$.

Let $M$ be an Archimedean quadratic module of $\mathbb{R}[x]$ and $f, u \in I$. Suppose that

(a) $f \geq 0$ on $S(M)$
(b) $\forall a \in S(M) : (f(a) = 0 \Rightarrow u(a) = 0)$
(c) $uM \subseteq M$
(d) $u$ is $F$-stably contained in $M$, i.e., $\forall f \in F : \exists \varepsilon > 0 : u + \varepsilon f \in M$
(e) $\forall f > 0$ for all zeros $a$ of $f$ on $S(M)$ and all test states $\Psi$ on $I$ for $M$ at a w.r.t. $u$.

Then, there is $\varepsilon > 0$ such that $f - \varepsilon u \in M$. In particular, $f \in M$. 

Let $M$ be a
Archimedean quadratic module
of $\mathbb{R}[x]$ and $f \in \mathbb{R}[x]$ with $f \geq 0$ on $S(M)$. Want to prove $f \in M$.

**Step-by-step strategy**

**Step 1.** Find "role-model element" $u$ of $M$ with (b) having the same zeros as $f$ on $S(M)$ and behaving similarly near them.
Theorem 11. Let $F \subseteq R[x]$ generate the ideal $I$. 

Let $M$ be an Archimedean quadratic module of $R[x]$ and $f, u \in I$. Suppose that:

(a) $f \geq 0$ on $S(M)$
(b) $\forall a \in S(M) : (f(a) = 0 \Rightarrow u(a) = 0)$
(c) $uM \subseteq M$
(d) $u$ is $F$-stably contained in $M$, i.e., $\forall f \in F : \exists \varepsilon > 0 : u \pm \varepsilon f \in M$
(e) $\Psi(f) > 0$ for all zeros $a$ of $f$ on $S(M)$ and all test states $\Psi$ on $I$ for $M$ at a wrt. $u$.

Then, there is $\varepsilon > 0$ such that $f - \varepsilon u \in M$. In particular, $f \in M$.

Step 1. Find "role-model" $u$ of $M$ with (b) having the same zeros as $f$ on $S(M)$ and behaving similarly near them.

Step 2. Identify $F = R[x]$ (the bigger the better) such that $f, u \in I$ such that (d) holds.
Let $F \subseteq \mathbb{R}[x]$ generate the ideal $I$.

Let $M$ be an Archimedean quadratic module of $\mathbb{R}[x]$ and $f, u \in I$. Suppose that

(a) $f \geq 0$ on $S(M)$
(b) $\forall a \in S(M) : (f(a) = 0 \Rightarrow u(a) = 0)$
(c) $uM \subseteq M$
(d) $u$ is $F$-stably contained in $M$, i.e., $\forall f \in F : \exists \varepsilon > 0 : u + \varepsilon f \in M$
(e) $\forall (f) > 0$ for all zeros $a$ of $f$ on $S(M)$ and all test states $\Psi$ on $I$ for $M$ at a wrt. $v$.

Then, there is $\varepsilon > 0$ such that $f - \varepsilon u \in M$. In particular, $f \in M$.

**Step 1.** Find "role-model" $u$ of $M$ with (b) having the same zeros as $f$ on $S(M)$ and behaving similarly near them.

**Step 2.** Identify $F \subseteq \mathbb{R}[x]$ (the bigger the better) such that $f, u \in I$ such that (d) holds.

**Step 3.** Prove (e) by using geometric arguments or algebraic identities inside the ideal $I$ or both.
Example 5 \[ f := (x_1^2 + \ldots + x_5^2)^2 - 4(x_1^2 x_3^2 + x_1^2 x_4^2 + x_2^2 x_4^2 + x_2^2 x_5^2 + x_3^2 x_5^2) \]

\[ \in R[x] = R[x_1, \ldots, x_5] \quad \text{Horn form} \]

\( f \geq 0 \) on \( \mathbb{R}^5 \)

\( f \) has infinitely many zeros on \( S^4 \).

