**Diplomarbeit**

# The Koopman Linearization of Dynamical Systems


KARI VALENTINA KÜSTER


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Betreuer

DR. ROLAND DERNDINGER

PROF. DR. RAINER NAGEL


Arbeitsbereich Funktionalanalysis

Mathematisches Institut

Eberhard-Karls-Universität Tübingen

# Erklärung

Hiermit erkläre ich, die vorliegende Arbeit selbstständig verfasst zu haben und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt zu haben.

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Kari Küster

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# Contents

# 1 Introduction

*„Nach unserer bisherigen Erfahrung sind wir zum Vertrauen berechtigt, dass die Natur die Realisierung des mathematisch denkbar Einfachsten ist.“*

Albert Einstein

In Mathematics and Physics, people have always tried to model natural phenomena. With so called *dynamical systems* – in general non-linear – processes from various fields such as Biology, Physics and Economy can be described. Often we are interested in the longtime behavior of a system, such as the development of a population size in time.

From a mathematical point of view, a dynamical system is a self-map

$$\varphi : S \to S$$

on a *state space* $S$ which contains all possible states the system can take. By repeated application of the dynamics we can draw conclusions on the evolution of a system in time.

Dynamical systems are crucial objects in the mathematical discipline *Ergodic Theory*, which has its origin in Mathematical Statistics and the considerations of Maxwell and Boltzmann in thermodynamics. It was born as a mathematical field in the 1930s due to the path-breaking work of von Neumann and Birkhoff. Today, characteristic for Ergodic Theory is its interplay with diverse other mathematical fields like Number Theory, Probability Theory, Topology, Operator Theory and Functional Analysis.

At the intersection of Ergodic Theory and Operator Theory lies the so-called *Koopman linearization* which is named after Birkhoff's student Bernard Koopman. With this less common kind of linearization of dynamical systems the perspective is changed from the state space to an *observable space*. We consider how observables on the state space change after application of the dynamical system, which induces a bounded linear operator on a Banach space, the *Koopman operator*.

We demonstrate this change in three steps with an example. We call the lowest level of modelling the *technical space*. An object in the technical space could, e.g., be a bar where we are interested in the heat distribution on it, or a simple box containing moving molecules of a gas. The question then becomes how to capture these objects in a mathematical language.

In the mathematical model the box is a subset of $\mathbb{R}^3$. Each of the $n$ particles of the gas we assign three position and velocity components which depend on the time of measurement. All states which are in principle possible, thus in this case all vectors in $\mathbb{R}^{6n}$ which consist of reasonable velocities and positions of all $n$ particles in the box, are collected in the state space $S$. For the bar $B$ which can be simplified to an interval in $\mathbb{R}$ a state space suggesting itself is the set $C(B)$ of all continuous functions on $B$ as set of heat distributions.

The change of the system is described by a dynamics $\varphi : S \to S$ which gives how a state in $S$ has developed after one time step. The snapshot of all particles in the box looks different after one time step and the heat distribution has gone over to another heat distribution, thus a new continuous function serves as its description. The development of the system after $n$ time steps is found in the iterate $\varphi^n$.

However, with the model of the state space we quickly reach a limit: The problem is that a precise identification of the states is often not possible or even nonsensical. For example, it may not be possible to determine the position and velocity components of each particle in the box. Therefore, instead of considering the whole system, we only consider certain key figures of a state such as the total heat of the bar or the temperature at some point. If the *observable* $f : S \to \mathbb{R}$ yields the temperature at a certain point of the bar, we thus consider the temperature $f(\varphi(x))$ after one time step.

We have now changed from the state space to the observable space, which usually is a vector space $\mathfrak{F}$ of yet unspecified functions

$$\mathfrak{F} := \{f : S \to \mathbb{R}\}.$$

On this space operates the Koopman operator

$$T_\varphi : \mathfrak{F} \to \mathfrak{F}, \quad f \mapsto f \circ \varphi$$

as a linear operator. Through the transition from $(S; \varphi)$ to $(\mathfrak{F}; T_\varphi)$, we change to a linear system. This is a global linearization, in contrast to conventional linearizations. Astonishingly – as we will see later – the Koopman system $(\mathfrak{F}; T_\varphi)$ contains for a suitable choice of $\mathfrak{F}$ in some sense all information about the dynamical system $(S; \varphi)$.

Because of these and other reasons the Koopman linearization seems to be quite promising for applications e.g. in fluid mechanics. Igor Mezic (University of California, Santa Barbara) propagates an „Applied Koopmanism“ (see [BMM12]), where conclusions are drawn from the Koopman operator and its eigenvalues to non-linear phenomena.

A relevant contribution from the standpoint of pure Mathematics is the forthcoming book „Operator Theoretic Aspects of Ergodic Theory“ [EFHN15] by Tanja Eisner (Universität Leipzig), Bálint Farkas (Bergische Universität Wuppertal), Markus Haase (Delft

University of Technology) and Rainer Nagel (Eberhard-Karls-Universität Tübingen), on which this diploma thesis is based.

A first step to build the bridge from pure Mathematics to applications is to better understand what happens in the change from a dynamical system to the corresponding Koopman system. Inspired by the lecture „Ergodentheorie“ by Dr. Roland Derndinger during the fall semester 2014/2015 at the University of Tübingen, this diploma thesis shall be a small contribution to this untertaking.

# 2 Dynamical systems and Koopman systems

## 2.1 Dynamical systems

We begin with an introduction into the mathematical theory of dynamical systems and determine some of their crucial properties.

**Definition 2.1.1.** A *topological dynamical system* is a pair $(K;\varphi)$ where $K$ is a nonempty compact space and $\varphi : K \to K$ a continuous mapping.

**Remark 2.1.2.** In this thesis we concentrate on topological dynamical systems and use the short form „dynamical system“ or just „system“ for a topological dynamical system. A *measure-preserving dynamical system* is a tuple $(X, \Sigma, \mu; \varphi)$ where $(X, \Sigma, \mu)$ is a probability space and $\varphi : X \to X$ a measurable mapping which is measure-preserving, i.e., $\mu(\varphi^{-1}(A)) = \mu(A)$ for every $A \in \Sigma$. For further information on measure-preserving dynamical systems we refer to [EFHN15, Chap. 5].

**Remark 2.1.3.** Note that for a dynamical system $(\mathrm{K};\varphi)$ with $\varphi$ bijective, also $(K;\varphi^{-1})$ is a dynamical system, called the *backward system*.

We give some first examples of dynamical systems.

**Example 2.1.4.** (i) Let $K := \{1, ..., n\}$, $n \in \mathbb{N}$, be a finite state space with the discrete topology. Then every mapping $\varphi : K \to K$ is continuous, thus $(\mathrm{K};\varphi)$ is a system.

(ii) For $K := [0,1]$ and $\varphi : K \to K$, $x \mapsto 4x(1-x)$ a so-called logistic map, $(\mathrm{K};\varphi)$ is a dynamical system. The interested reader may find further information on this system in [Dev03].

(iii) Let $K := \mathbb{T}$ be the torus in $\mathbb{C}$ and choose $a \in \mathbb{T}$. Then the rotation

$$\varphi : \mathbb{T} \to \mathbb{T},\ z \mapsto az$$

is a continuous mapping, thus $(\mathrm{K};\varphi)$ is a dynamical system also denoted by $(\mathbb{T}; a)$. The rotation is called *rational* if the rotation element $a$ is a root of unity and *irrational* if $a$ is not a root of unity.

(iv) Consider the finite dimensional Banach space $X := (\mathbb{C}^k; \|\cdot\|)$, let $A \in \mathcal{L}(X)$ be a contractive matrix, that is $\|A\| \le 1$, and consider its restriction $A\big|_{U_k}$ to the closed unit ball $U_k := \{x \in \mathbb{C}^k : \|x\| \le 1\}$. Then $\left(U_k; A\big|_{U_k}\right)$ is a dynamical system. This example will be studied in more detail in Section 3.1.

In the following we introduce several notions and properties which are fundamental to describe dynamical systems.
To understand a dynamical system, a major question to investigate is: How does the system evolve in time, i.e., what can we say about the iterates $\varphi, \varphi^2, \varphi^3, ...$? Therefore, a useful object to consider is the orbit of a point $x \in K$.

**Definition 2.1.5.** Let (K; $\varphi$) be a dynamical system and $x \in K$. Then the *orbit of* $x$ is the set of the iterates

$$\mathrm{orb}(x) := \{\varphi^n(x) : n \in \mathbb{N}_0\}.$$

The orbits of a dynamical system can be highly complex sets even if the map is simple. An important role also plays the closure of an orbit which we denote by

$$\overline{\mathrm{orb}}(x) := \overline{\mathrm{orb}(x)}.$$

Some dynamical systems have points from which basically the whole state space can be reached under repeated action of the dynamics $\varphi$. This behavior is expressed by the next definition.

**Definition 2.1.6.** Let $(K; \varphi)$ be a dynamical system. A point $x \in K$ is called *transitive* if its orbit $\mathrm{orb}(x)$ is dense in $K$, i.e. $\overline{\mathrm{orb}}(x) = K$.
A dynamical system $(K; \varphi)$ is called *transitive* if there is at least one transitive point.

**Example 2.1.7.** (i) For the one-point compactification $K_1 := \mathbb{N} \cup \{\infty\}$ and the two-point compactification $K_2 := \mathbb{Z} \cup \{\pm\infty\}$ consider the shifts

$$\varphi_1 : K_1 \to K_1,\, n \mapsto \begin{cases} n+1 & \text{for } n \in \mathbb{N} \\ \infty & \text{for } n = \infty \end{cases}$$

and

$$\varphi_2 : K_2 \to K_2,\, n \mapsto \begin{cases} n+1 & \text{for } n \in \mathbb{Z} \\ \pm\infty & \text{for } n = \pm\infty. \end{cases}$$

Then $(K_1; \varphi_1)$ is transitive since 1 is a transitive element, but $(K_2; \varphi_2)$ is not transitive.

