Globally Optimal Solution to Inverse Kinematics of 7DOF Serial Manipulator

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Abstract

The Inverse Kinematics (IK) problem is concerned with finding robot control parameters to bring the robot into a desired position under the kinematics and collision constraints. We present a global solution to the optimal IK problem for a general serial 7DOF manipulator with revolute joints and a quadratic polynomial objective function. We show that the kinematic constraints due to rotations can be all generated by the second-degree polynomials. This is an important result since it significantly simplifies the further step where we find the optimal solution by Lasserre relaxations of nonconvex polynomial systems. We demonstrate that the second relaxation is sufficient to solve a general 7DOF IK problem. Our approach is certifiably globally optimal. We demonstrate the method on the 7DOF KUKA LBR IIWA manipulator and show that we are in practice able to compute the optimal IK or certify infeasibility in 99.9 % tested poses. We also demonstrate that by the same approach, we are able to solve the IK problem for a random generic manipulator with seven revolute joints.

1 Introduction

The Inverse Kinematics (IK) problem is one of the most important problems in robotics [33]. The solution to the IK problem finds robot control parameters to bring the robot into a desired position under the kinematics and collision constraints [14].

The IK problem has been extensively studied in robotics and control [30, 31]. The classical formulation [30] of the problem for 6 degrees of freedom (6DOF) serial manipulators leads to solving a system of polynomial equations [6, 34]. This is, in general, a hard ("EXPSPACE complete" [26]) algebraic computational problem, but practical solving methods have been developed for 6DOF manipulators [30, 24, 9].

An important generalization of the IK problem aims at finding the optimal control parameters for an underconstrained mechanism, i.e., when the number of controlled joints in a manipulator is larger than six. Then, an algebraic computation problem turns into an optimization problem over an algebraic variety [6] of possible IK solutions. It is particularly convenient to choose a polynomial objective function to arrive at a semialgebraic optimization problem.

Semi-algebraic optimization problems are in general nonconvex, but they can be solved with certified global optimality [21] using the Lasserre hierarchy of convex optimization problems [20]. Computationally, however, semialgebraic optimization problems are in general extremely hard and were often considered too expensive to be used in practice. In this paper, we show that using "algebraic preprocessing", semialgebraic optimization methods become practical in solving the IK problem of general 7DOF serial manipulators with a polynomial objective function.

1.1 Contribution

Our main contributions are as follows.

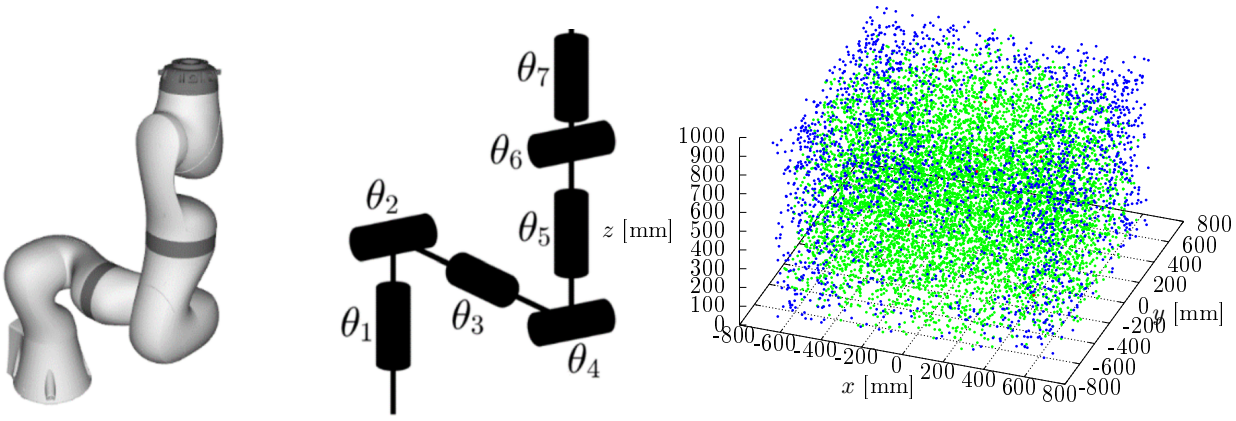

Figure 1: (left) 7DOF serial manipulator (*KUKA LBR IIWA*), and (middle) its kinematic model [17]. (right) We can optimally solve its inverse kinematics (green) or find it infeasible (blue) in 99.9 % of 10 000 tested poses.

1. We prove that the variety of IK solutions of all generic 7DOF revolute serial manipulators can be generated by the second-degree polynomials only (Theorem 1). This considerably reduces the complexity of semialgebraic optimization and makes it computationally feasible.

2. We provide a method for computing a globally optimal solution to the IK problem for a general 7DOF serial manipulator with a polynomial objective function.

3. We employ techniques from algebraic geometry [6] and polynomial optimization [21] to solve the 7DOF IK problem exactly (within the numerical accuracy of computation). Our approach is also able to certify the in-feasibility of solving when it happens.

4. We demonstrate that our approach works on a practical 7DOF *KUKA LBR IIWA* manipulator and allows us to solve 99.9 % configurations (Fig. 1) while the straightforward semi-algebraic optimization fails in approx. 28 % of cases.

5. We show that we can solve the IK problem for a randomly generated generic serial manipulator with seven revolute joints by the same approach.

2 Previous work

The first breakthrough in solving IK problems was the global solution to IK for a general 6DOF serial manipulator, which was given in [32, 24]. It leads to solving a polynomial system with 16 solutions. Another important result was the solution to the forward kinematics problem of the Stewart-Gough parallel manipulator platform [22], leading to a polynomial system with 40 solutions. See recent work [7] for the review of local and other approximate techniques for solving IK problems. We next review only the most relevant work.

2.1 The most relevant previous work

The closest previous works are related to solving IK for mechanisms, which are under constrained when considering positions of the final actuator only. The standard approach is to employ additional dynamics, time optimality, and collision constraints.

In [8], a technique for planning a dynamic whole-body motion of a humanoid robot has been developed. It solves IK as a part of motion planning by local optimization methods taking into account kinematics, dynamics, and collision models. The planning method requires a good initialization to converge, and, depending on the quality of the initialization, it may take from minutes to hours of

running time. Our approach provides a globally optimal solution for 7DOF kinematics subchains of more complex mechanisms and could be used to initialize the kinematic part of motion planning.

Work [17] presented an IK solution for 7DOF manipulators with zero link offsets, e.g., the *KUKA LBR IIWA* manipulators. The solution uses special kinematics of its class of manipulators to decompose the general IK problem into two simpler IK problems that can be solved in a closed form. The one-dimensional variety of self-motions becomes circular, and hence the paper proposes to parameterize it by the angle of a point of the circle. Our approach generalizes this solution to a general 7DOF manipulator and shows that it is feasible to solve the IK problem for completely general 7DOF manipulators and optimize over their self-motion varieties.

Paper [7] presents a global (but only approximate) solution to the IK for 7DOF manipulators. It formulates the IK problem as a mixed-integer convex optimization program. The key idea of the paper is to approximate the nonconvex space of rotations by piecewise linear functions on several intervals that partition the original space. This turns the original nonconvex problem into an approximate convex problem when a correct interval is chosen. Selecting the values of auxiliary binary variables to pick the actual interval of approximation leads to the integer part of the optimization. This was the first practical globally optimal approach, but it is only approximate and delivers solutions with errors in units of centimeters and units of degrees. It also fails to detect about 5 % of infeasible poses. Our approach solves the original problem with sub-10^{-6} mm and sub-10^{-3} degree error, and we can solve/decide the feasibility in all but 0.1 % of the tested cases. The computation times of [7] and our approach are roughly similar in units of seconds.

A global and precise solution to the IK problem for redundant serial manipulators is presented in [25]. It models the kinematic constraints as a distance geometry problem. Alongside a novel formulation of the joint limit constraints, it delivers quadratic constraints only. The final configuration is found as the nearest configuration to the given one while satisfying the quadratic constraints. This approach leads to a QCQP problem, which is solved by an SDP relaxation with a global optimality certificate and infeasibility detection. Their implementation is fast (2.5 ms per pose) and accurate (sub-10^{-2} mm position error) with the failure rate of less than 0.4 %. This formulation is restricted only to revolute joints for planar manipulators and spherical joints for three-dimensional ones. It does not take into account the full rotation of each link and can thus solve only for simplified situations. In contrast, our method is general and applicable to any serial manipulator with revolute joints. Moreover, any spherical joint can be modeled as three revolute joints with the advantage of finer control of the joint angle limits.