\( f \in M_{S^4}, f \notin M_{B^5} \)
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Theorem 12 Let \( d_1, \ldots, d_5 \in \mathbb{R}_{>0}. \) Then \( f \in \sum \mathbb{R}[x]^2 + \mathbb{R}[x](\sum_{i=1}^{5} d_i x_i^2 - 1). \)
Example 5 \[ f := (x_1^2 + \ldots + x_5^2)^2 - 4(x_1^2 x_3^2 + x_1^2 x_4^2 + x_2^2 x_4^2 + x_2^2 x_5^2 + x_3^2 x_5^2) \]
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Theorem 12 Let \( d_1, \ldots, d_5 \in \mathbb{R}_{>0} \). Then \( f \in \sum \mathbb{R}[x]^2 + \mathbb{R}[x] (\sum_{i=1}^5 d_i x_i^2 - 1). \)

Proof Step 1. \( u := (\sum_{i=1}^5 x_i^2) f \in \sum \mathbb{R}[x]^2 \)
\[ =: M \]
Example 5 \( f := (x_1^2 + \ldots + x_5^2)^2 - 4(x_1^2x_3^2 + x_1^2x_4^2 + x_1^2x_5^2 + x_2^2x_5^2 + x_3^2x_5^2) \) 
\( \in \mathbb{R}[x] = \mathbb{R}[x_1, \ldots, x_5] \) Horn form

\( f \geq 0 \) on \( \mathbb{R}^5 \)

\( f \) has infinitely many zeros on \( S^4 \).

\( f \in M_{S^4} \), \( f \notin M_{B^5} \)

---

Theorem 12 Let \( d_1, \ldots, d_5 \in \mathbb{R}_{>0} \). Then \( f \in \sum_{i} \mathbb{R}[x]^2 + \mathbb{R}[x](\sum_{i=1}^{5} d_i x_i^2 - 1) \).

Proof

Step 1. \( u := (\sum_{i=n}^{5} x_i^2) f \in \sum_{i} \mathbb{R}[x]^2 \) \( \equiv M \)

Step 2. \( F := \{ f \} \) (works by results of Laurent and Vargas)
Example 5 \( f := \left( x_1^2 + \ldots + x_5^2 \right)^2 - 4 \left( x_1^2 x_3^2 + x_1^2 x_4^2 + x_2^2 x_4^2 + x_2^2 x_5^2 + x_3^2 x_5^2 \right) \in \mathbb{R}[x] = \mathbb{R}[x_1, \ldots, x_5] \) Horn form

- \( f \geq 0 \) on \( \mathbb{R}^5 \)
- \( f \) has infinitely many zeros on \( S^4 \).
- \( f \in M_{S^4}, f \notin M_{B^5} \)

Theorem 12
Let \( d_1, \ldots, d_5 \in \mathbb{R} > 0 \). Then \( f \in \sum \mathbb{R}[x]^2 + \mathbb{R}[x] \left( \sum_{i=1}^{5} d_i x_i^2 - 1 \right) \).

Proof

Step 1. \( u := \left( \sum_{i=1}^{5} x_i^2 \right) f \in \sum \mathbb{R}[x]^2 \) \( \mathbb{R} \)

Step 2. \( F := \{ f \} \) (works by results of Laurent and Vargas)

Step 3. Let \( \gamma \) be a test state on \( I := \mathbb{R}[x]f \) for \( M \) at a zero \( a \) of \( f \) on the ellipsoid. Then \( \lambda = \gamma(u) = \left( \sum_{i=1}^{5} a_i^2 \right) \gamma(f) \). Hence \( \gamma(f) > 0 \). \( \square \)
Theorem 12 was the missing stone to show that each copositive matrix of size 5 is Reznick-certifiable.
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\[ C_n := \{ A \in S_n | \forall a \in \mathbb{R}^{\geq 0}^n : a^T A a \geq 0 \} \text{ copositive matrices} \]
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We call \( A \in C_n \) Reznick-certifiable if there exists \( r \in \mathbb{N}_0 \) such that \((x_1^2 + \ldots + x_n^2)^r (x_1^2 \ldots x_n^2) A \begin{pmatrix} x_1^2 \\ \vdots \\ x_n^2 \end{pmatrix} \in \Sigma \mathbb{R}[x]^2\). Even quartic psd form
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We call \( A \in C_n \) Reznick-certifiable if there exists \( r \in \mathbb{N}_0 \) such that

\[
(x_1^2 + \ldots + x_n^2)^r \begin{pmatrix} x_1^2 & \ldots & x_n^2 \\ \vdots & \ddots & \vdots \\ x_1^2 & \ldots & x_n^2 \end{pmatrix} A \begin{pmatrix} x_1^2 \\ \vdots \\ x_n^2 \end{pmatrix} \in \Sigma \mathbb{R}[x]^2
\]

even quartic psd form

**Theorem 13 (Reznick, 1995)** Let \( f \in \mathbb{R}[x]^2 \) be a pd form.

Then there is \( r \in \mathbb{N}_0 \) such that

\[
(x_1^2 + \ldots + x_n^2)^r f \in \Sigma \mathbb{R}[x]^2
\]
Proposition 14 (de Klerk, Laurent, Parrilo 2005)