(ii) Consider the symmetric *tent map* on $[0,1]$,

$$\varphi(x) := \begin{cases} 2x & \text{if } 0 \le x < \frac{1}{2} \\ 2-2x & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

Then $z \in [0,1]$ where $z$ contains all finite combinations of 0 and 1 in its representation in the binary numeral system, e.g. ordered the lexicographic order

$$z := (0, \underbrace{0}\,\underbrace{1}\,\underbrace{00}\,\underbrace{01}\,\underbrace{10}\,\underbrace{11}\,\underbrace{000}\,\underbrace{001}\,...)_2,$$

is a transitive element.

(iii) It is well-known that for $z \in \mathbb{T}$, the set of powers $\{z^n : n \in \mathbb{N}\}$ is dense in $\mathbb{T}$ if and only if $z$ is not a root of unity. This implies that the torus rotation is transitive if and only if the rotation is irrational (see e.g. [EFHN15, Ex. 2.37]).

The iterates of the irrational torus rotation at any point on the unit circle behave in a way that they never reach the same point twice, but come back arbitrarily close to any point. Points may show several ways of returning in some sense. Such behavior can be described by their orbits.

**Definition 2.1.8.** A point $x \in K$ is called

(i) *recurrent* if for any open neighborhood $U$ of $x$ there exists some $m \in \mathbb{N}$ such that $\varphi^m(x) \in U$.

(ii) *almost periodic* if for every open neighborhood $U$ of $x$ the set $M := \{m \in \mathbb{N} : \varphi^m(x) \in U\}$ of return times has bounded gaps, i.e. there is $N \in \mathbb{N}$ such that $M \cap [n, n+N] \neq \varnothing$ for all $n \in \mathbb{N}$.

(iii) *periodic* if there is some $m \in \mathbb{N}$ such that $\varphi^m(x) = x$.

A system $(\mathrm{K}; \varphi)$ is called *periodic* if there is $m \in \mathbb{N}$ such that $\varphi^m(x) = x$ for all $x \in K$. The *period of* $\varphi$ is $n := \min\{m \in \mathbb{N} : \varphi^m = \mathrm{id}\}$.

Obviously, a periodic point is almost periodic and an almost periodic point is recurrent, but the converse directions do not hold in general. A point $x \in K$ is periodic if and only if its orbit $\mathrm{orb}(x)$ consists only of finitely many elements. This is the case if and only if $x \in \mathrm{orb}(\varphi(x)) \subseteq \mathrm{orb}(x)$. It is recurrent if and only if $x \in \overline{\mathrm{orb}}(\varphi(x))$.

**Example 2.1.9.** (i) Let $X := (\mathbb{C}^k; \|\cdot\|)$ for some norm $\|\cdot\|$ and $A \in \mathcal{L}(X)$ be a contractive *asymptotic periodic* matrix. A matrix is called asymptotic periodic, if there is a *partially periodic* matrix $R \in \mathbb{C}^{k \times k}$, i.e., $R = \begin{pmatrix} R_0 & 0 \\ 0 & 0 \end{pmatrix}$ for a periodic matrix $R_0 \in \mathbb{C}^{l \times l}, 0 \le l \le k$, such that $\|A^n - R^n\| \xrightarrow{n \to \infty} 0$.
Then every point $x \in U$ is almost periodic for the restriction of $A$ to the unit ball $U$ in $X$.

*Proof.* Let $x \in U$ and $V_x$ an arbitrary open neighborhood of $x$. Then $B_\varepsilon(x) \subseteq V_x$ for some $\varepsilon > 0$. For this $\varepsilon$ there exists some $N \in \mathbb{N}$ such that for all $n \geq N$, $\|A^n x - R^n x\| < \varepsilon$. Hence $\{n \in \mathbb{N} : A^n x \in V_x\} \cap [n, n+N] \neq \varnothing$ for all $n \in \mathbb{N}$, i.e. $\{n \in \mathbb{N} : A^n x \in V_x\}$ has bounded gaps. □

(ii) The torus rotation is periodic if and only if the rotation element is a root of unity. It is always almost periodic and recurrent (see [EFHN15, Prop. 3.12 (d)]).

A particular role is played by *fixed points*, i.e. periodic points with period 1.

**Example 2.1.10.** Consider the map $\varphi(x) = x^3$ for $x \in [-1, 1]$. Obviously, $0, 1$ and $-1$ are fixed points and there are no other periodic points. Here, the fixed point 0 is an „attractive“ fixed point, thus $\varphi^n(x) = x^{3n} \overset{n\to\infty}{\longrightarrow} 0$ for $x \in (-1, 1)$, and $-1$ and $1$ are „repelling“ fixed points.

Also fundamental for the investigation of a dynamical system is the question: How does it „mix“ the points of its state space? For this purpose we call a set $S \subseteq K$ *invariant* if points in $S$ stay in $S$ under the action of $\varphi$, that is $\varphi(S) \subseteq S$. A closed $\varphi$-invariant set $S$ gives rise to a *subsystem* $\left(S; \varphi|_S\right)$. Of particular interest are systems that lack nontrivial subsystems.

**Definition 2.1.11.** A dynamical system $(K;\varphi)$ is called *minimal* if the only closed $\varphi$-invariant sets are $K$ and $\varnothing$.

**Example 2.1.12.** (i) The torus rotation has no nontrivial rotation invariant sets if and only if the rotation is irrational. If the torus element is a $k^{th}$ root of unity $\zeta_k$, then the set $\{1, \zeta_k, ..., \zeta_k^{k-1}\}$ is invariant under rotation.

(ii) The tent map (see Example 2.1.7 (ii)) is not minimal since for example $\{0\}$, $\{0, \frac{1}{2}, 1\}$, $\{0, \frac{1}{4}, \frac{1}{2}, 1\}$,... are closed $\varphi$-invariant sets.

**Remark 2.1.13.** If $(K;\varphi)$ is minimal, then $\varphi$ is surjective, that is $\varphi(K) = K$.

*Proof.* Assume that $\varphi(K) \subsetneq K$. Since $\varphi(\varphi(K)) \subseteq \varphi(K)$, $\varphi(K)$ is a nontrivial $\varphi$-invariant set and closed, since $\varphi$ is continuous and $K$ compact. This contradicts the minimality of (K; $\varphi$), hence $\varphi(K) = K$. □

The importance of minimal systems is justified by the next theorem.

**Theorem 2.1.14.** *Let* $(K;\varphi)$ *be a topological dynamical system. Then* $(K;\varphi)$ *has at least one minimal subsystem, i.e., there is some closed* $\varnothing \neq A \subseteq K$ *with* $\varphi(A) = A$ *and* $\left(A;\varphi|_A\right)$ *minimal.*

*Proof.* The proof is based on Zorn's lemma and can be found in [EFHN15, Thm. 3.5]. □

**Example 2.1.15.** For a torus rotation with rotation element $\zeta_k$ a $k^{th}$ root of unity, a minimal subsystem is, e.g., the system $(A;\zeta_k)$ with $A := \{1, \zeta_k, ..., \zeta_k^{k-1}\}$.

Having introduced these properties we ask how they are related to each other.

**Theorem 2.1.16.** *If a dynamical system* $(K;\varphi)$ *is minimal, then every point* $x \in K$ *is transitive and almost periodic.*

*Proof.* Assume there is a non-transitive point $x \in K$, thus $\overline{\mathrm{orb}}(x) \neq K$. Then $\varphi(\overline{\mathrm{orb}}(x)) \subseteq \overline{\mathrm{orb}}(x)$, hence $\overline{\mathrm{orb}}(x)$ is a nontrivial closed $\varphi$-invariant set, which contradicts the minimality of (K;$\varphi$). That every point is almost periodic see [EFHN15, Prop. 3.12 (b)]. □

**Remark 2.1.17.** Note that Theorem 2.1.16 also shows that every point in a minimal dynamical system is recurrent. This means that in a system that does not contain a nontrivial subsystem, every point returns arbitrarily close to itself under the repeated action of $\varphi$.

The next proposition reveals that the converse of Theorem 2.1.16 holds true.

**Proposition 2.1.18.** *If a topological system contains a transitive and almost periodic point, then it is minimal.*

*Proof.* See [EFHN15, 3.12 (a)]. □

Theorem 2.1.16 and Proposition 2.1.18 give the following.

**Lemma 2.1.19.** *A point* $x \in K$ *is almost periodic if and only if* $\left(\overline{\mathrm{orb}}(x);\varphi|_{\overline{\mathrm{orb}}(x)}\right)$ *is minimal.*

Theorem 2.1.16 and the existence of a minimal subsystem in Theorem 2.1.14 yields the following famous theorem.

**Theorem 2.1.20** (Birkhoff)**.** *Every topological dynamical system contains at least one almost periodic, hence recurrent point.*

## 2.2 Koopman systems

Every dynamical system gives rise to a „Koopman system“ yielding a linearization of the original dynamical system. This will be explained in more detail in the following sections. A major goal of this thesis is to achieve a better understanding of the change from a dynamical system to its Koopman system and vice versa. Having discussed dynamical systems in the previous section, we now introduce Koopman systems.

**Definition 2.2.1.** For a dynamical system $(K;\varphi)$ the *Koopman operator corresponding to $\varphi$ on the observable space $\mathfrak{F}$* is

$$T_\varphi : \mathfrak{F} \to \mathfrak{F},\ f \mapsto f \circ \varphi$$

for a subspace $\mathfrak{F} \subseteq \{f \,|\, f : K \to \mathbb{C}\}$, which is $T_\varphi$-invariant. The *Koopman system*, consisting of the observable space $\mathfrak{F}$ and the Koopman operator $T_\varphi$, is denoted by $(\mathfrak{F}; T_\varphi)$.

**Remark 2.2.2.** For a topological dynamical system (K; $\varphi$) the natural choice of the observable space is $\mathfrak{F} := C(K)$ where $C(K)$ denotes the space of continuous complex-valued functions on the compact set $K$.
For a measure-preserving dynamical system $(X, \Sigma, \mu; \varphi)$ the space $\mathfrak{F} := L^p[X, \mu]$ of $p$-integrable functions, $1 \le p \le \infty$, suggests itself as observable space.