3 Problem formulation

Here we formulate the IK problem for 7DOF serial manipulators as a semialgebraic optimization problem with a polynomial objective function.

The task is to find the joint coordinates of the manipulator in a way that the end-effector reaches the desired pose in space. The IK problem is called underconstrained for manipulators, which have more DOF than they require to execute the given task. In our case, to reach the desired pose in space, the manipulators require to have six DOFs, and therefore the IK problem for a 7DOF manipulator is underconstrained. The consequence is that the IK problem has an infinite number of solutions for reachable generic end-effector poses for such manipulators. This results in the self-motion property of these manipulators. A self-motion is a motion of a manipulator, which is not observed in the task space, i.e., the end-effector pose of the manipulator is constant while the links of the manipulator are moving. Therefore, moving the manipulator along a path consisting of joint configurations of different solutions of the IK problem for the same pose in space, will result in a self-motion of the manipulator.

The self-motion property provides the manipulator more adaptability since it allows, e.g., to avoid more obstacles in the path and to avoid singularities, which leads to a more versatile mechanism. On the other hand, increasing the number of degrees of freedom increases the difficulty of the IK problem computation dramatically. The IK problem has no longer a finite number of solutions, and thus, it is meaningful to formulate it as a constrained optimization problem choosing the optimal solution from the set of all feasible solutions.

3.1 The goal of our work

In this work, we present a general method for solving the IK problem for 7DOF serial manipulators. We aim at a method that solves the IK problem and that selects the globally optimal solution w.r.t. the given objective function from the infinite number of all feasible solutions. It is naturally more time-consuming to find the global solution than to find any solution, and therefore we do not expect our method to be an online method. For online methods, such as used in the control units of manipulators, the local methods are more suitable as they are fast and sufficiently accurate.

We see the application of our method in the design and exploration of the capabilities of the manipulators. The offline method suits these tasks well as we are not typically limited by computation time. Our method can be, e.g., used when designing new 7DOF serial manipulators and optimizing their parameters, such as the manipulability in regions of interest of the Cartesian space. We develop our method for 7DOF serial manipulators, which are currently the most common redundant manipulators in the industry.

We next show how the IK problem for 7DOF serial manipulators can be modeled as a polynomial optimization problem (POP).

3.2 Forward kinematics

We describe manipulators by the Denavit-Hartenberg (D-H) convention [12] to construct D-H transformation matrices $M_i(\theta_i) \in \mathbb{R}^{4\times 4}$ from link i to $i-1$. D-H matrices are parameterized by joint angles θ_i. The product of the D-H matrices for i from 1 to 7 gives us the transformation matrix M, which represents the transformation from the end-effector coordinate system to the base coordinate system

$$\prod_{i=1}^{7} M_i(\theta_i) = M. \tag{1}$$

The matrix M consists of the position vector $t \in \mathbb{R}^3$ and the rotation matrix $R \in SO(3)$, which together represent the end-effector pose w.r.t. the base coordinate system. When knowing the joint angles θ_i, a straightforward evaluation of Eqn. (1) gives the end-effector pose in the base coordinate system.

Due to kinematic constraints, manipulators come with joint limits, i.e., with restrictions on the joint angles θ_i. Typically, the maximal θ_i^{High} and minimal θ_i^{Low} values of joint angles are given as

$$\theta_i^{Low} \leq \theta_i \leq \theta_i^{High},\ i = 1, \ldots, 7. \tag{2}$$

3.3 Inverse kinematics problem

The forward kinematics problem is very easy to solve for serial manipulators. On the other hand, the IK problem is much more difficult for serial manipulators since it leads to solving systems of polynomial equations. To solve the IK problem, we set up our desired pose of the end-effector in the form of matrix M and then solve matrix Eqn. (1) for the joint coordinates θ_i.

For redundant manipulators, there is an infinite number of solutions, and therefore we introduce an objective function to select a solution for which the value of the objective function is minimal. In our case, we prefer the solutions that minimize the weighted sum of the distances of the joint angles $\boldsymbol{\theta} = [\theta_1, \ldots, \theta_7]^\top$ from their preferred values $\hat{\boldsymbol{\theta}} = [\hat{\theta}_1, \ldots, \hat{\theta}_7]^\top$

$$\min_{\boldsymbol{\theta} \in \langle -\pi, \pi)^7} \sum_{i=1}^{7} w_i \left| \operatorname{angdiff}(\theta_i, \hat{\theta}_i) \right|, \tag{3}$$

where $w_i \geq 0$, $\sum_{i=1}^{7} w_i = 1$ and the function $\operatorname{angdiff}(\alpha, \beta)$ calculates the difference $\alpha - \beta$ and wraps it on the interval $\langle -\pi, \pi)$[1]. This objective function is widely used in the literature, e.g., in [28]. In practice, the preferred values $\hat{\boldsymbol{\theta}}$ can be set to the previous configuration of the manipulator, to minimize the total movement of the manipulator to reach the desired pose.

[1] The output of the function $\operatorname{angdiff}(\alpha, \beta)$ can be computed as $\big((\alpha - \beta + \pi) \bmod 2\pi\big) - \pi$.

Next, we add the joint limits to obtain the following optimization problem

$$\begin{aligned} &\min_{\boldsymbol{\theta}\in\langle-\pi,\pi\rangle^7} \sum_{i=1}^{7} w_i \left|\text{angdiff}(\theta_i, \hat{\theta}_i)\right| \\ &\text{s.t. } \textstyle\prod_{i=1}^{7} M_i(\theta_i) = M \\ &\qquad \theta_i^{Low} \leq \theta_i \leq \theta_i^{High} \quad (i = 1, \ldots, 7) \end{aligned} \tag{4}$$

To be able to use the techniques of polynomial optimization, we need to remove the trigonometric functions that appear in Eqn. (1). We do that by introducing new variables $\boldsymbol{c} = [c_1, \ldots, c_7]^\top$ and $\boldsymbol{s} = [s_1, \ldots, s_7]^\top$, which represent the cosines and sines of the joint angles $\boldsymbol{\theta} = [\theta_1, \ldots, \theta_7]^\top$, respectively. Then, we can rewrite Problem (4) in the new variables. To preserve the structure of the problem, we need to add the trigonometric identities

$$q_i(\boldsymbol{c}, \boldsymbol{s}) = c_i^2 + s_i^2 - 1 = 0, \ i = 1, \ldots, 7. \tag{5}$$

Matrix Eqn. (1) contains 12 trigonometric equations and can be directly rewritten as 12 polynomial equations of degrees up to seven in the newly introduced variables. To lower the maximal degree of the equations, we use fact that the inverse of a rotation matrix is its transpose, i.e., it is a linear function of the original rotation matrix, and rewrite Eqn. (1) as

$$\prod_{i=3}^{5} M_i(\theta_i) - M_2^{-1}(\theta_2) M_1^{-1}(\theta_1) M M_7^{-1}(\theta_7) M_6^{-1}(\theta_6) = 0. \tag{6}$$

It reduces the maximal degree of the polynomials in unknowns $\boldsymbol{c}$ and $\boldsymbol{s}$ to four. We denote these polynomials in Eqn. (6) as

$$p_j(\boldsymbol{c}, \boldsymbol{s}) = 0, \ j = 1, \ldots, 12. \tag{7}$$

The next step is to change the objective function (3) to a polynomial in the new variables $\boldsymbol{c}, \boldsymbol{s}$. Instead of evaluating the distance between the joint angles and their preferred values, we can do the same in the space of their cosines and sines to reach the same goal, i.e., to get $\boldsymbol{\theta}$ as close as possible to $\hat{\boldsymbol{\theta}}$. Choosing the proper ℓ_p norm for the problem at hand may lead to a more straightforward solution to the problem (e.g., the ℓ_∞ norm is often used in multiple view geometry problems [15] to obtain a convex relaxation of the original problem). We have decided to use the squared ℓ_2 norm on the cosines and sines since it leads to an objective function, which is linear in the new variables $\boldsymbol{c}$ and $\boldsymbol{s}$:

$$\min_{\boldsymbol{c}\in\langle-1,1\rangle^7,\ \boldsymbol{s}\in\langle-1,1\rangle^7} \sum_{i=1}^{7} w_i \Big((c_i - \cos\hat{\theta}_i)^2 + (s_i - \sin\hat{\theta}_i)^2 \Big) \tag{8}$$

$$= \min_{\boldsymbol{c}\in\langle-1,1\rangle^7,\ \boldsymbol{s}\in\langle-1,1\rangle^7} \sum_{i=1}^{7} 2w_i (1 - c_i \cos\hat{\theta}_i - s_i \sin\hat{\theta}_i). \tag{9}$$