For every form \( p \in \mathbb{R}[x] \) of even degree,

\[
\rho \in M_{2n-1} \iff \exists r \in \mathbb{N}_0 : \ \left( \sum_{i=1}^{n} x_i^2 \right)^r \rho \in \sum_{r} \mathbb{R}[x]^2.
\]
Proposition 14 (de Klerk, Laurent, Parrilo 2005)

For every form $p \in \mathbb{R}[x]$ of even degree,

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For every form $p \in \mathbb{R}[x]$ of even degree,

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Proposition 14 (de Klerk, Laurent, Parrilo 2005)

For every form \( p \in \mathbb{R}[x] \) of even degree, \( p \in M_{n-1} \) \( \iff \exists r \in \mathbb{N}_0 : (\sum_{i=1}^{n} x_i^2)^r p \in \sum \mathbb{R}[x]^2 \).

Thm. 12 says exactly that all positive diagonal scalings of the Horn form are in \( M_{n-1} \).
Proposition 14 (de Klerk, Laurent, Parrilo 2005)

For every form $p \in \mathbb{R}[x]$ of even degree,

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Proposition 14 (de Klerk, Laurent, Parrilo 2005)

For every form $p \in \mathbb{R}[x]$ of even degree,

$$p \in M_{3n-1} \iff \exists r \in \mathbb{N}_0 : \left( \sum_{i=1}^{n} x_i^2 \right)^r p \in \sum \mathbb{R}[x]^2.$$ 

Thm. 12 says exactly that all positive diagonal scalings of the Horn form are in $M_{3n-1}$. Taking into account Prop. 14, it says that DHD is Reznick-certifiable for all pd diagonal $

\begin{pmatrix}
1 & 1 & -1 & -1 & 1 \\
1 & 1 & 1 & -1 & -1 \\
-1 & -1 & 1 & 1 & 1 \\
-1 & -1 & -1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1
\end{pmatrix}$

is the Horn matrix.
Proposition 14 (de Klerk, Laurent, Parrilo 2005)

For every form \( p \in \mathbb{R}[x] \) of even degree,

\[
p \in M_{\frac{n(n-1)}{2}} \iff \exists r \in \mathbb{N}_0 : (\sum_{i=1}^{n} x_i^2)^r \cdot p \in \Sigma \mathbb{R}[x]^2.
\]

Thm. 12 says exactly that all positive diagonal scalings of the Horn form are in \( M_{\frac{n(n-1)}{2}} \). Taking into account Prop. 14, it says that DHD is Reznick-certifiable for all pd diagonal \( D \in S_5 \). Here,

\[
H := \begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 \\
1 & 1 & 1 & -1 & 1 \\
1 & 1 & -1 & 1 & 1 \\
1 & -1 & 1 & 1 & 1
\end{pmatrix}
\]

is the Horn matrix.

Hildebrand classified in 2012 the extreme rays of \( C_5 \) and these DHD span those extreme rays that Laurent and Vargas could not handle. ...
Theorem 15  Every copositive matrix of size 5 is Renick-certifiable.
Theorem 15  Every copositive matrix of size 5 is Reznick-certifiable. As seen, the Reznick exponent $r$ cannot be bounded uniformly for $\mathbb{S}_5$. 
Theorem 15  Every copositive matrix of size 5 is Reznick-certifiable.

As seen, the Reznick exponent $r$ cannot be bounded uniformly for $C_5$. Bodirsky, Kummer and Thom showed recently much more:

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For size $> 5$, the same holds for certain matrices associated to graphs: If $G$ is a graph, $A_G$ its adjacency matrix and $\kappa(G)$ its stability number, set $M_G := \kappa(G)(I+A_G) - J$.  

[Identity matrix] [All ones matrix]
Theorem 15: Every copositive matrix of size 5 is Reznick-certifiable.

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$H = M$ identity matrix all ones matrix
Computing the stability number of a graph is NP-hard. Already Motzkin and Straus in 1965 knew that
\[ \alpha(G) = \min \{ t \in \mathbb{R} | t(I + A_G) - J \in C_n \} \]
for any graph $G$ on $n \geq 1$ vertices.
Computing the stability number of a graph is \( \text{NP}- \text{hard.} \) Already Motzkin and Straus in 1965 knew that

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for any graph \( G \) on \( n \geq 1 \) vertices. By Reznick’s Thm. 13, this gives rise to a convergent semidefinite hierarchy for computing the stability number. De Klerk and Pasechnik proposed and investigated this hierarchy and conjectured that it converges after \( \alpha(G) - 1 \) steps. We show at least finite convergence:

**Theorem 16** For any graph \( G \), \( M_G \) is Reznick-certifiable.
Laurent and Vargas reduced the proof of Thm. 16 to showing that Reznick-certifiability is preserved by adding an isolated node to the graph.
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So let $G$ be a graph on $n-1$ vertices and let $H$ be the graph arising from $G$ by adding an isolated $n$-th vertex.
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So let $G$ be a graph on $n-1$ vertices and let $H$ be the graph arising from $G$ by adding an isolated $n$-th vertex. Suppose that $M_G$ is Reznick-certifiable. We show that $M_H$ also is.
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So let $G$ be a graph on $n-1$ vertices and let $H$ be the graph arising from $G$ by adding an isolated $n$-th vertex.