**Remark 2.2.3.** Sometimes the Koopman operator is also referred to as *composition operator*.

The change from the dynamical system $(K;\varphi)$ to its Koopman system $(C(K); T_\varphi)$ consists of two steps: The transition from the state space $K$ to the observable space $C(K)$ and the transition from the dynamics $\varphi$ to the operator $T_\varphi$. We first give some insights into what is gained by going from $K$ to $C(K)$.

### 2.2.1 The observable space $C(K)$

We consider the space

$$C(K) := \{f : K \to \mathbb{C} \,|\, f \text{ continuous}\}$$

for a compact set $K$ as observable space and outline some of its crucial properties.
If we endow $C(K)$ with the supremum norm $\|\cdot\|$, defined by $\|f\| := \sup_{s\in K} |f(s)|$ for every $f \in C(K)$, then it becomes a Banach space. But $C(K)$ is even a so-called $C^*$-algebra.

**Definition 2.2.4.** A vector space $(\mathfrak{A}, +)$ over $\mathbb{R}$ or $\mathbb{C}$ is called an *algebra* if there is a multiplication

$$\mathfrak{A} \times \mathfrak{A} \to \mathfrak{A},\ (a,b) \mapsto ab$$

which is associative and bilinear, i.e., it satisfies

$$(ab)c = a(bc),$$

$$a(b+c) = ab + ac,$$

$$(b+c)a = ba + ca$$

and

$$(\lambda a)b = a(\lambda b)$$

for every $a, b, c \in \mathfrak{A}$ and $\lambda \in \mathbb{C}$. Moreover, there is a unit element $e \in \mathfrak{A}$ satisfying

$$ea = ae = a$$

for every $a \in \mathfrak{A}$.
An algebra $\mathfrak{A}$ which is also a Banach space for some norm $\|\cdot\|$ is called a *Banach algebra* if

$$\|ab\| \leq \|a\| \|b\|$$

for all $a, b \in \mathfrak{A}$.
A complex Banach algebra $\mathfrak{A}$ with an involution, that is, a mapping $* : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A},\ a \mapsto a^*$ satisfying $(a^*)^* = a$, $(a+b)^* = a^* + b^*$, $(\lambda a)^* = \bar{\lambda} a^*$ and $(ab)^* = b^* a^*$ for all $a, b \in \mathfrak{A}$, $\lambda \in \mathbb{C}$, is called a *$C^*$-algebra* if

$$\|a\|^2 = \|a^* a\|$$

for all $a \in \mathfrak{A}$.

**Remark 2.2.5.** In literature you find definitions of an algebra both with and without a unit element. We shall consider only algebras with a unit element.

We see that $C(K)$ satisfies all these properties for the pointwise addition and multiplication. It is a Banach space, has the unit element $\mathbb{1}_K$, for $f, g \in C(K)$ also $f \cdot g \in C(K)$ and

$$\|fg\| = \sup_{s \in K} |f(s) g(s)| \leq \sup_{s \in K} |f(s)| \sup_{s \in K} |g(s)| = \|f\| \|g\|.$$

Moreover, the pointwise complex conjugation is an involution with

$$\|\overline{f} f\| = \sup_{s \in K} |\overline{f}(s) f(s)| = \sup_{s \in K} \left| |f(s)|^2 \right| = \sup_{s \in K} |f(s)|^2 = \left( \sup_{s \in K} |f(s)| \right)^2 = \|f\|^2.$$

Therefore, $C(K)$ is a commutative $C^*$-algebra.

In the following we introduce ideals as important objects in $C^*$-algebras and characterize them in $C(K)$.

**Definition 2.2.6.** An *(algebra) ideal* is a subspace $I \subseteq \mathfrak{A}$ of a commutative Banach algebra $\mathfrak{A}$ such that for $a \in I, b \in \mathfrak{A}$ also $ab \in I$.
An ideal $I \neq \mathfrak{A}$ is called *maximal* if for any ideal $J \subseteq \mathfrak{A}$ with $I \subseteq J$, either $J = I$ or $J = \mathfrak{A}$.

In Section 2.3.3 we will see the importance of closed ideals in $C(K)$ for the connection between a dynamical system and its Koopman operator.
The closed ideals in $C(K)$ are readily determined: For a closed subset $M \subseteq K$ define

$$I_M := \{f \in C(K) : f \equiv 0 \text{ on } M\}.$$

Then $I_M$ is a closed ideal. The next theorem tells us that this gives *all* closed ideals in $C(K)$.

**Theorem 2.2.7.** *For every closed ideal $I \subseteq C(K)$ there exists a closed subset $M \subseteq K$ such that $I = I_M$.*

*Proof.* See [EFHN15, Thm. 4.8]. □

As a corollary we determine the maximal ideals in $C(K)$.

**Corollary 2.2.8.** *An ideal $I \subseteq C(K)$ is maximal if and only if there is some $x \in K$ such that $I = I_{\{x\}}$.*

*Proof.* See [EFHN15, Lemma 4.9]. □

We know that $C(K)$ is a commutative $C^*$-algebra. The Gelfand-Naimark theorem states that, conversely, *every* commutative $C^*$-algebra is - up to isomorphy - a $C(K)$ for some compact space $K$.

**Theorem 2.2.9** (Gelfand-Naimark)**.** *Let $\mathfrak{A}$ be a commutative $C^*$-algebra. Then there is a unique compact space $K$ and an isometric $*$-isomorphism such that*

$$\mathfrak{A} \cong C(K).$$

*Proof.* See [EFHN15, Thm. 4.23]. □

**Example 2.2.10.** Consider the space $X := (B(S), \|\cdot\|)$, where $B(S)$ is the space of bounded complex-valued functions on a set $S$ with the usual supremum norm and pointwise addition and multiplication. Clearly, $X$ is a commutative $C^*$-algebra. Therefore there exists some $K$ such that $B(S) \cong C(K)$. This is remarkable since $S$ is an arbitrary set without any topological structure.

The Gelfand-Naimark theorem is very useful since there are many important functional analytic results known about $X := (C(K), \|\cdot\|)$, like the Arzelà-Ascoli theorem, the Weierstraß theorem and its generalization, the Stone-Weierstraß theorem. We will briefly recall them to emphasize the rich structure of $C(K)$.

**Theorem 2.2.11** (Arzelà-Ascoli)**.** *Let $K$ be a compact space and $M \subseteq C(K)$. Then the following assertions are equivalent:*

*(i) The set $M$ is relatively compact, i.e., $\overline{M}$ is compact.*

*(ii) The set $M$ is bounded and equicontinuous.*

*Proof.* The proof can be found in [DS58, Thm. IV.6.7] □

**Theorem 2.2.12** (Stone-Weierstraß)**.** *For a compact space $K$, let $\mathfrak{A} \subseteq C(K)$ be a conjugation invariant subalgebra of $C(K)$ such that $\mathbb{1} \in \mathfrak{A}$ and $\mathfrak{A}$ separates the points of $K$. Then $\mathfrak{A}$ is dense in $C(K)$.*

*Proof.* See [Wer11, Thm. VIII.4.7]. □

**Remark 2.2.13.** Besides the $C^*$-algebra structure, $C(K)$ also carries the structure of a *Banach lattice*. A Banach lattice is a *vector lattice*, i.e., a vector space $X$ with an order „$\leq$“ such that $\sup(f,g)$, $\inf(f,g)$ exist for every $f,g \in X$ and $|f| := \sup(f,-f)$, which is also a Banach space such that $|f| \leq |g|$ implies $\|f\| \leq \|g\|$ for all $f,g \in X$.

We conclude that we can pass from a compact space $K$ to the highly structured space $C(K)$ which provides more mathematical tools to be used. This goes at the expense of changing to an infinite dimensional, hence very large vector space. Moreover, we pass from „states“ – which potentially have a physical meaning – to observables.

### 2.2.2 The Koopman operator $T_\varphi$

Having discussed the transition from the state space $K$ to the observable space $C(K)$, we now turn towards the change from the dynamics $\varphi$ to the Koopman operator $T_\varphi$. First, we specify the basic properties of $T_\varphi$.

**Theorem 2.2.14.** *Let* $(\mathrm{K};\varphi)$ *be a dynamical system and* $(C(K);T_\varphi)$ *the corresponding Koopman system, that is,* $T_\varphi : C(K) \to C(K)$, $f \mapsto f \circ \varphi$. *Then the Koopman operator* $T_\varphi$ *has the following properties:*

*(i)* $T_\varphi$ *is linear and multiplicative.*

*(ii)* $T_\varphi$ *is contractive.*

*(iii)* $T_\varphi$ *is a lattice operator, i.e.,* $T_\varphi|f| = |T_\varphi f|$ *for every* $f \in C(K)$.

*(iv)* $T_\varphi$ *is a Markov operator, i.e.,* $T_\varphi \mathbb{1}_K = \mathbb{1}_K$.

*Proof.* The Koopman operator $T_\varphi$ is linear since

$$T_\varphi(f+g) = (f+g)\circ\varphi = f\circ\varphi + g\circ\varphi = T_\varphi f + T_\varphi g$$

for all $f, g \in C(K)$ and

$$T_\varphi(\lambda f) = \lambda f \circ \varphi = \lambda T_\varphi f$$

for $\lambda \in \mathbb{C}$. It is multiplicative since

$$T_\varphi(f\cdot g) = T_\varphi f \cdot T_\varphi g$$

for all $f, g \in C(K)$ and a contraction since $\|T_\varphi\| = \sup_{\|f\|\le 1}\|T_\varphi f\| = \sup_{\|f\|\le 1}\|f\circ\varphi\| \le 1$. Moreover,

$$T_\varphi|f| = |f|\circ\varphi = |f\circ\varphi| = |T_\varphi f|$$

for every $f \in C(K)$ and

$$T_\varphi \mathbb{1}_K = \mathbb{1}_K \circ \varphi = \mathbb{1}_K.$$

□

**Remark 2.2.15.** From property (iv) in Theorem 2.2.14 follows that 1 always belongs to the point spectrum $P\sigma(T_\varphi)$ with $\mathbb{1}_K$ as a corresponding eigenfunction. This implies that the fixed space of $T_\varphi$ is one-dimensional if and only if $\mathrm{Fix}(T_\varphi) = \langle \mathbb{1}_K \rangle$.