After rewriting the joint limit inequalities into a polynomial form, we obtain the following final polynomial optimization problem

$$\begin{aligned} &\min_{\boldsymbol{c}\in\langle-1,1\rangle^7,\ \boldsymbol{s}\in\langle-1,1\rangle^7} \sum_{i=1}^{7} 2w_i (1 - c_i \cos\hat{\theta}_i - s_i \sin\hat{\theta}_i) \\ &\text{s.t.} \qquad\qquad p_j(\boldsymbol{c}, \boldsymbol{s}) = 0 \quad (j = 1, \ldots, 12) \\ &\qquad\qquad q_i(\boldsymbol{c}, \boldsymbol{s}) = 0 \quad (i = 1, \ldots, 7) \\ &\qquad -(c_i + 1)\tan\tfrac{\theta_i^{Low}}{2} + s_i \geq 0 \quad (i = 1, \ldots, 7) \\ &\qquad (c_i + 1)\tan\tfrac{\theta_i^{High}}{2} - s_i \geq 0 \quad (i = 1, \ldots, 7) \end{aligned} \tag{10}$$

We next show how to solve this polynomial optimization problem in a general way such that any objective function can be chosen as long as it can be expressed as a low degree polynomial in sines and cosines of the joint angles. Despite different objective functions will be chosen for different tasks, we demonstrate the presented approach with a classical objective function (9).

After solving Problem (10), we recover $\boldsymbol{\theta}$ from $\boldsymbol{c}$ and $\boldsymbol{s}$ by function atan2, which takes into account signs of the arguments.

4 Polynomial optimization

Here we describe the polynomial optimization methods we use to solve Problem (10).

Polynomial optimization problems (POPs) are generally nonconvex, but they can be solved with global optimality certificates with the help of convex optimization, as surveyed in [21]. The idea consists of building a hierarchy of convex optimization problems of increasing size whose values converge to the value of the POP. The convergence proof is based on the results of real algebraic geometry, namely, on the representation of positive polynomials, or Positivstellensatz (PSatz for short). One of the most popular PSatz is due to Putinar [29], and it expresses a polynomial positive on a compact basic semialgebraic set as a weighted sum of squares (SOS).

Finding this SOS representation amounts to solving a semidefinite programming (SDP) problem, a particular convex optimization problem that can be solved efficiently numerically with interior point algorithms. By increasing the degree of the SOS representation, we increase the size of the SDP problem, thereby constructing a hierarchy of SDP problems. Dual to this polynomial positivity problem is the problem of characterizing the moments of measures supported on a compact basic semialgebraic set. This also admits an SDP formulation, called moment relaxations, yielding a dual hierarchy indexed by the so-called relaxation order.

The primal-dual hierarchy is called the moment-SOS hierarchy, or the Lasserre hierarchy since it was first proposed in [20] in the context of POP with convergence and duality proofs. As the relaxation order increases, the Lasserre hierarchy generates a monotone sequence of superoptimal bounds on the global optimum of a given POP. Eventually, the result on the moment problem can be used to certify the exactness of the bound for the current relaxation order. This solves the original nonconvex POP at the price of solving a relaxed convex SDP problem of typically (quite) bigger size than was the original problem. A Matlab package GloptiPoly [13] has been designed to construct the SDP problems in the hierarchy and solve them with a general-purpose SDP solver.

As observed in many applications, the main limitation of the Lasserre hierarchy (in its original form) is its poor scalability as a function of the number of variables and the degree of the POP. This is balanced by the practical observation that, very often, global optimality is certified by the second or third-order relaxation. As our experiments reveal, for the degree 4 POP studied in our paper, the third-order relaxation is out of reach of state-of-the-art SDP solvers. It becomes hence critical to investigate reformulation techniques to reduce the degree as much as possible. This is the topic of the next section.

5 Symbolic reduction of the POP

Here we provide the description of the algebraic geometry technique we use to reduce the degree of our POP problem to obtain a practical solving method. See [6] for algebraic-geometric notation and concepts.

Let us assume that our POP is constrained by polynomial equations

$$f_1 = \cdots = f_s = 0 \tag{11}$$

of degree 4 in $\mathbb{Q}[x_1, \ldots, x_n]$. Observe that one can replace these polynomial equations in the POP formulation with any other set of polynomial equations

$$g_1 = \cdots = g_t = 0 \tag{12}$$

as long as both systems of equations have the same solution set. Natural candidates for g_i's are polynomials in the *ideal* generated by $f_1, \ldots, f_s$, i.e., in the set of algebraic combinations $I = \{\sum_i q_i f_i \mid q_i \in \mathbb{Q}[x_1, \ldots, x_n]\}$, which we denote as $I = \langle f_1, \ldots, f_s \rangle$. It is clear that if all f_i's vanish simultaneously at a point, any polynomial g in this set I will vanish at that point.

The difficulty is how to understand the structure of this set and find a nice finite representation of it that would allow many algebraic operations (such as deciding whether a given polynomial lies in this set). Solutions have been brought by symbolic computation, aka computer algebra, through the development of algorithms computing *Gröbner bases*, which were introduced by Buchberger, see [6].

These are finite sets, depending on a monomial ordering [6], which generate I as input equations do, but from which the whole structure of I can be read off.

Modern algorithms for computing Gröbner bases ($F4$ and $F5$ algorithms), which significantly improved by several orders of magnitude the state-of-the-art, were introduced next by J. C. Faugère [10, 11]. These latter algorithms bring a linear algebra approach to Gröbner bases computations. In particular, noticing that the intersection of I with the subset of polynomials in $\mathbb{Q}[x_1, \ldots, x_n]$ of degree $\leq d$ is a vector space of finite dimension is the key to reduce Gröbner bases computations to exact linear algebra operations.

Hence, Gröbner bases provide bases of such vector spaces when one uses monomial orderings which filter monomials w.r.t. degree first. Finally, going back to our problem, a Gröbner basis computation allows us to discover if I contains degree 2 polynomials (and is generated by such quadrics).

While this is never the case when starting with a generic POP of degree 4, observe that there are many relations between the coefficients of the degree 4 equations of our POP. Hence, we are not facing a generic situation here, and we will see further that a Gröbner basis computation provides a set of quadrics that can replace our initial set of constraints. Note also that since Gröbner basis algorithms rely on exact linear algebra, such a property holds for every generic instance of our POP if it holds for a randomly chosen one (the trace of the computation will always be the same, giving rise to polynomials of degree ≤ 2).

6 Solving the IK problem

To solve the IK problem, we need to solve the optimization problem (10). First, we apply the implementation GloptiPoly [13] of the method described in Section 4 directly on the Problem (10).

6.1 Direct application of polynomial solver

Since the original Problem (10) contains polynomials of degree four, we start with the first relaxation of order two. It means we substitute each monomial in the original 14 variables up to degree four by a new variable, and therefore the resulting SDP program will have 3060 variables.

Solving the first relaxation typically does not yield the solution for this parameterization of the problem, and therefore it is required to go higher in the relaxation hierarchy. Unfortunately, the relaxation order three for a polynomial problem in 14 variables leads to an SDP problem in 38 760 variables. Such a huge problem is still often solvable on contemporary computers, but it often takes hours to finish.

An example of using the GloptiPoly package [13] to solve a polynomial problem is shown in Listing 1. In the first part (lines 1–5), we create the unknowns `c` and `s` and load the kinematic parameters of the manipulator from a file. In the second part (lines 7–15), based on the kinematic parameters, function `FKT` generates the polynomials from the forward kinematics (Eqn. (7)), and we add to them the trigonometric identities (Eqn. (5)). Then, function `jointLimits`, also based on the kinematic parameters of the manipulator, generates the inequalities coming from the joint limits. We define the objective function `f`, which depends on the weights `w` and preferred values of the joint angles `thHat`. In the third part (lines 17–21), we configure GloptiPoly. We use the YALMIP toolbox [23] to define the SDP problems and the MOSEK solver [2] to solve them. In the fourth part (lines 23–27), we use GloptiPoly to create the SDP problem from the POP for the given relaxation order `relaxOrder` and solve it. In the last part (lines 29–34), we extract the solutions if we have succeeded in solving. The variable `status` is set to -1 if the POP is infeasible, to 0 if GloptiPoly can not certify the globality of the found solution, or to 1 if the found optimum was certified as global. If it is set to 1, then in the variable `obj` is the value of the objective function evaluated on the solution, and we can recover the joint angles from their sines and cosines by the function `atan2`.