Suppose that $M_G$ is Reznick-certifiable. We show that $M_H$ also is.

By Prop. 14, this means we have to show $h \in M_{S^{n-1}}$ where

$$h := (x_1^2, \ldots, x_n^2)M_H \begin{pmatrix} x_1^2 \\ \vdots \\ x_n^2 \end{pmatrix}.$$
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So let $G$ be a graph on $n-1$ vertices and let $H$ be the graph arising from $G$ by adding an isolated $n$-th vertex.

Suppose that $M_G$ is Reznick-certifiable. We show that $M_H$ also is.

By Prop. 14, this means we have to show $h \in M_{S^{n-1}}$ where

$$h := (x_0^2 \ldots x_n^2) M_H \begin{pmatrix} x_0^2 \\ \vdots \\ x_n^2 \\ x_{n-1}^2 \end{pmatrix}.$$ 

Choose $r \in \mathbb{N}_0$ such that $(x_0^2 + \ldots + x_{n-1}^2)^r g \in \Sigma_{Rxy}^2$

where $g := (x_0^2 \ldots x_{n-1}^2) M_G \begin{pmatrix} x_0^2 \\ \vdots \\ x_{n-1}^2 \end{pmatrix}$. 


Laurent and Vargas reduced the proof of Thm. 16 to showing that Reznick-certifiability is preserved by adding an isolated node to the graph.

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$$h := (x_1^2, \ldots, x_n^2) M_H \left( \begin{array}{c} x_1^2 \\ \vdots \\ x_n^2 \end{array} \right),$$

choose $r \in \mathbb{N}_0$ such that $(x_1^2 + \cdots + x_{n-1}^2)^r g \in \Sigma \mathcal{R}(x^2)$

where $g := (x_1^2, \ldots, x_{n-1}^2) M_G \left( \begin{array}{c} x_1^2 \\ \vdots \\ x_{n-1}^2 \end{array} \right)$. It turns out that $h = p^2 + cg$ for some $p \in \mathcal{R}(x)$ and $c > 0$. 
Suppose that $M_G$ is Reznick-certifiable. We show that $M_H$ also is. By Prop. 14, this means we have to show $h \in M_{S^{n-1}}$ where

$$h := (x_1^2 \ldots x_n^2) M_H \left( \begin{array}{c} x_1^2 \\ \vdots \\ x_{n-1}^2 \end{array} \right).$$

Choose $r \in \mathbb{N}_0$ such that $(x_1^2 + \ldots + x_{n-1}^2)^r g \in \Sigma^{\mathbb{R}G}_{j^2}$ where

$$g := (x_1^2 \ldots x_{n-1}^2) M_G \left( \begin{array}{c} x_1^2 \\ \vdots \\ x_{n-1}^2 \end{array} \right).$$

It turns out that $h = r^2 + cg$ for some $p \in \mathbb{R}[x]$ and $c > 0$. 
where \( g = (x_2, \ldots, x_n)M \). It turns out that
\[
 h = (x_2, \ldots, x_n)M \quad (x_2, \ldots, x_n) \quad \text{for some } p \in R[x]\text{ and } c > 0.
\]

Choose \( r \in M \), such that \( (x^2 + \cdots + x_n^2)g \in 2R[x] \).

\[
 h := (x_2, \ldots, x_n)M \quad (x_2, \ldots, x_n)
\]

By Prop. \( \ref{prop} \), this means we have to show \( h \in M_{n-1} \), where

Suppose that \( M \) is Reznick-certifiable. We show that \( M \) also is.

Then, there is \( \varepsilon > 0 \) such that \( f \in \text{EVEN} \). In particular, \( f \in \text{EVEN} \).
Suppose that $M_G$ is Reznick-certifiable. We show that $M_H$ also is.