**Example 2.2.16.** Consider the system $(\mathrm{K};\varphi)$ and its Koopman system $(C(K);T_\varphi)$, where

$$K := \{1,\dots,n\}$$

is a finite state space and

$$\varphi : \{1,\dots,n\} \to \{1,\dots,n\}.$$

Since $K$ is finite, $C(K) \cong \mathbb{C}^n$, thus every $f \in C(K)$ can be seen as a vector $f = (f(1),\dots,f(n))$. The action of the Koopman operator is $T_\varphi f = (f(\varphi(1)),\dots,f(\varphi(n)))$, hence $T_\varphi$ can be represented by a matrix with entries 0 and 1 with exactly one 1 in every row. Conversely, a matrix corresponds to a Koopman operator if and only if it is of this form. If $\varphi$ is a permutation, then $T_\varphi$ is a permutation matrix. Clearly, $\dim \mathrm{Fix}(T_\varphi) = 1$ if and only if $\varphi$ is a cyclic permutation.

As seen above, $T_\varphi$ is an algebra homomorphism on $C(K)$, i.e. it is linear, multiplicative and $T_\varphi \mathbb{1}_K = \mathbb{1}_K$. Conversely, *every* algebra homomorphism is a Koopman operator.

**Theorem 2.2.17.** *Let $T : C(K) \to C(K)$ be a bounded linear operator. Then $T$ is an algebra homomorphism if and only if there is a continuous mapping $\varphi : K \to K$ such that $T = T_\varphi$. In this case, $\varphi$ is uniquely determined and $\|T\| = 1$.*

*Proof.* The proof is based on Gelfand's theorem and can be found in [EFHN15, Thm. 4.13]. □

This theorem implies that important information is preserved when going from a dynamical system to its Koopman system. For more details see Section 2.3.1.
The following lemma gives a first correspondence between properties of $\varphi$ and properties of $T_\varphi$.

**Lemma 2.2.18.** *Let $(K;\varphi)$ be a dynamical system with corresponding Koopman system $(C(K);T_\varphi)$. Then*

*(i) $\varphi$ is surjective if and only if $T_\varphi$ is injective.*

*(ii) $\varphi$ is injective if and only if $T_\varphi$ is surjective.*

*Proof.* (i) Let $f,\ g \in C(K)$ with $f(x) \neq g(x)$ for some $x \in K$. If $\varphi$ is surjective, then there is $y \in K$ such that $\varphi(y) = x$, and the injectivity of $T_\varphi$ follows from

$$T_\varphi f(y) = f(\varphi(y)) = f(x) \neq g(x) = g(\varphi(y)) = T_\varphi g(y).$$

Conversely, assume that there is $x \in K$ such that $x \notin \operatorname{Im} \varphi$. By Urysohn's lemma there is $f \in C(K)$ with $f\Big|_{\varphi(K)} = 0$ and $f(x) \neq 0$. Then $T_\varphi f = f \circ \varphi = 0$, which contradicts the injectivity of $T_\varphi$.

(ii) If $\varphi$ is not injective, i.e. $\varphi(x) = \varphi(y)$ for some $x \neq y \in K$, then for $f \in \operatorname{Im} T_\varphi$

$$f(x) = T_\varphi g(x) = g(\varphi(x)) = g(\varphi(y)) = T_\varphi g(y) = f(y)$$

for some $g \in C(K)$, thus $\operatorname{Im} T_\varphi \neq C(K)$.
If $\varphi$ is injective it has a left inverse, i.e. $\varphi_l^{-1}$ with $\varphi_l^{-1} \circ \varphi = \mathrm{id}_K$. For an arbitrary $f \in C(K)$ choose $g \in C(K)$ with $g\Big|_{\varphi(K)} = f \circ \varphi_l^{-1}$. Then

$$T_\varphi g = g \circ \varphi = f \circ \varphi_l^{-1} \circ \varphi = f,$$

hence $T_\varphi$ is surjective. □

Depending on the dynamics $\varphi$, the Koopman operator $T_\varphi$ can have the following useful property.

**Definition 2.2.19.** The Koopman operator $T_\varphi$ is called *mean ergodic* if the *Cesàro means* $A_n := \frac{1}{n}\sum_{i=0}^{n-1} {T_\varphi}^i$ are convergent in the strong operator topology, i.e., $A_n f$ is convergent in $\|\cdot\|$ for every $f \in C(K)$.

We refer to [EFHN15, Chap. 8 and 10] for a systematic study of mean ergodic operators on Banach spaces.

**Remark 2.2.20.** If $T$ is mean ergodic, then the limit in the strong operator topology $P := \lim_{n\to\infty} A_n$ is a projection and $\operatorname{Im} P = \operatorname{Fix}(T_\varphi)$.

*Proof.* For $f \in C(K)$,

$$T_\varphi Pf = \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} {T_\varphi}^{i+1} f = \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} {T_\varphi}^{i} f - \frac{1}{n} f + \frac{1}{n} {T_\varphi}^n f = Pf,$$

thus $\operatorname{Im} P \subseteq \operatorname{Fix}(T_\varphi)$. Conversely, for $f \in \operatorname{Fix}(T_\varphi)$, $A_n f = f$ for all $n \in \mathbb{N}$, thus $Pf = f$. This also implies $P^2 = P$, thus $P$ is a projection. □

**Example 2.2.21.** Consider the rotation system $(\mathbb{T}; a)$ for some $a \in \mathbb{T}$ and its corresponding Koopman system $(C(\mathbb{T}); T_a)$. We show that $T_a$ is mean ergodic (cf. [EFHN15, Prop. 10.10]):
If $a = \zeta_k$ for a $k^{th}$ root of unity, then we obtain for any $n := mk + l$ with $m \in \mathbb{N}_0$ and $0 \le l \le k-1$

$$\begin{aligned} A_n &= \frac{1}{n}\sum_{i=0}^{n-1} T_a^i = \frac{1}{mk+l}\left(m\sum_{i=0}^{k-1} T_a^i + \sum_{i=0}^{l-1} T_a^i\right) \\ &= \frac{1}{k+\frac{l}{m}}\sum_{i=0}^{k-1} T_a^i + \frac{1}{mk+l}\sum_{i=0}^{l-1} T_a^i. \end{aligned}$$

Then $A_n \overset{n\to\infty}{\longrightarrow} \frac{1}{k}\left(\sum_{i=0}^{k-1} T_a^i\right)$ even in the operator norm.

If $a$ is not a root of unity, then consider the monoms $g_k : \mathbb{T} \to \mathbb{T}$, $z \mapsto z^k$ for $k \in \mathbb{Z}$. Then $\operatorname{lin}\{g_k : k \in \mathbb{Z}\}$ is a conjugation invariant subalgebra which contains $\mathbb{1}_\mathbb{T}$ and separates the points of $\mathbb{T}$, thus it is dense in $C(\mathbb{T})$ by the Stone-Weierstraß theorem. Since $T_\varphi$ is a contraction, also each $A_n$ is a contraction and it is therefore sufficient to show that the $A_n$ converge on $\operatorname{lin}\{g_k : k \in \mathbb{Z}\}$. Indeed,

$$\|A_n g_k\| = \sup_{z\in\mathbb{T}} \left|\frac{1}{n}\sum_{i=0}^{n-1} T_a^i g_k(z)\right| = \sup_{z\in\mathbb{T}} \left|\frac{1}{n}\sum_{i=0}^{n-1} a^{ik} z\right| = \sup_{z\in\mathbb{T}} \left|\frac{1}{n}\frac{1-a^{kn}}{1-a^k} z\right| \overset{n\to\infty}{\longrightarrow} 0.$$

## 2.3 From a dynamical system to a Koopman system and back

### 2.3.1 Isomorphy of dynamical systems and Koopman systems

In this section we shall discuss in which sense information about a dynamical system (K;$\varphi$) is preserved in the corresponding Koopman system $(C(K); T_\varphi)$. Before doing so, we have to specify which kind of properties of a dynamical system are relevant. Dynamical systems that are „isomorphic", i.e. basically the same, should have the same properties. This means that properties are considered as relevant if they are invariant under isomorphy. What isomorphy means shall be be specified now.

**Definition 2.3.1.** Two topological dynamical systems $(K;\varphi)$ and $(L;\psi)$ are called isomorphic if there is a homeomorphism $\theta : K \to L$ such that the diagram

$$\begin{tikzcd} L \arrow[r, "\psi"] & L \\ K \arrow[u, "\theta"] \arrow[r, "\varphi"'] & K \arrow[u, "\theta"'] \end{tikzcd}$$

commutes. We denote the isomorphy by (K;$\varphi$) $\cong (L;\psi)$.

**Example 2.3.2.** Consider the tent map

$$\varphi(x) := \begin{cases} 2x & \text{if } 0 \le x < \frac{1}{2} \\ 2 - 2x & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

and the logistic map

$$\psi : [0,1] \to [0,1],\ \psi(x) := 4x(1-x).$$

Consider $\theta : [0,1] \to [0,1]$, $\theta(x) := \left(\sin(\frac{\pi x}{2})\right)^2$ which is a continuous mapping with continuous inverse $\theta^{-1} : [0,1] \to [0,1]$, $\theta^{-1}(x) := \frac{2}{\pi}\arcsin\sqrt{x}$, thus $\theta$ is an isomorphism.

Then for any $x \in [0,1]$ the addition theorem yields

$$\begin{aligned} \psi(\theta(x)) &= 4\left(\sin(\frac{\pi x}{2})\right)^2 \left(1 - \left(\sin(\frac{\pi x}{2})\right)^2\right) = \left(2\sin(\frac{\pi x}{2})\cos(\frac{\pi x}{2})\right)^2 \\ &= \left(\sin(2\frac{\pi x}{2})\right)^2 = (\sin(\pi x))^2. \end{aligned}$$

Moreover,

$$\theta(\varphi(x)) = \begin{Bmatrix} (\sin(\pi x))^2 & \text{if } 0 \le x < \frac{1}{2} \\ (\sin(\pi - \pi x))^2 & \text{if } \frac{1}{2} \le x \le 1 \end{Bmatrix} = (\sin(\pi x))^2 ,$$

thus $\psi \circ \theta = \theta \circ \varphi$, hence $([0,1];\varphi) \cong ([0,1];\psi)$.