6.2 Symbolic reduction

In the view of the previous paragraph, we aim at simplifying the original polynomial problem to be able to obtain solutions even for the relaxation of order two, which takes seconds to solve.

Listing 1: Demonstration of the usage of the GloptiPoly [13] package in Matlab to solve a polynomial optimization problem arising from an IKT.

```
% initialize the GloptiPoly unknowns
mpol('c', 7);
mpol('s', 7);
% load the kinematic parameters of the manipulator
manipulator = load('manipulator.mat');

%% PREPARE THE POLYNOMIALS
% polynomials from the forward kinematics problem (Eqn. (7))
eq = FKT(manipulator, c, s);
% add the trigonometric identites (Eqn. (5))
eq = [eq; c.^2 + s.^2 - 1];
% polynomials representing the joint limits inequalities
ineq = jointLimits(manipulator, c, s);
% polynomial objective function
f = sum(w.*2*(-c.*cos(thHat) - s.*sin(thHat) + 1));

%% CONFIGURE GLOPTIPOLY
% use YALMIP [23] toolbox for the definition of the SDP problem
mset('yalmip', true);
% set MOSEK [2] solver as the external SDP problem solver
mset(sdpsettings('solver', 'mosek'));

%% SOLVE THE POLYNOMIAL PROBLEM
% create the SDP problem from the POP
SDP = msdp(min(f), eq == 0, ineq >= 0, relaxOrder);
% solve it
[status, obj] = msol(SDP);

%% EXTRACT THE SOLUTIONS
% check that the POP is feasible
if status == 1
  % recover the angles from their sines and cosines
  theta = atan2(double(s), double(c));
end
```

Here is our main result that allows us to do it. We claim that polynomials p_j and q_i of degrees up to four in Problem (10) can be reduced to polynomials of degree two.

Theorem 1. *The ideal generated by the kinematics constraints (7) for a generic serial manipulator with seven revolute joints and for generic pose M with the addition of the trigonometric identities (5) can be generated by a set of degree two polynomials.*

Proof. The proof is computational. We generate generic instances of serial manipulators and generic poses. Then a Gröbner basis G [5] of polynomials p_j and q_i is computed for each instance of the manipulator and pose. We select a subset S of degree two polynomials from the basis G and, by computing a new Gröbner basis G' from S, we verify that S generates the same ideal as the original set of polynomials. The structure of the computational proof is shown in Fig. 2. See Maple code in Listing 2. The polynomials p_j and q_i are put into the variable `eq`. The last command of the code will be evaluated to `True` if the bases G and G' are equal, and therefore generate the same ideal. □

We want to point out that a simple selection of the second-degree polynomials from the original polynomials p_j and q_i is not enough. They do not generate the same ideal as all polynomials p_j and q_i. First, the Gröbner basis G of all polynomials p_j and q_i has to be computed. Then, the second-degree polynomials can then be extracted from G and used as a replacement for polynomials p_j and q_i.

Listing 2: Maple code for the proof of Theorem 1.

```
# compute the reduced Groebner basis from polynomials pj and qi (stored in the variable
    eq)
G := Basis(eq, tdeg(op(indets(eq)))):
# select degree two polynomials from the basis and compute a new reduced Groebner basis
idxDegTwo := SearchAll(2, map(degree, G)):
eqPrime := G[[idxDegTwo]]:
GPrime := Basis(eqPrime, tdeg(op(indets(eq)))):
# compare the two bases
evalb(G = GPrime);
                                                    true
```

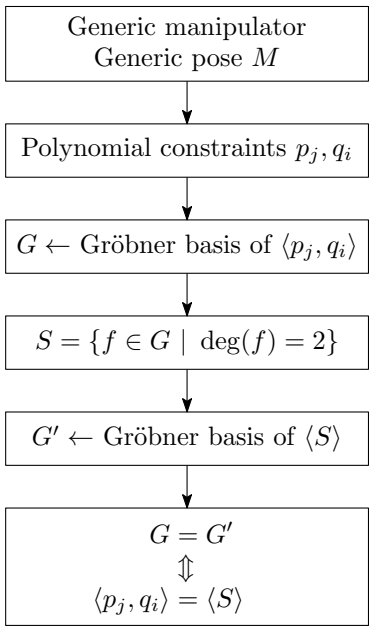

Figure 2: Diagram describing the procedure of the proof of Theorem 1.

6.3 Solving the reduced polynomial optimization problem

We exploit Theorem 1 in our approach to solve the IK problem. First, we compute a Gröbner basis G of the kinematics constraints (7) and (5), and we select only the subset S of polynomials of degree two in G. Then, we construct the Problem (10) but with the degree two polynomial constraints S only. We solve the problem by using the hierarchy of semidefinite programs as described above.

Reducing the degree of polynomials from four to two allows us to start with SDP relaxation of order one. The size of this SDP problem, in terms of the number of variables, is now 120. Practical experiments have shown that the first relaxation is not tight enough to yield the solution. On the other hand, the second relaxation gives a solution for almost all poses, see Tab. 3.

6.4 Overcoming numerical issues

It may happen that the POP solver is not able to certify that the optimum found is indeed global. The certification is based on the computation of the rank of a matrix of floats returned by the SDP solver, and it may fail because of two reasons.

First, it may happen that the relaxation order was not high enough, and we have to go higher in the hierarchy. Secondly, the relaxation is actually tight, but the numerical rank of the matrix returned by the SDP is hard to compute correctly. This is because of the numerical issues of the SDP solver caused by the significant number of variables and constraints that typically have SDPs originated from the Lasserre hierarchies. Currently, we do not distinguish these two cases and whenever we are unable to certify the optimum, we say that the method has failed for the given end-effector pose.

Naturally, we want to minimize the number of end-effector poses that our method fails to compute. For a given relaxation order, we can thus only reduce the number of failing poses by addressing the numerical issues of the SDP solver used.

From our experience and as we show in Section 7, SDPs with sharper objective function at the global minimum have a lower failure rate. By a sharper function at a point we mean that it has a greater absolute value of its second derivative at that point. Therefore, we find a new objective function that attains the global minimum at the same point as Eqn. (9), but it is sharper at the global minimum. If the original objective function is nonnegative, its higher powers do not change the argument of the minimum but typically make them sharper. In particular, the second derivative at any critical point x_0 of a function $\xi^n(x)$ is multiplied by factor $n\xi^{n-1}(x_0)$ w.r.t. the second derivative of function $\xi(x)$ at the same critical point for $n \in \mathbb{N}$:

$$(\xi^n)''(x_0) = n\,\xi^{n-1}(x_0)\xi''(x_0). \tag{13}$$

As long as the multiplication factor $n\,\xi^{n-1}(x_0) \geq 1$, the function ξ^n is sharper at the point x_0. This is fulfilled for any $n \in \mathbb{N}$ when $\xi(x_0) \geq 1$ for the critical point x_0.

In our case, the objective function (9), here denoted as L, is a weighted sum of cosine functions shifted in phase and along the vertical axis:

$$L(\boldsymbol{c}, \boldsymbol{s}, \hat{\boldsymbol{\theta}}, \boldsymbol{w}) = \sum_{i=1}^{7} 2w_i \ell(c_i, s_i, \hat{\theta}_i), \tag{14}$$

where

$$\ell(c_i, s_i, \hat{\theta}_i) = 1 - c_i \cos\hat{\theta}_i - s_i \sin\hat{\theta}_i. \tag{15}$$

The range of ℓ is $\langle 0, 2\rangle$ and has only one minimum with value 0 at point $\hat{\theta}_i$. To fulfill the condition above, we shift ℓ vertically by $+1$:

$$\bar{\ell}(c_i, s_i, \hat{\theta}_i) = \ell(c_i, s_i, \hat{\theta}_i) + 1. \tag{16}$$

Then, we can take higher powers of $\bar{\ell}$ to get sharper objective functions without a change of the argument of the minimum.

Our new sharper objective function replacing of Eqn. (9) becomes

$$\min_{\boldsymbol{c}\in\langle -1,1\rangle^7,\ \boldsymbol{s}\in\langle -1,1\rangle^7} \sum_{i=1}^{7} 2w_i(2 - c_i\cos\hat{\theta}_i - s_i\sin\hat{\theta}_i)^n \tag{17}$$

for any $n \in \mathbb{N}$. Even though the argument of the minimum of ℓ and $\bar{\ell}^n$ is the same, it is not true for their weighted sums. Therefore, Eqn. (17) has a different argument of the minimum than Eqn. (9), but still Eqn. (17) keeps the idea of the original non-polynomial objective function (3).