By Prop. 14, this means we have to show $h \in M_{S^{n-1}}$ where

$$h := (x_1^2, \ldots, x_n^2) M_H \left( \begin{array}{c} x_1^2 \\ \vdots \\ x_n^2 \end{array} \right).$$

Choose $r \in \mathbb{N}_0$ such that $(x_1^2 + \ldots + x_{n-1}^2)^r g \in \Sigma_{\mathbb{R}[x]}$. It turns out that $h = p^2 + cg$ for some $p \in \mathbb{R}[x]$ and $c > 0$. 

**Theorem 14**

Let $F \subseteq \mathbb{R}[x]$ generate the ideal $I$. Let $M$ be an Archimedean quadratic module of $\mathbb{R}[x]$ and $f, u \in I$. Suppose that

(a) $f \geq 0$ on $S(M)$

(b) $\forall a \in S(M) : (f(a) = 0 \Rightarrow u(a) = 0)$

(c) $u M \subseteq M$

(d) $u$ is $F$-stably contained in $M$, i.e., $\forall f \in F : \exists \epsilon > 0 : u \pm \epsilon f \in M$

(e) $y(f) > 0$ for all zeros $a$ of $f$ on $S(M)$ and all test states $y$ on $I$ for $M$ at $a$ wrt. $u$.

Then, there is $\epsilon > 0$ such that $f - \epsilon u \in M$, in particular, $f \in M$. 

Suppose that $M_G$ is Reznick-certifiable. We show that $M_H$ also is.
Let \( F \subseteq R[x] \) generate the ideal \( I \).

Let \( M \) be an Archimedean quadratic module of \( R[x] \) and \( f, u \in I \). Suppose that

(a) \( f \geq 0 \) on \( S(M) \)
(b) \( \forall a \in S(M) : (f(a) = 0 \implies u(a) = 0) \)
(c) \( u \in M \)
(d) \( u \) is \( F \)-stably contained in \( M \), i.e., \( \forall f \in F : \exists \epsilon > 0 : u \pm \epsilon f \in M \)
(e) \( \forall f \geq 0 \) for all zeros \( a \) of \( f \) on \( S(M) \) and all test states \( y \) on \( I \) for \( M \) at \( a \) wrt. \( u \).

Then, there is \( \epsilon > 0 \) such that \( f - f \epsilon u \in M \), in particular, \( f \in M \).

\[ f := h, \ M := M_{S_{n-1}} \]

**Step 1.** \( u := p^2 + c \left(x_1^2 + \ldots + x_{n-1}^2\right)^{2r} g \in \sum R[x]^2 \]

Suppose that \( M_G \) is Reznick-certifiable. We show that \( M_H \) also is.

By Prop. 14, this means we have to show \( h \in M_{S_{n-1}} \) where

\[ h := (x_1^2 \ldots x_n^2) M_H \left( \begin{array}{c} x_1^2 \\ \vdots \\ x_n^2 \end{array} \right) \]

Choose \( r \in \mathbb{N}_0 \) such that \( (x_1^2 + \ldots + x_{n-1}^2)^r g \in \sum R[x]^2 \)

where \( g := (x_1^2 \ldots x_{n-1}^2) M_G \left( \begin{array}{c} x_1^2 \\ \vdots \\ x_{n-1}^2 \end{array} \right) \).

It turns out that \( h = p^2 + c g \) for some \( p \in R[x] \) and \( c > 0 \).
Let $F \subseteq R[x]$ generate the ideal $I$. Let $M$ be an Archimedean quadratic module of $R[x]$ and $f, u \in I$. Suppose that

- $f \geq 0$ on $S(M)$
- $\forall a \in S(M): (f(a) = 0 \Rightarrow u(a) = 0)$
- $uM \subseteq M$
- $u$ is $F$-stably contained in $M$, i.e., $\forall f \in F : \exists \epsilon > 0 : u \not\in \epsilon f \in M$
- $\forall (f) > 0$ for all zeros $a$ of $f$ on $S(M)$ and all test states $y$ on $I$ for $M$ at a w.r.t. $u$.

Then, there is $\epsilon > 0$ such that $f - \epsilon u \in M$, in particular, $f \in M$.

**Suppose that $M_G$ is Reznick-certifiable.** We show that $M_H$ also is.

By Prop. 14, this means we have to show $h \in M_{S^{n-1}}$ where

$$h = (x_1^2 \ldots x_n^2) M_H \left( \begin{array}{c} x_1^2 \\ \vdots \\ x_n^2 \end{array} \right)$$

Choose $r \in \mathbb{N}_0$ such that $(x_1^2 + \ldots + x_{n-1}^2)^r g \in \Sigma R[x]^2$

where $g = (x_1^2 \ldots x_{n-1}^2) M_G \left( \begin{array}{c} x_1^2 \\ \vdots \\ x_{n-1}^2 \end{array} \right)$. It turns out that $h = p^2 + cg$ for some $p \in R[x]$ and $c > 0$. 