It follows a list of properties that are invariant under isomorphy.

**Example 2.3.3.** Let $(\mathrm{K};\varphi) \cong (L;\psi)$ be two dynamical systems with corresponding homeomorphism $\theta : K \to L$. Then the following assertions hold.

(i) $(\mathrm{K};\varphi)$ is minimal if and only if $(L;\psi)$ is minimal.

(ii) $(\mathrm{K};\varphi)$ is periodic if and only if $(L;\psi)$ is periodic.

(iii) A point $x \in K$ is periodic if and only if $\theta(x) \in L$ is periodic.

(iv) A point $x \in K$ is almost periodic if and only if $\theta(x) \in L$ is almost periodic.

(v) A point $x \in K$ is recurrent if and only if $\theta(x) \in L$ is recurrent.

(vi) A point $x \in K$ is transitive if and only if $\theta(x) \in L$ is transitive.

*Proof.* (i) Let $(\mathrm{K};\varphi)$ be minimal. Then for $B \subseteq L$ with $\psi(B) \subseteq B$, also $\theta \circ \varphi \circ \theta^{-1}(B) \subseteq B$, hence also $\varphi(\theta^{-1}(B)) \subseteq \theta^{-1}(B)$. Since $(\mathrm{K};\varphi)$ is minimal, we have $\theta^{-1}(B) = K$ or $\theta^{-1}(B) = \varnothing$ which implies $B = \theta(K) = L$ or $B = \theta(\varnothing) = \varnothing$. Therefore, also $(L;\psi)$ is minimal.

(ii) If $\varphi^m = id$ for some $m \in \mathbb{N}$, then $\psi^m \circ \theta = \theta \circ \varphi^m = \theta$ which implies that also $\psi^m = \mathrm{id}$.

(iii) If $\varphi^m(x) = x$ for some $m \in \mathbb{N}$, then $\psi^m(\theta(x)) = \theta(\varphi^m(x)) = \theta(x)$.

(iv) - (vi) are shown similarly.

□

Now we need to clarify what we mean by isomorphy in the case of two Koopman systems.

**Definition 2.3.4.** Two Koopman systems $(C(K); T_\varphi)$ and $(C(L); T_\psi)$ are called isomorphic if there is an $C^*$-algebra isomorphism $T : C(L) \to C(K)$ such that the diagram

$$\begin{tikzcd} C(L) \arrow[r, "T_\psi"] \arrow[d, "T"'] & C(L) \arrow[d, "T"] \\ C(K) \arrow[r, "T_\varphi"'] & C(K) \end{tikzcd}$$

is commutative.

We relate these two notions of isomorphy.

**Theorem 2.3.5.** *Two dynamical systems $(K; \varphi)$ and $(L; \psi)$ are isomorphic if and only if the corresponding Koopman systems $(C(K); T_\varphi)$ and $(C(L); T_\psi)$ are isomorphic.*

*Proof.* If $\theta \circ \varphi = \psi \circ \theta$, choose $T := T_\theta$. Then

$$T_\varphi \circ T_\theta f = f \circ \theta \circ \varphi = f \circ \psi \circ \theta = T_\theta \circ T_\psi f$$

for all $f \in C(L)$, hence $T_\varphi \circ T_\theta = T_\theta \circ T_\psi$.
Conversely, if there is a $C^*$-algebra isomorphism $T : C(L) \to C(K)$, Theorem 2.2.17 implies that there are continuous $\theta : K \to L$ and $\tilde{\theta} : L \to K$ such that $T = T_\theta$ and $T^{-1} = T_{\tilde{\theta}}$. Then

$$f \circ \tilde{\theta} \circ \theta = T_\theta \circ T_{\tilde{\theta}} f = T \circ T^{-1} f = f$$

for every $f \in C(L)$ and

$$g \circ \theta \circ \tilde{\theta} = T_{\tilde{\theta}} \circ T_\theta g = T^{-1} \circ T g = g$$

for every $g \in C(K)$, thus $\tilde{\theta} = \theta^{-1}$.
Moreover,

$$f \circ \theta \circ \varphi = T_\varphi \circ T_\theta f = T_\theta \circ T_\psi f = f \circ \psi \circ \theta$$

for every $f \in C(L)$. Then by Urysohn's lemma $\theta \circ \varphi = \psi \circ \theta$. □

This theorem reveals that two dynamical systems are „the same“ if and only if their corresponding Koopman systems are „the same“ which means that by considering $(C(K); T_\varphi)$ in place of (K; $\varphi$) no important information is lost. This is remarkable and allows us to switch from the non-linear system $(K;\varphi)$ to the linear system $(C(K);T_\varphi)$. Thus, we can work with the well-developed theory on linear operators offering a large toolbox of deep results. Note that the advantage of the Koopman linearization in comparison to other linearizations is that it is not only local but global.

It is our goal to investigate how dynamical properties that are preserved under isomorphy (see Example 2.3.3) are translated from "downstairs" to "upstairs", that is, from the dynamical system to the Koopman system, and the other way round.

### 2.3.2 From $C(K)$ back to $K$

In Section 2.2.1 we explained what is gained by going from $K$ to $C(K)$. Now we want to investigate the other direction, that is, going back from $C(K)$ to $K$. Indeed, $K$ can be recovered if $C(K)$ is known. How to accomplish this is described in the following.

Consider the *evaluation functionals* $\delta_x \in C(K)'$, where

$$\delta_x : C(K) \to \mathbb{C},\ \langle f, \delta_x\rangle := f(x)$$

for $f \in C(K)$ and some $x \in K$. Since $C(K)$ separates the points of $K$, i.e. for every $x, y \in K$, $x \neq y$, there exists some $f \in C(K)$ such that $f(x) \neq f(y)$, the map

$$\delta : K \to C(K)',\ x \mapsto \delta_x$$

is injective. Moreover, $\delta$ is continuous with respect to the weak* topology in $C(K)'$, hence $\delta$ is a homeomorphism onto its image. Therefore, $K$ can be represented by the evaluation functionals,

$$K \cong \{\delta_x : x \in K\}. \tag{2.1}$$

We notice that every $\delta_x$ is an algebra homomorphism between $C(K)$ and $\mathbb{C}$. The following lemma gives the converse.

**Lemma 2.3.6.** *A nonzero mapping $\psi : C(K) \to \mathbb{C}$ is linear and multiplicative if and only if there exists $x \in K$ such that $\psi = \delta_x$.*

*Proof.* We refer to [EFHN15, Lemma 4.10]. □

By this we can now state 2.1 as

$$K \cong \{\gamma : C(K) \to \mathbb{C} : \gamma \text{ algebra homomorphism}\}, \tag{2.2}$$

where $\{\gamma : C(K) \to \mathbb{C} : \gamma \text{ algebra homomorphism}\}$ is endowed with the weak* topology.

For another characterization of $K$ by means of $C(K)$, note that any maximal ideal $I_{\{x\}} = \{f \in C(K) : f(x) = 0\}$ for $x \in K$ can be expressed as $I_{\{x\}} = \ker \delta_x$.
Then the map $\Phi : \{\delta_x : x \in K\} \to \{I_{\{x\}} \subseteq C(K) : x \in K\}$, $\delta_x \mapsto I_{\{x\}}$ is clearly a bijection and a homeomorphism if we define $M \subseteq \{I_{\{x\}} \subseteq C(K) : x \in K\}$ as open if and only if $\Phi^{-1}(M)$ is open in the weak* topology.
This leads to the representation

$$K \cong \{I_{\{x\}} \subseteq C(K) : x \in K\} \tag{2.3}$$

or alternatively to

$$K \cong \{I \subseteq C(K) : I \text{ maximal ideal in } C(K)\}. \tag{2.4}$$

These characterizations mean that by knowing either the maximal ideals in $C(K)$ or the algebra homomorphisms between $C(K)$ and $\mathbb{C}$, we get back $K$.

### 2.3.3 Transference of dynamical properties

In Section 2.3.1 we gave a list of properties of dynamical systems that are invariant under isomorphy. We now investigate how these properties are reflected in the corresponding Koopman system and the other way around.
Before doing so we point out the role of the action of the Koopman operator on the closed ideals in $C(K)$ which – as described in the previous section – give back the state space $K$. Indeed, from the action of $T_\varphi$ on the closed ideals we obtain the complete action of $\varphi$ on K.

**Lemma 2.3.7.** *Let $n \in \mathbb{N}$ and $A, B \subseteq K$ closed. Then*

*(i)* $\varphi^n(A) \subseteq B$ *if and only if* $I_B \subseteq T_\varphi^{-n} I_A$.

*(ii)* $A \subseteq \varphi^n(B)$ *if and only if* $T_\varphi^{-n} I_B \subseteq I_A$.

*(iii)* $\varphi^n(A) = B$ *if and only if* $I_B = T_\varphi^{-n} I_A$.

*Proof.* $(i)$ follows from

$$T_\varphi^{-n} I_A = \{f \in C(K) : f(\varphi^n(A)) \equiv 0\} \supseteq \{f \in C(K) : f(B) \equiv 0\} = I_B$$

if and only if $\varphi^n(A) \subseteq B$. $(ii)$ follows from

$$T_\varphi^{-n} I_B = \{f \in C(K) : f(\varphi^n(B)) \equiv 0\} \subseteq \{f \in C(K) : f(A) \equiv 0\} = I_A$$

if and only if $A \subseteq \varphi^n(B)$. $(i)$ and $(ii)$ then imply $(iii)$. □

Considering images of closed ideals under $T_\varphi$ instead of preimages yields the following weaker conditions.