Since we use the second relaxation order in the Lasserre hierarchies, we can use an objective function of up to degree four without the need for enlargement of the relaxation order. Because $\bar{\ell}$ has degree one, we can use n up to four.

We show that taking higher power of the objective function improves the failure rate of our method in Section 7.

6.5 Rational approximation

The end-effector pose M consists of translation vector $t \in \mathbb{R}^3$ and rotation matrix $R \in SO(3)$, which are, as well as the D-H parameters of the manipulator, in practice given in their floating-point representation. This is a common approach as these values are typically an outcome of some planning algorithm (the end-effector pose) or measured and calibrated (the parameters of the manipulator). Moreover, the identities that must hold for rotation matrices are often violated not only because of the floating-point representation but also because the algorithms producing them neglect the importance of the identities.

Table 1: Kinematic parameters of the *KUKA LBR IIWA* manipulator.

$\boldsymbol{i}$	$\boldsymbol{d_i}$ [mm]	$\boldsymbol{\theta_i}$	$\boldsymbol{a_i}$ [mm]	$\boldsymbol{\alpha_i}$ [rad]	$\boldsymbol{\theta_i^{Low}}$ [°]	$\boldsymbol{\theta_i^{High}}$ [°]
1	340	par.	0	$-\frac{\pi}{2}$	−170	170
2	0	par.	0	$\frac{\pi}{2}$	−120	120
3	400	par.	0	$-\frac{\pi}{2}$	−170	170
4	0	par.	0	$\frac{\pi}{2}$	−120	120
5	400	par.	0	$-\frac{\pi}{2}$	−170	170
6	0	par.	0	$\frac{\pi}{2}$	−120	120
7	126	par.	0	0	−175	175

When a numerical method (such as GloptiPoly) is used to find the solution of the IK task, everything works smoothly as long as the problem is well-conditioned. For us, this is the case when we directly apply the polynomial solver, as described in Section 6.1.

On the other hand, symbolic methods require to compute exactly. Therefore, if we want to use a symbolic method, e.g., as in Section 6.3, we need to pass from floating-point numbers to exact rational numbers and ensure that all the identities, following from sines, cosines and rotations that have to hold, are valid. Otherwise, the feasible set of our equations and inequalities for such data will be empty, even if it is nonempty on exact data.

The input for the symbolic reduction method (Section 6.3) is the D-H parameters of the manipulator and the end-effector pose M, which are floating points and need to be approximated by rational numbers. The D-H parameters are a) the lengths a_i and d_i, which we approximate by rounding them to 2κ digits to the right of the decimal point, where $\kappa \in \mathbb{N}$ and treat them as rational numbers, and b) the angles α_i. In Eqn. (6), we only need rational values of $\sin\alpha_i$ and $\cos\alpha_i$ such that the trigonometric identities $\sin^2\alpha_i + \cos^2\alpha_i = 1$ hold. How to obtain such rational representation in an optimal way w.r.t. its bit size has been proposed, e.g., by [4]. For simplicity, we have used a nonoptimal approach in this paper. To provide the rational representation, we round $\tan\frac{\alpha}{2}$ to κ digits to the right of the decimal point, which we denote as τ_i and treat as rational. Then, the sines and cosines of α_i are replaced by their approximation followingly:

$$\cos\alpha_i = \frac{1-\tau_i^2}{1+\tau_i^2}, \tag{18}$$

$$\sin\alpha_i = \frac{2\tau}{1+\tau_i^2}, \tag{19}$$

which are rational functions of τ_i, and therefore also rational.

We approximate the rotational part R and the translational part t of the end-effector pose M independently. The translation vector t is approximated element-wise by rounding to 2κ digits to the right of the decimal point. As for the rotation matrix R, we need to find its approximation in the form of a rational orthonormal matrix to ensure that $R^\top = R^{-1}$ and $\det R = 1$. [27] have introduced several algorithms for finding an optimal approximation of R w.r.t. the bit size of its elements. We use a more straightforward and easier-to-implement method in this paper. To find a rational approximation of R, we convert it to a quaternion q, which we round element-wise to κ digits to the right of the decimal point and denote it as $\bar{q}$. The rounding, of course, violates the condition $\|\bar{q}\| = 1$, and we can not divide $\bar{q}$ by non-rational $\|\bar{q}\|$ to get a rational quaternion. We overcome the issue simply by constructing a rotation matrix from non-unit rational $\bar{q}$, which we then divide by $\|\bar{q}\|^2$, which is rational. The result is a rational rotation matrix, which we use as a replacement for R.

The output of the symbolic reduction method is the degree two polynomials. They have rational coefficients, so we evaluate them using floating-point arithmetic to convert them to floating-point numbers. Then, we can use them in the numerical method (GloptiPoly) to find the solution of the IK task.

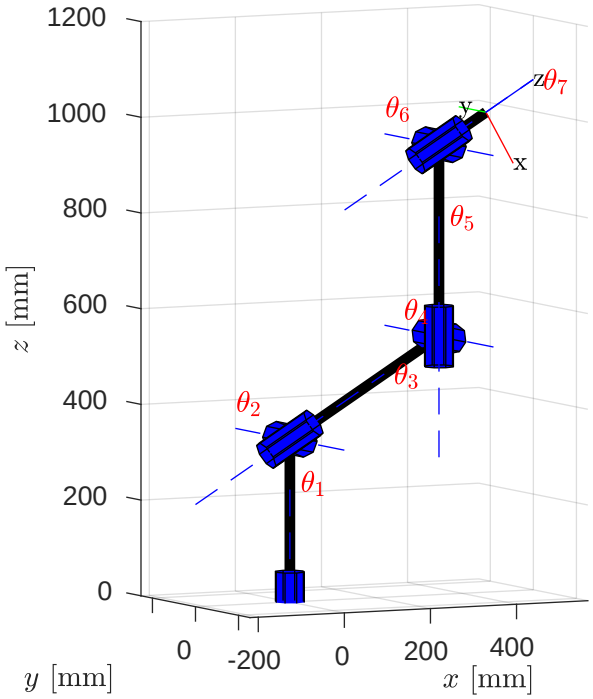

Figure 3: Kinematic model of the *KUKA LBR IIWA* manipulator.

Table 2: Overview of the execution times and accuracy of the presented methods. Results for both methods are for the relaxation order $r = 2$ and the power of the objective function (17) $n = 4$. Methods have been evaluated on the *KUKA LBR IIWA* manipulator.

Method	**Average execution time**		**Median error**		**% of failed poses**
	Reduction step	**GloptiPoly**	**Translation**	**Rotation**	
Naïve (Section 7.1.2)	—	12.9 s	$1.69 \cdot 10^{-5}$ mm	$2.41 \cdot 10^{-6}$ deg	28.4 %
With symbolic reduction (Section 7.1.3	2.6 s	3.6 s	$1.22 \cdot 10^{-6}$ mm	$5.56 \cdot 10^{-3}$ deg	0.07 %

7 Experiments

We demonstrate our method on the IK problem for the *KUKA LBR IIWA* arm with seven revolute joints, which is simple to solve. Then, we modify the *KUKA LBR IIWA* manipulator to obtain a more general manipulator, which is much harder to solve. We show that we are able to solve the IK problem successfully for this modified, but still not fully generic, manipulator. Finally, we randomly generate a completely generic serial manipulator with seven revolute joints and solve the IK problem for it.

7.1 The *KUKA LBR IIWA* manipulator

The manipulator structure is designed in a special way such that the IK problem is simple to compute. There are three sequences of three consecutive revolute joints whose axes of motion intersect in a single point. Namely, they are the joints $(1, 2, 3)$, $(3, 4, 5)$, and $(5, 6, 7)$, see the kinematic model of the manipulator in Fig. 3 based on the kinematic parameters from Tab. 1. Each of these triplets can be substituted by a single spherical joint. Such property makes the manipulator a very nongeneric serial manipulator.