Theorem 11. Let $f \in \mathbb{R}[x]$ generate the ideal $I$. Let $M$ be an Archimedean quadratic module of $\mathbb{R}[x]$ and $f, u \in I$. Suppose that:

(a) $f \geq 0$ on $S(M)$
(b) $\forall a \in S(M): (f(a) = 0 \Rightarrow u(a) = 0)$
(c) $u M \subseteq M$
(d) $u$ is $F$-stably contained in $M$, i.e., $\forall f \in F: \exists e > 0: u \pm e f \in M$
(e) $y(f) > 0$ for all zeros $a$ of $f$ on $S(M)$ and all test states $y$ on $I$ for $M$ at $a$ with $u$.

Then, there is $\varepsilon > 0$ such that $f - \varepsilon u \in M$. In particular, $f \in M$.

Step 1. $u := p^2 + c (x_1^2 + \ldots + x_{n-1}^2)^2 g \in \sum \mathbb{R}[x]^2$
Step 2. $F := \{ p^2, g \}$ (very tricky, two pages)
Step 3. Let $\Psi$ be a test state on $I := \mathbb{R}[x] p^2 + \mathbb{R}[x] g$ for $M$ at an $S^{n-1}$ wrt. $u$.

Suppose that $M_\Psi$ is Reznick-certifiable. We show that $M_H$ also is.

By Prop. 14, this means we have to show $h \in M_{S^{n-1}}$ where

$$h := (x_1^2 \ldots x_n^2) M_H \left( \begin{array}{c} x_1^2 \\ \vdots \\ x_n^2 \end{array} \right),$$

$$g := (x_1^2 \ldots x_{n-1}^2) M_G \left( \begin{array}{c} x_1^2 \\ \vdots \\ x_{n-1}^2 \end{array} \right).$$

Choose $r \in \mathbb{N}_0$ such that $(x_1^2 + \ldots + x_{n-1}^2)^r g \in \sum \mathbb{R}[x]^2$.

It turns out that $h = p^2 + c g$ for some $p \in \mathbb{R}[x]$ and $c > 0$. 


Let \( F \subseteq \mathbb{R}[x] \) generate the ideal \( I \).
Let \( M \) be an Archimedean quadratic module of \( \mathbb{R}[x] \) and \( f, u \in I \). Suppose that
(a) \( f \geq 0 \) on \( S(M) \)
(b) \( \forall a \in S(M) : (f(a) = 0 \implies u(a) = 0) \)
(c) \( uM \subseteq M \)
(d) \( u \) is \( F \)-stably contained in \( M \), i.e., \( \forall f \in F : \exists \varepsilon > 0 : u \pm \varepsilon f \in M \)
(e) \( \psi(f) > 0 \) for all zeros \( a \) of \( f \) on \( S(M) \) and all test states \( \psi \) on \( I \) for \( M \) at \( a \) wrt. \( u \).

Then, there is \( \varepsilon > 0 \) such that \( f - \varepsilon u \in M \), in particular, \( f \in M \).

Then \( \lambda = \psi(u) = \psi(p^2) + c(a_1^2 + \ldots + a_n^2)^{2r} \psi(g) \).

By Prop. 14, this means we have to show \( h \in M_{S^{n-1}} \) where

\[
h := (x_1^2 \ldots x_n^2) M_H \left( \begin{array}{c} x_1^2 \\ \vdots \\ x_n^2 \end{array} \right),
\]

Choose \( r \in \mathbb{N}_0 \) such that \( (x_1^2 + \ldots + x_{n-1}^2)^r g \in \Sigma R[x]^2 \)

where \( g := (x_1^2 \ldots x_{n-1}^2) M_G \left( \begin{array}{c} x_1^2 \\ \vdots \\ x_{n-1}^2 \end{array} \right) \). It turns out that \( h = p^2 + cg \) for some \( p \in \mathbb{R}[x] \) and \( c > 0 \).
Theorem 11. Let $F \in \mathbb{R}[x]$ generate the ideal $I$. Let $M$ be an Archimedean quadratic module of $\mathbb{R}[x]$ and $f, u \in I$. Suppose that

(a) $f \geq 0$ on $S(M)$
(b) $\forall a \in S(M): (f(a) = 0 \Rightarrow u(a) = 0)$
(c) $uM \subseteq M$
(d) $u$ is $F$-stably contained in $M$, i.e., $\forall f \in F: \exists \varepsilon > 0: u \pm \varepsilon f \in M$
(e) $\gamma(f) > 0$ for all zeros $a$ of $f$ on $S(M)$ and all test states $\gamma$ on $I$ for $M$ at $a$ wrt. $u$.