**Lemma 2.3.8.** *Let $n \in \mathbb{N}$ and $A, B \subseteq K$ closed. Then*

*(i) if ${T_\varphi}^n I_B \subseteq I_A$, then $\varphi^n(A) \subseteq B$,*

*(ii) if $A \subseteq \varphi^n(B)$, then $I_B \subseteq {T_\varphi}^n I_A$,*

*(iii) if $A = \varphi^n(B)$, then $I_B = {T_\varphi}^n I_A$.*

*Proof.* The condition ${T_\varphi}^n I_B \subseteq I_A$ means that for any $f \in I_B$ it follows that $f \circ \varphi^n \in I_A$, thus $\varphi^n(A) \subseteq B$ which proves $(i)$. Clearly, for $A \subseteq \varphi^n(B)$, we obtain for any $f \in C(K)$ with $f(B) \equiv 0$ that $f(\varphi^n(A)) \equiv 0$, hence $I_B \subseteq {T_\varphi}^n I_A$ which shows $(ii)$. Since for $\varphi^n(B) = A$ it follows ${T_\varphi}^n I_A = \{f \circ \varphi^n \in C(K) : f(A) \equiv 0\} = \{f \in C(K) : f(B) \equiv 0\} = I_B$ we obtain $(iii)$. □

The following lemma, based on Lemma 2.3.7 and 2.3.8, gives more insight how to recover $\varphi$ from $T_\varphi$.

**Lemma 2.3.9.** *(i) For any $x, y \in K$, $\varphi(x) = y$ if and only if $T_\varphi I_{\{y\}} = I_{\{x\}}$ if and only if $I_{\{y\}} = {T_\varphi}^{-1} I_{\{x\}}$.*

*(ii) For every $n \in \mathbb{N}$, ${T_\varphi}^{-n} I_{\{x\}} = I_{\{\varphi^n(x)\}}$ and $I_{\{\varphi^n(x)\}} \subseteq {T_\varphi}^n I_{\{x\}}$.*

In Section 2.1 we pointed out that invariant sets play an important role. They correspond to invariant ideals as a consequence of Lemma 2.3.7 and 2.3.8.

**Lemma 2.3.10.** *For $A \subseteq K$ the following assertions are equivalent.*

*(i) $A$ is $\varphi$-invariant, i.e., $\varphi(A) \subseteq A$.*

*(ii) The closed ideal $I_A$ is $T_\varphi$-invariant, that is $T_\varphi I_A \subseteq I_A$.*

*(iii) $I_A \subseteq {T_\varphi}^{-1} I_A$.*

*Proof.* It remains to show that $(i)$ implies $(ii)$. If $A$ is $\varphi$-invariant, then for each $f \in I_A$ we obtain $T_\varphi f(A) = f(\varphi(A)) \equiv 0$, thus also $T_\varphi f \in I_A$. □

We now turn towards the translation of the dynamical properties. Closely connected to the notion of invariant sets are minimal dynamical systems. They are expressed by means of the Koopman system as the following.

**Lemma 2.3.11.** *A system $(K;\varphi)$ is minimal if and only if $\{0\}$ and $C(K)$ are the only closed $T_\varphi$-invariant ideals.*

*Proof.* By Lemma 2.3.10, $K$ and $\varnothing$ are the only $\varphi$-invariant sets if and only if $I_K$ and $I_\varnothing$ are the only $T_\varphi$-invariant closed ideals. The assertion follows from $I_K = \{0\}$ and $I_\varnothing = C(K)$. □

**Remark 2.3.12.** Such Koopman operators are called *irreducible*. For further information we refer to [Sch74, Chap. III, §8].

Also transitivity of a dynamical system can be characterized by the Koopman operator.

**Lemma 2.3.13.** *A system $(K;\varphi)$ is transitive if and only if there is some $x \in K$ with $\bigcap_{n\in\mathbb{N}_0} T_\varphi{}^{-n} I_{\{x\}} = \{0\}$ for the maximal ideal $I_{\{x\}} \subseteq K$.*

*Proof.* First remark that
$$\bigcap_{n\in\mathbb{N}_0} T_\varphi{}^{-n} I_{\{x\}} = \{0\}$$
if and only if
$$\bigcap_{n\in\mathbb{N}_0} I_{\varphi^n(x)} = \{f \in C(K) : f(\varphi^n(x)) = 0 \text{ for all } n \in \mathbb{N}_0\} = \{0\}$$
and this holds if and only if for every $0 \neq f \in C(K)$ there is some $n \in \mathbb{N}_0$ such that $f(\varphi^n(x)) \neq 0$.
Assume that $(K,\varphi)$ is not transitive, i.e., there exists some $y \in K \setminus \overline{\mathrm{orb}}(x)$. Then by Urysohn's Lemma we find $f \in C(K)$ with $f(y) = 1$ and $f(\overline{\mathrm{orb}}(x)) = 0$, thus $f(\varphi^n(x)) = 0$ for every $n \in \mathbb{N}_0$. This is a contradiction, hence $(K,\varphi)$ is transitive.
Conversely, if $(K;\varphi)$ is transitive, assume that there is some $0 \neq f \in C(K)$ such that $f(\varphi^n(x)) = 0$ for all $n \in \mathbb{N}_0$, that is $f(\mathrm{orb}(x)) \equiv 0$. Hence $f$ vanishes on a dense set in $K$, thus, since $f$ is continuous, $f \equiv 0$ on $K$. This contradicts $f \neq 0$ and therefore $\bigcap_{n\in\mathbb{N}_0} T_\varphi{}^{-n} I_{\{x\}} = \{0\}$. □

A crucial implication of transitivity is the following.

**Lemma 2.3.14.** *(i) If $(K;\varphi)$ is transitive, then the fixed space* $\mathrm{Fix}(T_\varphi)$ *is one-dimensional.*

*(ii) If* $(\mathrm{K};\varphi)$ *is bijective and there exists some $x \in K$ such that the set $\{\varphi^m(x) : m \in \mathbb{Z}\}$ is dense in $K$, then the fixed space* $\mathrm{Fix}(T_\varphi)$ *is one-dimensional.*

*Proof.* To show $(i)$, consider $f \in \mathrm{Fix}(T_\varphi)$ and $x \in K$. Then

$$f(\varphi^n(x)) = {T_\varphi}^n f(x) = f(x)$$

for every $n \geq 0$, that is, $f$ is constant on $\mathrm{orb}(x)$.
If $\overline{\mathrm{orb}}(x) = K$ for some $x \in K$, then since $f$ is continuous also $f(\mathrm{orb}(x))$ is dense in $f(K)$. Hence every fixed function must be constant, which implies that $\mathrm{Fix}(T_\varphi)$ is one-dimensional. $(ii)$ follows similarly. □

To show that $(ii)$ in 2.3.14 is a weaker condition than $(i)$, consider the following example.

**Example 2.3.15.** Let $K := \mathbb{Z} \cup \{\pm\infty\}$ be the two-point compactification of $\mathbb{Z}$. Consider the left-shift

$$\varphi : K \to K,\ z \mapsto \begin{cases} z+1 & \text{if } z \in \mathbb{Z} \\ z & \text{if } z \in \{\pm\infty\} \end{cases}$$

which is continuous and bijective.
Then $\overline{\{\varphi^m(z) : m \in \mathbb{Z}\}} = K$ but $\overline{\mathrm{orb}}(z) = \{x \in \mathbb{Z} : x \geq z\} \cup \{\pm\infty\} \neq K$ for $z \in \mathbb{Z}$.

**Remark 2.3.16.** Theorem 2.1.16 implies that for a minimal dynamical system the fixed space $\mathrm{Fix}(T_\varphi)$ is one-dimensional.

The opposite direction in Lemma 2.3.14 is not true in general as illustrated by the following example.

**Example 2.3.17.** Consider $K := [0,1]$ and $\varphi(x) = x^2$ for $x \in [0,1]$. Then $\varphi$ is clearly not transitive since $\mathrm{orb}(x) \subseteq [0,x]$ for any $x \in [0,1)$, thus $\mathrm{orb}(x)$ is not dense in $[0,1]$. The point 1 is not transitive since it is a fixed point.
Any fixed function $f \in \mathrm{Fix}(T_\varphi)$ is constant on all orbits, $f|_{\overline{\mathrm{orb}}(x)} \equiv f(x)$ for all $x \in K$. Since $0 \in \overline{\mathrm{orb}}(x)$ for all $x \neq 1$, this implies $f(x) = f(0)$ for all $x \in [0,1)$. The continuitiy of $f$ then implies that $f$ is constant on $[0,1]$, thus that $\dim \mathrm{Fix}(T_\varphi) = 1$.

If $T_\varphi$ is mean ergodic, then the following characterization of a one-dimensional fixed space holds true.

**Proposition 2.3.18.** *Let* $(\mathrm{K};\varphi)$ *be a dynamical system and* $T_\varphi$ *be mean ergodic. Then the following assertions are equivalent:*

*(i)* $\dim \mathrm{Fix}(T_\varphi) = 1$

*(ii)* $\overline{\mathrm{orb}}(x) \cap \overline{\mathrm{orb}}(y) \neq \varnothing$ *for every* $x, y \in K$.