Moreover, it is designed in such a special way that the joint angle θ_4 is constant within the self-motion for a fixed end-effector pose. This allows for a geometrical derivation of a closed-form solution

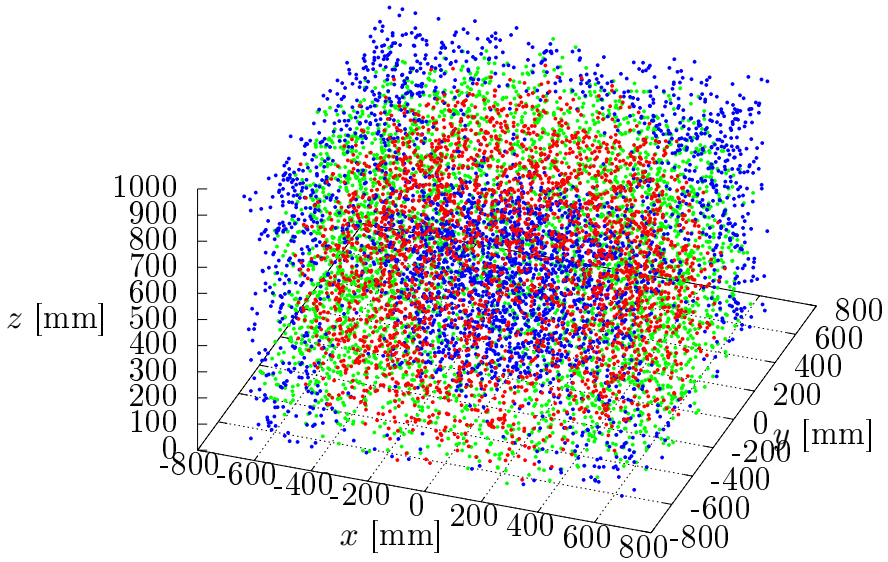

Figure 4: Generated poses for the *KUKA LBR IIWA* manipulator. Green dots are poses marked as feasible by direct solving with GloptiPoly, blue as infeasible, and for the red ones, the computation failed (29.2 %). The illustration is for the relaxation order $r = 2$ and the power of the objective function (17) $n = 4$.

Table 3: Comparison of failure rates and average execution times (end-to-end) of our proposed methods for different relaxation orders and powers of the objective function. Methods have been evaluated on the *KUKA LBR IIWA* manipulator.

Method	**Relaxation order**	**Power of the objective function (17)**							
		$n = 1$		$n = 2$		$n = 3$		$n = 4$	
Naïve (Section 7.1.2)	$r = 2$	34.1 %	14.1 s	30.6 %	13.6 s	29.2 %	14.4 s	28.4 %	13.6 s
With symbolic reduction (Section 7.1.3)	$r = 1$	38.3 %	3.6 s	34.7 %	3.6 s	—		—	
With symbolic reduction (Section 7.1.3)	$r = 2$	1.3 %	7.5 s	0.5 %	7.1 s	0.09 %	7.1 s	**0.07 %**	**7.2 s**

to the IK problem, such as [17]. The authors introduce a new angle parameter δ that fixes the left DOF of the IK problem.

Another approach is to solve the problem by local nonlinear optimization techniques [3]. However, such methods do not provide global optima, and the found solution is highly dependent on the initial guess.

Solving the IK problem globally is more computationally challenging. To be able to tackle the problem in a matter of seconds, relaxations of the problem were developed in the past. Dai et al. in [7] proposed mix-integer convex relaxation of the nonconvex rotational constraints. Their method finds all classes of solutions that are in correspondence with a different set of active binary variables. However, they are unable to select a global optima w.r.t. an objective function.

7.1.1 Polynomial optimization problem for *KUKA LBR IIWA*

We directly parameterize Problem (10) by the D-H parameters of the *KUKA LBR IIWA* manipulator (Tab. 1). We set the weights equally to $w_i = \frac{1}{7}$, and we set the preferred values of $\hat{\theta}_i$ to zero, which is in the middle of the joint allowed interval. This leads to POP in 14 variables and with polynomials p_j of degrees up to four.

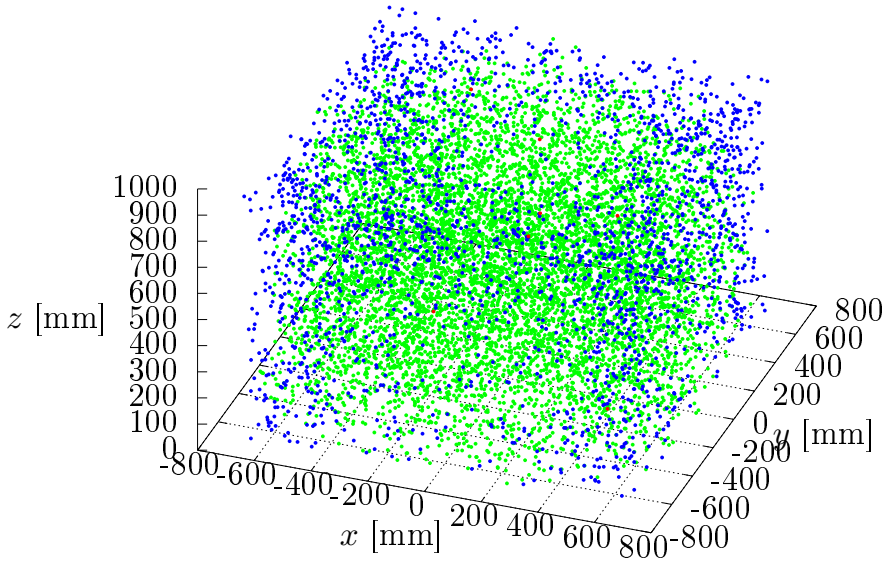

Figure 5: Generated poses for the *KUKA LBR IIWA* manipulator. Green dots are poses marked as feasible by GloptiPoly after symbolic simplification, blue as infeasible, and for the red ones, the computation failed (0.09 %). The illustration is for the relaxation order $r = 2$ and the power of the objective function (17) $n = 4$.

Table 4: Comparison of mean translation error of our proposed methods for different relaxation orders and powers of the objective function. Methods have been evaluated on the *KUKA LBR IIWA* manipulator.

Method	**Relaxation order**	**Power of the objective function (17)**			
		$n = 1$	$n = 2$	$n = 3$	$n = 4$
Naïve (Section 7.1.2)	$r = 2$	$3.14 \cdot 10^{-4}$ mm	$8.91 \cdot 10^{-5}$ mm	$3.07 \cdot 10^{-5}$ mm	$1.69 \cdot 10^{-5}$ mm
With symbolic reduction (Section 7.1.3)	$r = 1$	$1.57 \cdot 10^{-4}$ mm	$4.95 \cdot 10^{-5}$ mm	—	—
With symbolic reduction (Section 7.1.3)	$r = 2$	$4.77 \cdot 10^{-5}$ mm	$9.26 \cdot 10^{-6}$ mm	$3.10 \cdot 10^{-6}$ mm	$1.22 \cdot 10^{-6}$ mm

7.1.2 Direct application of the polynomial solver

First, we solve Problem (10) with objective function (17) with powers n from one to four directly by polynomial optimization toolbox GloptiPoly [13] for relaxation order two with the use of MOSEK [2] as the semidefinite problem solver.

Our dataset consists of 10 000 randomly chosen poses within and outside of the working space of the manipulator, as shown in Fig. 4. For poses marked by red color, GloptiPoly failed to compute the solution or report infeasibility. That is mainly due to the small relaxation order of the semidefinite relaxation of the POP. For the best choice of the power of the objective function $n = 4$ (see Tab. 3), there is 28.4 % of such poses, which makes this approach quite impractical. Computations for the next relaxation of order three are still often feasible on contemporary computers but take hours to finish.

7.1.3 POP with symbolic reduction

Since the performance of GloptiPoly highly depends on the number of variables of the POP and the relaxation degree, which grows with the degrees of the polynomials contained in the POP, we first symbolically reduce polynomials p_j and q_i and then solve the resulting POP by GloptiPoly. To be able to use symbolic computation, we first have to approximate the inputs, which are given in the floating-point representation. We do that according to Section 6.5 with the precision set to $\kappa = 4$.

Firstly, we use the advantage of the simple structure of the *KUKA LBR IIWA* manipulator, i.e., that the joint angle θ_4 is constant within the self-motion. Therefore, it plays no role in the objective

Table 5: Comparison of mean rotation error of our proposed methods for different relaxation orders and powers of the objective function. Methods have been evaluated on the *KUKA LBR IIWA* manipulator.