Then, there is $\varepsilon > 0$ such that $f - \varepsilon u \subseteq M$, in particular, $f \in M$.

Let $f := h$, $M := M_{\mathbb{R}^{n-1}}$

Step 1. $u := p^2 + c \left( x_1^2 + \ldots + x_n^2 \right)^{2r} g \in \Sigma \mathbb{R}[x]^2$

Step 2. $F := \{ p^2, g \}$ (very tricky, two pages)

Step 3. Let $\gamma$ be a test state on $I := \mathbb{R}[x] p^2 + \mathbb{R}[x] g$ for $M$ at a zero $a$ of $f$ on $S^{n-1}$ wrt. $u$.

Then $\lambda = \gamma(u) = \gamma(p^2) + c \left( x_1^2 + \ldots + x_n^2 \right)^{2r} \gamma(g)$.

at least one of these two positive!

$h := (x_1^2, \ldots, x_n^2) M_H \left( \begin{array}{c} x_1^2 \\ x_2^2 \\ \vdots \\ x_n^2 \end{array} \right)$.

Choose $r \in \mathbb{N}$ so that $(x_1^2 + \ldots + x_n^2)^r g \in \Sigma \mathbb{R}[x]^2$

where $g := (x_1^2, \ldots, x_n^2) M_G \left( \begin{array}{c} x_1^2 \\ x_2^2 \\ \vdots \\ x_n^2 \end{array} \right)$.

It turns out that $h = p^2 + cg$ for some $p \in \mathbb{R}[x]$ and $c > 0$. 
Theorem 11. Let $f \in R[x]$ generate the ideal $I$. Let $M$ be an Archimedean quadratic module of $R[x]$ and $f, u \in I$. Suppose that

(a) $f \geq 0$ on $S(M)$
(b) $\forall a \in S(M): (f(a) = 0 \Rightarrow u(a) = 0)$
(c) $uM \subseteq M$
(d) $u$ is $f$-stably contained in $M$, i.e., $\forall f \in F: \exists \varepsilon > 0: u \pm \varepsilon f \in M$
(e) $\forall f \geq 0$ for all zeros $a$ of $f$ on $S(M)$ and all test states $u$ on $I$ for $M$ at $a$ wrt. $u$.

Then, there is $\varepsilon > 0$ such that $f - \varepsilon u \in M$. In particular, $f \in M$.

Let $f = h$, $M = M_{S^{n-1}}^n$.

**Step 1.** $u = p^2 + c (x_{n-1}^2 + \ldots + x_n^2)^2 \in \sum R[x]^2$

**Step 2.** $F = \{ p^2, g \}$ (very tricky, two pages)

**Step 3.** Let $Y$ be a test state on $I = R[x]p^2 + R[x]g$ for $M$ at a zero $a$ of $f$ on $S^{n-1}$ wrt. $u$.

Then, $\lambda = \Phi(u) = \Phi(p^2) + c (a_1^2 + \ldots + a_n^2) \geq 0 \Phi(g)$.

Moreover, $0 \leq \Phi((x_1^2 + \ldots + x_{n-1}^2)^2g) = (a_1^2 + \ldots + a_{n-1}^2)^2 \Phi(g)$.

Choose $r \in \mathbb{N}_0$ such that $(x_1^2 + \ldots + x_{n-1}^2)^r g \in \sum R[x]^2$.

where $g = (x_n^2 \ldots x_{n-1}^2) M G \left( \begin{array}{c} x_1^2 \\ \vdots \\ x_{n-1}^2 \end{array} \right)$. It turns out that $h = p^2 + cg$ for some $p \in R[x]$ and $c > 0$. 

at least one of these two positive!
Theorem 11 Let \( f \in R[x] \) generate the ideal \( I \).
Let \( M \) be an Archimedean quadratic module of \( R[x] \)
and \( f, g \in I \). Suppose that
\[
\begin{align*}
(a) & \quad f \geq 0 \text{ on } S(M) \\
(b) & \quad \forall a \in S(M) : (f(a) = 0 \Rightarrow g(a) = 0) \\
(c) & \quad \forall M \in M \\
(d) & \quad u \text{ is } F\text{-stably contained in } M, \text{ i.e., } \forall f \in F : \exists e > 0 : u \pm ef \in M \\
(e) & \quad \forall (f) > 0 \text{ for all zeros } a \text{ of } f \text{ on } S(M) \\
\end{align*}
\]
and all test states \( \psi \) on \( I \) for \( M \) at \( u \).