*Proof.* For $\dim \mathrm{Fix}(T_\varphi) = 1$ assume that there exist $x, y \in K$ such that $\overline{\mathrm{orb}}(x) \cap \overline{\mathrm{orb}}(y) = \varnothing$. By Urysohn's lemma there is some $f \in C(K)$ such that $f\big|_{\overline{\mathrm{orb}(x)}} \equiv 0$ and $f\big|_{\overline{\mathrm{orb}(y)}} \equiv 1$. Then

$$A_n f(x) = \frac{1}{n}\sum_{i=0}^{n-1} f(\varphi^i(x)) = 0$$

and

$$A_n f(y) = \frac{1}{n}\sum_{i=0}^{n-1} f(\varphi^i(y)) = 1,$$

hence for $P \coloneqq \text{stop} - \lim_{n\to\infty}$ also $Pf\big|_{\mathrm{orb}(x)} \equiv 0$ and $Pf\big|_{\mathrm{orb}(y)} \equiv 1$.
Since $\operatorname{Im} P = \mathrm{Fix}(T_\varphi)$ (confer Remark 2.2.20) this means that there is some $g \in \mathrm{Fix}(T_\varphi)$ with $g\big|_{\mathrm{orb}(x)} \equiv 0$ and $g\big|_{\mathrm{orb}(y)} \equiv 1$. This contradicts $\mathrm{Fix}(T_\varphi) = \langle \mathbb{1} \rangle$.
Conversely, let $f \in \mathrm{Fix}(T_\varphi)$. Clearly, $f\big|_{\overline{\mathrm{orb}}(x)}$ is constant for every $x \in K$, thus since $\overline{\mathrm{orb}}(x) \cap \overline{\mathrm{orb}}(y) \neq \varnothing$ for every $x, y \in K$, $f$ is constant on $K$. □

**Example 2.3.19.** Consider $K \coloneqq \{\pm\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\} \subseteq \mathbb{R}$ which is compact in the subspace topology and the dynamics

$$\varphi(\pm\frac{1}{n}) \coloneqq \pm\frac{1}{n+1}$$

for all $n \in \mathbb{N}$ and

$$\varphi(0) \coloneqq 0.$$

Clearly $\lim_{i\to\infty} T_\varphi^i f(s) = f(\varphi^i(s)) = f(0)$ for every $f \in C(K)$ and $s \in K$. The continuity of $f$ in 0 implies

$$\|T_\varphi^i f\| = \sup_{s\in K} |f(\varphi^i(s)) = \sup_{s\in K,\, s\le\frac{1}{i+1}} |f(s)| \xrightarrow{i\to\infty} f(0).$$

Thus, $\left(T_\varphi{}^i f\right)_{i\in\mathbb{N}}$ is convergent in $\|\cdot\|$ for every $f \in C(K)$, hence also the means $\frac{1}{n}\sum_{i=0}^{n-1} T_\varphi{}^i f$ are convergent, i.e. $T_\varphi$ is mean ergodic.
If $f \in C(K)$ is a fixed function of $T_\varphi$, then $f(\pm\frac{1}{n}) = f(\pm\frac{1}{n+1})$ for all $n \in \mathbb{N}$ and the continuity of $f$ implies that $f\big|_{\overline{\mathrm{orb}}(\pm\frac{1}{n})}$ is constant. For every $n \in \mathbb{N}$, $0 \in \overline{\mathrm{orb}}(\pm\frac{1}{n})$, thus $\mathrm{Fix}(T_\varphi) = \langle \mathbb{1} \rangle$ by Proposition 2.3.18.
Clearly, there is no transitive element in $(\mathrm{K};\varphi)$.

Another connection between mean ergodicity and a one-dimensional fixed space is the following.

**Proposition 2.3.20.** *Let* $(\mathrm{K};\varphi)$ *be a dynamical system with corresponding Koopman system* $(C(K);T_\varphi)$*. Then the following assertions are equivalent.*

*(i)* $(\mathrm{K};\varphi)$ *is minimal and* $T_\varphi$ *is mean ergodic.*

*(ii)* $\mathrm{Fix}(T'_\varphi) = \langle\mu\rangle$ *for a strictly positive probability measure* $\mu$*.*

*Proof.* We refer to [EFHN15, Cor. 10.9]. □

To give another characterization of a one-dimensional fixed space, consider a dynamical system $(\mathrm{K};\varphi)$ and denote by $\mathcal{T}$ the topology on $K$. Define a coarser topology $\mathcal{T}' := \{A \in \mathcal{T} : \varphi(A) \subseteq A\}$.

**Theorem 2.3.21.** *The following assertions are equivalent:*

*(i)* $\mathrm{Fix}(T_\varphi) = \langle\mathbb{1}\rangle$

*(ii) Every continuous function* $f : (K;\mathcal{T}') \to \mathbb{C}$ *is constant.*

*Proof.* We show

$$\mathrm{Fix}(T_\varphi) = \{f : K \to \mathbb{C} : f \text{ continuous with respect to } \mathcal{T}'\}.$$

Take $f \in \mathrm{Fix}(T_\varphi)$. For $A \in \mathcal{T}'$ also $f^{-1}(A) \in \mathcal{T}$ since $f$ is continuous with respect to $\mathcal{T}$. From $T_\varphi f = f$ follows $f(\varphi(B)) = f(B)$ for every $B \subseteq K$. Thus for the choice $B := f^{-1}(A)$ we obtain $f(\varphi(f^{-1}(A))) = f(f^{-1}(A)) \subseteq A$, hence $f^{-1}(f(\varphi(f^{-1}(A)))) \subseteq f^{-1}(A)$.
From $\varphi(f^{-1}(A)) \subseteq f^{-1}(f(\varphi(f^{-1}(A))))$ then follows $\varphi(f^{-1}(A)) \subseteq f^{-1}(A)$. Therefore $f^{-1}(A) \in \mathcal{T}'$ and $f$ continuous with respect to $\mathcal{T}'$.

Conversely, take any $\mathcal{T}'$-continuous $f \in C(K)$, $\alpha \in \mathbb{C}$ and $x \in f^{-1}(\{\alpha\})$. Consider for $n \in \mathbb{N}$ the open ball $B_{\frac{1}{n}}(\alpha) \subseteq \mathbb{C}$. Then the preimage $f^{-1}(B_{\frac{1}{n}}(\alpha))$ is open with respect to $\mathcal{T}'$, thus it is $\varphi$-invariant. This implies $\varphi(x) \in f^{-1}(B_{\frac{1}{n}}(\alpha))$ for all $n \in \mathbb{N}$, thus

$$\varphi(x) \in \bigcap_{n\in\mathbb{N}} f^{-1}(B_{\frac{1}{n}}(\alpha)) = f^{-1}(\bigcap_{n\in\mathbb{N}} B_{\frac{1}{n}}(\alpha)) = f^{-1}(\{\alpha\}).$$

Hence $\varphi(f^{-1}(\{\alpha\})) \subseteq f^{-1}(\{\alpha\})$. This implies that for any $x \in K$ with $f(x) = \alpha$ also $f(\varphi(x)) = \alpha$, i.e. $T_\varphi f(x) = \alpha$. Since $\alpha \in \mathbb{C}$ was arbitrary, we obtain $f \in \mathrm{Fix}(T_\varphi)$.
□

We turn to further properties of $\varphi$ introduced in Section 2.1.

**Lemma 2.3.22.** *Let* $(\mathrm{K};\varphi)$ *be a topological dynamical system and* $(C(K), T_\varphi)$ *the corresponding Koopman system. Then the orbits of* $T_\varphi$ *are relatively strongly compact if and only if* $\{\varphi^n : n \in \mathbb{N}_0\}$ *is equicontinuous.*

*Proof.* By the Arzelà-Ascoli theorem, $\overline{\{T_\varphi^n f : n \in \mathbb{N}_0\}} = \overline{\{f \circ \varphi^n : n \in \mathbb{N}_0\}}$ is strongly compact for every $f \in C(K)$ if and only if it is bounded and equicontinous for every $f \in C(K)$. Since $T_\varphi$ is a contraction and by the continuity of $f$, this implies that $\overline{\{T_\varphi^n f : n \in \mathbb{N}_0\}}$ is strongly compact if and only if $\{\varphi^n : n \in \mathbb{N}_0\}$ is equicontinuous. □

The different notions that describe the „returning“ of a point – recurrent, almost periodic and periodic – are found in the Koopman system as the following.

**Lemma 2.3.23.** *A point* $x \in K$ *is recurrent if and only if* $\bigcap_{n\in\mathbb{N}} T_\varphi{}^{-n} I_{\{x\}} \subseteq I_{\{x\}}$.

*Proof.* First notice that for $f \in C(K)$, $f\big|_{\mathrm{orb}(\varphi(x))} = 0$ if and only if $f\big|_{\overline{\mathrm{orb}}(\varphi(x))} = 0$.
Then we have

$$\begin{aligned}
\bigcap_{n\in\mathbb{N}} T_\varphi{}^{-n} I_{\{x\}} &= \bigcap_{n\in\mathbb{N}} I_{\varphi^n(x)} \\
&= \{f \in C(K) : f\big|_{\mathrm{orb}(\varphi(x))=0}\} \\
&= \{f \in C(K) : f\big|_{\overline{\mathrm{orb}}(\varphi(x))=0}\} \\
&= I_{\overline{\mathrm{orb}}(\varphi(x))} \\
&\subseteq I_{\{x\}}
\end{aligned}$$

if and only if $\{x\} \subseteq \overline{\mathrm{orb}}(\varphi(x))$, that is, $x$ is recurrent. □

**Lemma 2.3.24.** *A point* $x \in K$ *is almost periodic if and only if* $I = \{0\}$ *and* $I = C(L)$ *are the only* $T_{\varphi|_L}$*-invariant closed ideals contained in* $C(L)$ *for* $L := \overline{\mathrm{orb}}(x)$.

*Proof.* This follows from Lemma 2.3.11. □

**Lemma 2.3.25.** *A point* $x \in K$ *is periodic with period* $m \in \mathbb{N}$ *if and only if* $T_{\varphi^m} I_{\{x\}} = I_{\{x\}}$.

*Proof.* For a periodic point $x \in K$ the set $\{x\}$ is closed and $\varphi^m$-invariant, thus $T_{\varphi^m} I_{\{x\}} \subseteq I_{\{x\}}$ by Lemma 2.3.10. For the other inclusion, let $f \in I_{\{x\}}$, then $T_{\varphi^m} f(x) = f(\varphi^m(x)) = f(x) = 0$. Hence also $T_{\varphi^m} f \in I_{\{x\}}$, which implies $I_{\{x\}} \subseteq T_{\varphi^m} I_{\{x\}}$.
Conversely, if $T_{\varphi^m} I_{\{x\}} = I_{\{x\}}$ for some $x \in K$, assume that $\varphi^m(x) \neq x$. Then by Urysohn's lemma there is some $f \in I_{\{x\}}$ with $f(x) = 0$ and $f(\varphi^m(x)) \neq 0$, hence $T_{\varphi^m} f(x) \neq 0$. This is a contradiction, thus $x$ is a periodic point of $\varphi$. □

The periodicity of the map $\varphi$, that is $\varphi^m = \mathrm{id}$ for some $m \in \mathbb{N}$, is directly translated to the Koopman operator and backwards.