Method	**Relaxation order**	**Power of the objective function (17)**			
		$n = 1$	$n = 2$	$n = 3$	$n = 4$
Naïve (Section 7.1.2)	$r = 2$	$4.98 \cdot 10^{-5}$ deg	$1.38 \cdot 10^{-5}$ deg	$4.68 \cdot 10^{-6}$ deg	$2.41 \cdot 10^{-6}$ deg
With symbolic reduction (Section 7.1.3)	$r = 1$	$5.62 \cdot 10^{-3}$ deg	$5.59 \cdot 10^{-3}$ deg	—	—
With symbolic reduction (Section 7.1.3)	$r = 2$	$5.60 \cdot 10^{-3}$ deg	$5.57 \cdot 10^{-3}$ deg	$5.56 \cdot 10^{-3}$ deg	$5.56 \cdot 10^{-3}$ deg

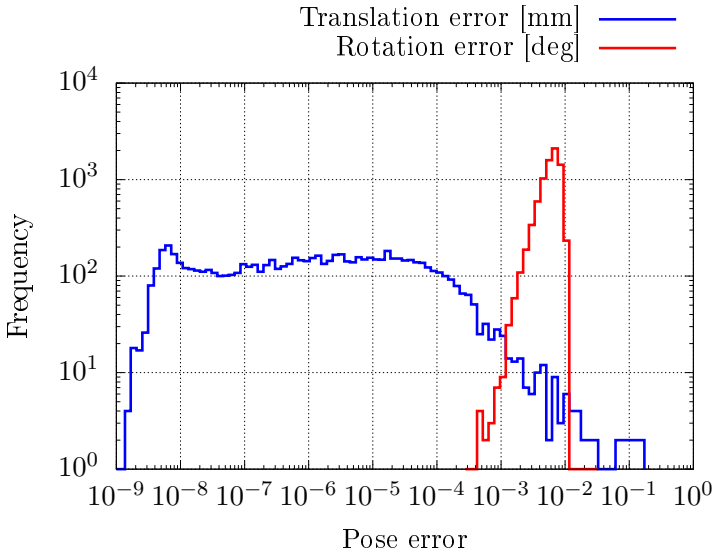

Figure 6: Histogram of translation and rotation error of the poses computed from the forward kinematics on found solutions w.r.t. the desired poses. Showed results are evaluated on the *KUKA LBR IIWA* manipulator by the method with the symbolic reduction step (Section 7.1.3) and for the relaxation order $r = 2$ and the power of the objective function $n = 4$. There are 4 zero translation errors and 0 zero rotation errors.

function (3). That allows us to eliminate the variables c_4 and s_4 from the equations. Secondly, we reduce the polynomials p_j and q_j symbolically with the use of Theorem 1.

In this way, we have reduced the number of variables from 14 to 12, and we have reduced the degrees of the polynomials to two, which significantly speeds up the SDP solver. Practical experiments showed that GloptiPoly is now able to compute IK for more poses with the same relaxation order two than by the naïve approach used before, see Fig. 5. Again, this approach performs best for the highest possible power of the objective function, i.e., $n = 4$ (see Tab. 3). Now only 0.07 % of poses failed to be solved on the same dataset as in Section 7.1.2.

7.1.4 Results

To verify the numerical stability of the solver, we have computed the forward kinematics problem based on the found joint angles from the IK problem. Then, we have computed the translation error and rotation error of this pose w.r.t. the desired pose. The comparison of the errors between the proposed approaches for various values of the relaxation order r and the power of the objective function n can be found in Tab. 4 for the translation error and in Tab. 5 for the rotation error. The drop of precision between the naïve approach and the approach with the symbolic reduction step, which can be noted especially in the rotational part, is mainly due to the "rational approximation" of the end-effector pose and the parameters of the manipulator as described in Section 6.5. The histogram of the translation and rotation error using the best value $r = 2$ and $n = 4$ can be seen in Fig. 6.

From Tab. 3, we can see that it is worth taking higher powers of the objective function as explained in Section 6.4. We have shown that with the increasing power of the objective function, both the

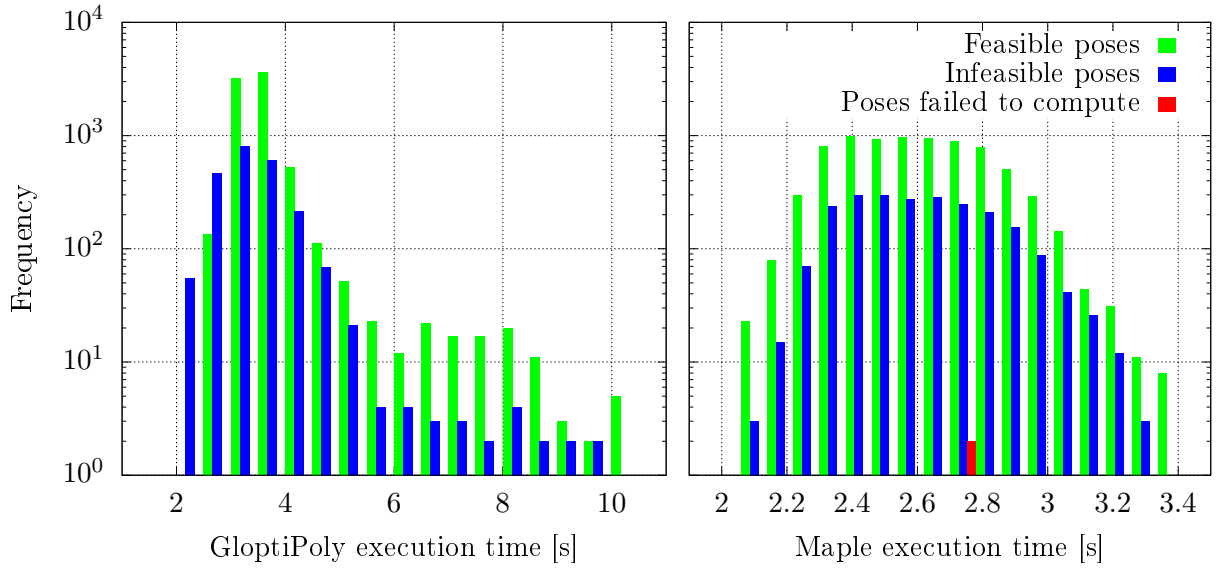

Figure 7: Histograms of execution time. Left: execution time of the online phase of GloptiPoly. Right: execution time of the symbolic reduction and elimination in Maple. Showed results are evaluated on the *KUKA LBR IIWA* manipulator by the method with the symbolic reduction step (Section 7.1.3) and for the relaxation order $r = 2$ and the power of the objective function $n = 4$.

failure rate and the execution time of the methods improve. Both methods perform best for values $r = 2$ and $n = 4$. The overall comparison of the methods for these values can be seen in Tab. 2.

For practical applications, the execution time of this method is essential. In Fig. 7, we show the histograms of the execution time of the online phase of GloptiPoly as well as of the symbolic reduction of the initial polynomials to degree two polynomials. We observe that our execution times are comparable to the computation times in [7] when using off-the-shelf POP and GB computation tools. We next plan to develop optimized solvers leading to considerable speedup, as it was done in solving polynomial systems in computer vision [18, 19].

7.2 Modified *KUKA LBR IIWA* manipulator

To take a step towards solving the IK problem of a generic serial manipulator, we have slightly modified the *KUKA LBR IIWA* manipulator. In particular, we have set the previously zero parameter $a_2 = 100$ mm. See the modified kinematic model in Fig. 8. This change broke the first spherical joint as the axes of rotation of joints 1, 2, and 3 no longer all intersect in one point. Now, it does not hold anymore that the joint angle θ_4 is constant for a fixed end-effector pose, and therefore we can not readily eliminate it from the equations. However, the equations will still be simpler than for a generic manipulator as many of the kinematic parameters are zero.

Again, we set the weights to $w_i = \frac{1}{7}$, the preferred values of the joint angles $\hat{\theta}_i$ to zero, relaxation order to $r = 2$, the power of the objective function to $n = 4$, and the precision of the rounding to $\kappa = 4$. Our dataset of the end-effector poses consists of 500 randomly chosen poses from inside and outside of the working space of the manipulator.

We compare two approaches: direct solving of the IK problem by the baseline POP solver and our symbolically reducing the degree of the polynomials to two and then solving it by the POP solver. The average execution time, failure rate, and the median translation and rotation error for both approaches can be found in Tab. 6.