Then, there is \( \varepsilon > 0 \) such that \( f \pm \varepsilon u \in M \).
In particular, \( f \notin M \).

Let \( I = I_f = R[x]p^2 + R[x]g \) for \( M \) at a zero \( a \) of \( f \) on \( S^{n-1} \) wrt. \( u \).

Thus \( \eta = \psi(u) = \psi(p^2) + \left( a_1^2 + \ldots + a_{n-1}^2 \right)^{2r} \psi(g) \).

Moreover \( 0 \leq \psi((x_1^2 + \ldots + x_{n-1}^2)g) \leq \left( a_1^2 + \ldots + a_{n-1}^2 \right)^{2r} \psi(g) \).

Choose \( r \in \mathbb{N} \_ \) such that \( (x_1^2 + \ldots + x_{n-1}^2)^r g \in \Sigma R[x] \).

It turns out that \( h = p^2 + cg \) for some \( p \in R[x] \) and \( c > 0 \).
Let $f \in \mathbb{R}[x]$ generate the ideal $I$. Let $M$ be an Archimedean quadratic module of $\mathbb{R}[x]$ and $u,v \in I$. Suppose that

(a) $f \geq 0$ on $S(M)$
(b) $\forall a \in S(M) : (f(a) = 0 \Rightarrow u(a) = 0)$
(c) $u,M \in M$
(d) $u$ is $F$-stably contained in $M$, i.e., $\forall f \in F : \exists \varepsilon > 0 : u \pm \varepsilon f \in M$
(e) $\varphi(f) > 0$ for all zeros $a$ of $f$ on $S(M)$ and all test states $\varphi$ on $M$ at a w.r.t. $u$. Then, there is $\varepsilon > 0$ such that $f - \varepsilon u \in M$. In particular, $f \in M$.

Then \( \lambda = \varphi(u) = \varphi(p^2 + c(a_n^2 + \ldots + a_{n-1}^2)) > 0 \).

Moreover, \( 0 \leq \varphi((x_1^2 + \ldots + x_{n-1}^2)g) = (a_n^2 + \ldots + a_{n-1}^2)^{2r} \varphi(g) \).

Finally, \( \varphi(f) = \varphi(h) = \varphi(p^2 + c \varphi(g)) > 0 \).

It turns out that $h = p^2 + cg$ for some $p \in \mathbb{R}[x]$ and $c > 0$. 

\( \text{Step 1. } u := p^2 + c(x_1^2 + \ldots + x_{n-1}^2)^{2r} g \in \Sigma \mathbb{R}[x]^2 \\
\text{Step 2. } F := \{p^2, g\} \text{ (very tricky, two pages)} \\
\text{Step 3. } \text{Let } \varphi \text{ be a test state on } \mathbb{R}[x]^2 \text{ for } M \text{ at a zero } a \text{ of } f \text{ on } S^{n-1} \text{ wrt. } u. \)
Theorem 14. Let \( f \in \mathbb{R}[x] \) generate the ideal \( I \).

Let \( M \) be an Archimedean quadratic module of \( \mathbb{R}[x] \) and \( f, u \in I \). Suppose that

(a) \( f \geq 0 \) on \( S(M) \)
(b) \( \forall a \in S(M) : (f(a) = 0 \implies u(a) = 0) \)
(c) \( uM \subseteq M \)
(d) \( u \) is \( F \)-stably contained in \( M \), i.e., \( \forall f \in F : \exists \varepsilon > 0 : u + \varepsilon f \in M \)
(e) \( \forall f \) such that \( f \) is a zero of \( f \) on \( S(M) \) and all test states \( Y \) on \( I \) for \( M \) at \( u \).

Then there is \( \varepsilon > 0 \) such that \( f - \varepsilon u \in M \), in particular, \( f \in \mathbb{R}[x] \).

Then \( \lambda = \mathcal{Y}(u) = \mathcal{Y}(p^2) + c(\alpha_1^2 + \ldots + \alpha_{n-1}^2)^{2r} \mathcal{Y}(g) \).

Moreover \( 0 \leq \mathcal{Y}(\sum(x_1^2 + \ldots + x_{n-1}^2)g) = (\alpha_1^2 + \ldots + \alpha_{n-1}^2)^{2r} \mathcal{Y}(g) \). Hence \( \mathcal{Y}(p^2) \geq 0 \) and \( \mathcal{Y}(g) \geq 0 \).

Finally, \( \mathcal{Y}(f) = \mathcal{Y}(h) = \mathcal{Y}(p^2) + c \mathcal{Y}(g) > 0 \).

It turns out that \( h = p^2 + cg \) for some \( p \in \mathbb{R}[x] \) and \( c > 0 \).