We can see that now the symbolic reduction step requires much more time than the POP solver. Therefore, the naïve approach is faster as it does not need this reduction step. On the other hand, the naïve approach fails for 26.4 % of poses, but we were able to solve all 500 poses from the dataset by the method with the symbolic reduction step. The rotation and translation errors of both methods are small enough for most practical applications. Again, the drop of precision of the method with the

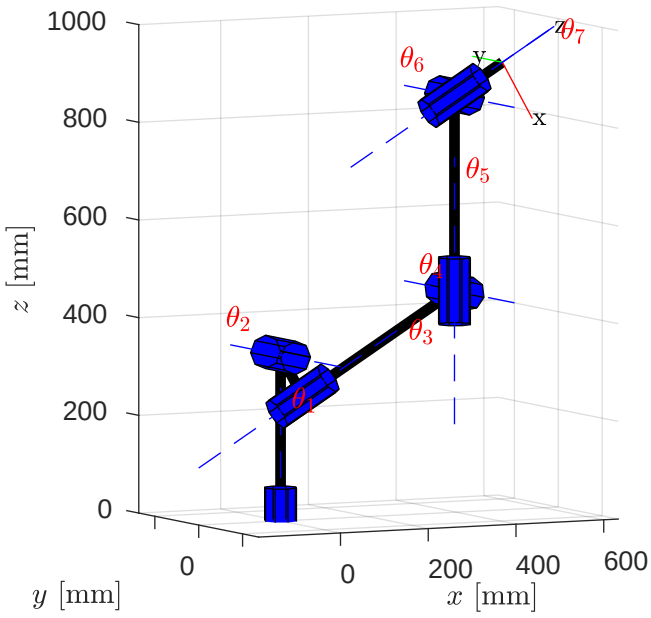

Figure 8: Kinematic model of the modified *KUKA LBR IIWA* manipulator.

Table 6: Overview of the execution times and accuracy of the presented methods. Methods have been evaluated on the modified *KUKA LBR IIWA* manipulator.

Method	**Average execution time**		**Median error**		**% of failed**
	Reduction step	**GloptiPoly**	**Translation**	**Rotation**	**poses**
Naïve	—	12.6 s	$2.23 \cdot 10^{-5}$ mm	$4.18 \cdot 10^{-6}$ deg	26.4 %
With symbolic reduction step	33.9 s	9.9 s	$1.28 \cdot 10^{-6}$ mm	$5.34 \cdot 10^{-3}$ deg	0 %

symbolic reduction step w.r.t. the naïve approach is mainly due to the "rational approximation" of the end-effector poses, which is essential for symbolic computation. On the other hand, this error can be reduced by increasing the κ number of digits used to represent the elements of rational rotation matrices. Alternatively, a few steps of a local optimization, e.g. Newton's method, will dramatically reduce this error too.

7.3 A generic 7DOF serial manipulator

Here we show that we are able to solve the IK problem of a randomly generated fully generic serial manipulator with seven revolute joints.

We randomly generate the kinematic parameters of the manipulator, see Tab. 7 and the corresponding kinematic model in Fig. 9. The values of d_i and a_i were generated as integers from 10 to 100 mm. We have set the allowance interval for all the joints angles θ_i to $\langle -3, 3 \rangle$ rad. From the same interval, we have generated the angles α_i, for which we have found rational representations of their sines and cosines by the same approach as in Section 6.5 with $\kappa = 1$.

The dataset of poses used to verify the IK problem solver consists of 100 poses. The poses were generated from inside and outside of the working space of the manipulator and were then rounded to the rational representation as described in Section 6.5 with $\kappa = 1$.

Contrary to the previous experiments, we rounded the kinematic parameters of the manipulator and the end-effector poses to fractions in advance. This is because the rounding with higher values of κ would lead to very long coefficients in the Gröbner basis computation, which would significantly increase

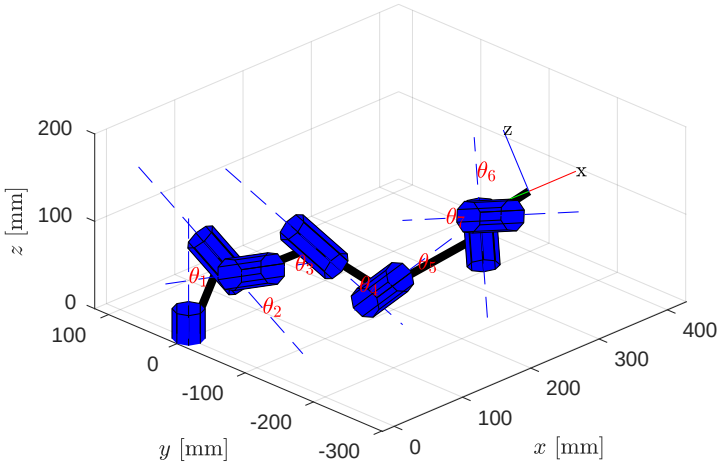

Figure 9: Kinematic model of the randomly generated generic manipulator.

Table 7: Kinematic parameters of the randomly generated generic manipulator.

i	d_i [mm]	θ_i	a_i [mm]	α_i [rad]	$\cos\alpha_i$	$\sin\alpha_i$	θ_i^{Low} [rad]	θ_i^{High} [rad]
1	74	par.	47	2.078	$-\frac{189}{389}$	$\frac{340}{389}$	-3	3
2	26	par.	22	-0.927	$\frac{3}{5}$	$-\frac{4}{5}$	-3	3
3	11	par.	82	0.761	$\frac{21}{29}$	$\frac{20}{29}$	-3	3
4	86	par.	19	-1.081	$\frac{8}{17}$	$-\frac{15}{17}$	-3	3
5	81	par.	85	-0.927	$\frac{3}{5}$	$-\frac{4}{5}$	-3	3
6	16	par.	15	-1.830	$-\frac{69}{269}$	$-\frac{260}{269}$	-3	3
7	35	par.	89	1.350	$\frac{9}{41}$	$\frac{40}{41}$	-3	3

the execution time. However, doing the rounding in the symbolic method directly with $\kappa = 1$ would mean that we would be computing the IK problem for very different (rounded rational representation) kinematic parameters and end-effector poses than we are evaluating the errors for (the original floating-point representation). Therefore, we have decided to use the dataset with rational representations of the end-effector poses.

We again compare two approaches: direct solving of the IK problem by the POP solver and symbolically reducing the degree of the polynomials to two and then solving it by the POP solver. We set all the parameters to the same values as in Section 7.2. The average execution time, failure rate, and the median translation and rotation error for both approaches can be found in Tab. 8.

From the results, we can see that by the naïve approach, we were unable to solve 51.0 % of the end-effector poses. The approach with the symbolic reduction step is able to solve all poses from the dataset, but it takes more than 2 hours on average to symbolically process the equations. In this case, the best approach would be to try to solve a query end-effector pose by the naïve approach, which takes seconds and has approx. a 50 % chance to succeed. If it fails, then one would need to preprocess the equations by the symbolic reduction step. This approach may significantly reduce the required computation time but still keep the failure rate near zero.

8 Conclusions

We presented the first practical method for globally solving the 7DOF IK problem with a polynomial objective function. Our solution is accurate and can solve/decide infeasibility in 99.9 % of cases out of 10 000 cases tested on the *KUKA LBR IIWA* manipulator. We have shown that the method is general

Table 8: Overview of the execution times and accuracy of the presented methods. Methods have been evaluated on the randomly generated generic manipulator.

Method	**Average execution time**		**Median error**		**% of failed**
	Reduction step	**GloptiPoly**	**Translation**	**Rotation**	**poses**
Naïve	—	14.0 s	$1.02 \cdot 10^{-6}$ mm	0 deg	51.0 %
With symbolic reduction step	7331 s	11.0 s	$9.62 \cdot 10^{-8}$ mm	0 deg	0 %

and therefore can be used to solve the IK problem of a generic 7DOF serial revolute manipulator. The code is open-sourced at `https://github.com/PavelTrutman/Global-7DOF-IKT`.

For future work, we consider two interesting directions. First, in the case that the POP constraints are incompatible (i.e., the feasible set of admissible parameters is empty), it would be desirable to return a certificate of infeasibility. This certificate can be either numerical (obtained by solving the moment-SOS hierarchy with an SDP solver) or symbolic (obtained by the Gröbner basis method). It can be obtained, e.g., by computing an SOS representation for the polynomial -1 (or any other negative polynomial) on the quadratic module corresponding to the feasible set. See, e.g., [16] in the specific case of certifying emptiness of spectrahedra (SDP feasibility sets).

Secondly, it would be interesting to exploit the specific structure of the POP studied in this paper to prove (maybe under some assumptions on the data) the exactness of the first or the second SDP relaxation in the moment-SOS hierarchy, i.e., that solving this relaxation always solves the original POP. For Euclidean distance POP arising in computer vision, this was achieved in [1] by arguing on the curvature properties of the Lagrangian and its SOS representation in the quadratic module.

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