**Lemma 2.3.26.** *The map $\varphi$ is periodic if and only if $T_\varphi$ is periodic.*

*Proof.* For some $m \in \mathbb{N}$ we have ${T_\varphi}^m f = \varphi^m \circ f = f$ for any $f \in C(K)$ if and only if $\varphi^m = \mathrm{id}$. □

# 3 Spectral properties of a Koopman operator

In this section we discuss how spectral properties of a Koopman operator are reflected in the underlying dynamical system and vice versa. Before doing so, we give a brief introduction to the spectral properties of a Koopman operator.
First, we recall the definition of the spectrum of a linear bounded operator and some important parts of it.

**Definition 3.0.1.** Let $T : X \to X$ be a linear bounded operator on a Banach space $X$. Then the *spectrum of* $T$ is the set

$$\sigma(T) := \{\lambda \in \mathbb{C} : \lambda - T \text{ is not bijective}\}$$

which is always nonempty and closed.
The most important subset is the *point spectrum of* $T$,

$$P\sigma(T) := \{\lambda \in \sigma(T) : \lambda - T \text{ is not injective}\}$$

and, more general, the *approximative point spectrum of* $T$ is defined as

$$A\sigma(T) := \{\lambda \in \mathbb{C} : \exists\ (x_n)_{n\in\mathbb{N}} \subseteq X \text{ with } \|x_n\| = 1 \text{ for all } n \in \mathbb{N} \text{ and } (\lambda - T)\, x_n \overset{n\to\infty}{\longrightarrow} 0\}.$$

Elements in $P\sigma(T)$ are called *eigenvalues*. The *spectral radius of* $T$ is

$$r(T) := \sup\{|\lambda| : \lambda \in \sigma(T)\}.$$

We give several observations concerning the spectrum of a Koopman operator.

**Theorem 3.0.2.** *Let* $(\mathrm{K}; \varphi)$ *be a dynamical system with corresponding Koopman system* $(C(K); T_\varphi)$*. Then the following assertions hold true.*

*(i)* $r(T_\varphi) = 1$*, hence the spectrum* $\sigma(T_\varphi)$ *is contained in the unit disk* $D$*.*

*(ii) Let* $T_\varphi$ *be bijective. Then* $\sigma(T_\varphi)$ *is a cyclic closed subset of* $\mathbb{T}$*.*

*(iii) Let* $T_\varphi$ *be not bijective. Then* $\sigma(T_\varphi) = \{0\} \cup M$ *for* $M \subseteq \mathbb{T}$ *cyclic or* $\sigma(T_\varphi) = D$*.*

*Proof.* (i) Clearly, $r(T_\varphi) \le \| T_\varphi \| = 1$. From $T_\varphi \mathbb{1}_K = \mathbb{1}_K$ follows that $1 \in P\sigma(T_\varphi)$, thus $r(T_\varphi) = 1$.

(ii) By assumption, $0 \notin \sigma(T_\varphi)$. The injectivity of $T_\varphi$ implies that $\varphi$ is surjective which yields that $T_\varphi$ is isometric. Now assume that there is some $\lambda \in \sigma(T_\varphi)$ such that $0 < |\lambda| < 1$. Since $\partial\sigma(T_\varphi) \subseteq A\sigma(T_\varphi)$ (see e.g. [Sch74, above Thm. V.1.4]), assume without loss of generality that $\lambda \in A\sigma(T_\varphi)$. By definition, there is a sequence $(f_n)_{n\in\mathbb{N}} \subseteq C(K)$ with $\|f_n\| = 1$ for every $n \in \mathbb{N}$ such that

$$0 \xleftarrow{n\to\infty} \| T_\varphi f_n - \lambda f_n\| \ge \big|\| T_\varphi f_n\| - |\lambda| \cdot \|f_n\|\big| = \big|1 - |\lambda|\big| > 0.$$

This contradiction implies $\sigma(T_\varphi) \subseteq \mathbb{T}$. Since $T_\varphi$ is a lattice operator, $\sigma(T_\varphi)$ is cyclic (see [Sch74, Thm. V.4.4]).

(iii) If $T_\varphi$ is not bijective, then clearly $0 \in \sigma(T_\varphi)$.
If $\varphi$ is surjective, then $T_\varphi$ is isometric. The same argument as in the proof of (ii) gives $\partial\sigma(T_\varphi) \subseteq \mathbb{T}$. As $\sigma(T_\varphi)$ is closed, $\sigma(T_\varphi) = D$.
For the proof of the case that $\varphi$ is not surjective, we refer to [Sch71, Thm. 2.7]. □

Theorem 3.0.2 reveals that there are only very few possibilities for $\sigma(T_\varphi)$. This prevents it from being particularly useful for the distinction of different Koopman operators. Therefore, we now turn towards the point spectrum $P\sigma(T_\varphi)$ which can give more insights.

**Theorem 3.0.3.** *(i)* $1 \in P\sigma(T_\varphi)$.

*(ii) If* $\lambda \in P\sigma(T_\varphi)$, *then* $\lambda^k \in P\sigma(T_\varphi)$ *for every* $k \in \mathbb{N}_0$.

*(iii)* $P\sigma(T_\varphi)$ *is cyclic, that is, for every* $\lambda = |\lambda|e^{i\varphi_\lambda} \in P\sigma(T_\varphi)$ *also* $|\lambda|e^{i\varphi_\lambda k} \in P\sigma(T_\varphi)$ *for all* $k \in \mathbb{Z}$.

*Proof.* (i) follows from $T_\varphi \mathbb{1}_K = \mathbb{1}_K$. The multiplicativity of $T_\varphi$ implies (ii), since if $0 \ne f \in C(K)$ is an eigenfunction corresponding to $\lambda$, then $0 \ne f^k$ is an eigenfunction corresponding to $\lambda^k$. (iii) follows from the fact that $T_\varphi$ is a lattice operator (see [Sch74, Cor. V.4.2]) □

The next theorem concerns the number of eigenvalues on the unit circle.

**Theorem 3.0.4.** *Let* $(\mathrm{K};\varphi)$ *be a dynamical system with corresponding Koopman system* $(C(K);T_\varphi)$*. If* $K$ *is metrizable, then* $T_\varphi$ *has at most a countable number of eigenvalues on the unit circle* $\mathbb{T}$*.*

*Proof.* Since $K$ is a metrizable compact space if and only if $C(K)$ is separable (see [EFHN15, Thm. 4.7]) and $T_\varphi$ is powerbounded, the assertion follows by the Jamison Theorem (see [Jam65]). □

**Remark 3.0.5.** Since $T_\varphi$ is a lattice operator, i.e., $T_\varphi|f| = |T_\varphi f|$ for every $f \in C(K)$, we obtain for a unimodular eigenvalue $\lambda \in \mathbb{T}$ of $T_\varphi$ with corresponding eigenfunction $g \in C(K)$ that

$$T_\varphi|g| = |T_\varphi g| = |\lambda g| = |g|.$$

This is interesting since it reveals a connection between the eigenspaces of unimodular eigenvalues and the fixed space $\mathrm{Fix}(T_\varphi)$.

**Example 3.0.6.** (i) Clearly, for the system $(K;\mathrm{id}_K)$ the corresponding Koopman system is $\big(C(K);\mathrm{Id}_{C(K)}\big)$, thus $P\sigma(T_\varphi) = \{1\}$ which is the smallest possible point spectrum of a Koopman operator.

(ii) Consider $\ell^\infty(\mathbb{N}) := \{x := (x_i)_{i\in\mathbb{N}}$ sequence in $\mathbb{C} : \|x\|_\infty = \sup_{i\in\mathbb{N}}|x_i| < \infty\}$ and the left shift

$$T : \ell^\infty(\mathbb{N}) \to \ell^\infty(\mathbb{N}),\ (x_i)_{i\in\mathbb{N}} \mapsto (x_{i+1})_{i\in\mathbb{N}}.$$

Clearly, $\ell^\infty(\mathbb{N})$ is a commutative $C^*$-algebra. Thus by the Gelfand-Naimark theorem there exists a compact space $K$ such that $\ell^\infty(\mathbb{N}) \cong C(K)$ where $K$ is isomorphic to the Stone-Čech compactification of $\mathbb{N}$, see e.g. [Wer11, IX.5]. Since $T$ is a $C^*$-algebra homomorphism, there is a dynamics $\varphi : K \to K$ such that $T = T_\varphi$ (see Theorem 2.2.17).
Take any $\lambda \in \mathbb{C}$ with $|\lambda| \le 1$. Then $(x_i)_{i\in\mathbb{N}} := (\lambda^i)_{i\in\mathbb{N}} \in \ell^\infty(\mathbb{N})$ is an eigenfunction in $\ell^\infty(\mathbb{N})$ corresponding to $\lambda$, thus $P\sigma(T_\varphi) = D$ which is the largest possible point spectrum of a Koopman operator. Note that this is not a contradiction to Theorem 3 since $\ell^\infty(\mathbb{N})$ is not separable.

(iii) Now consider $\ell^\infty(\mathbb{Z}) := \{x := (x_i)_{i\in\mathbb{Z}} \subseteq \mathbb{C} : \|x\|_\infty = \sup_{i\in\mathbb{Z}}|x_i| < \infty\}$ and the left shift

$$T : \ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z}),\ (x_i)_{i\in\mathbb{Z}} \mapsto (x_{i+1})_{i\in\mathbb{Z}}.$$

Analogously to (ii), there exists a compact space $K$ – which is isomorphic to the Stone-Čech compactification of $\mathbb{Z}$ – such that $\ell^\infty(\mathbb{Z}) \cong C(K)$ and $T = T_\varphi$ for some $\varphi : K \to K$. By Theorem 3.0.2, $\sigma(T_\varphi) \subseteq \mathbb{T}$. By means of eigenfunctions constructed as in (ii), we see $\lambda \in P\sigma(T_\varphi)$ for all $\lambda \in \mathbb{T}$. Thus $P\sigma(T_\varphi) = \sigma(T_\varphi) = \mathbb{T}$.

## 3.1 Spectral properties of a Koopman operator of an affine finite dimensional dynamical system

In this section we always take $X := (\mathbb{C}^k; \|\cdot\|)$ and a contractive matrix $A \in \mathcal{L}(X)$, i.e., $\|A\| \le 1$ for the induced operator norm. Consider the restriction of $A$ to the closed unit ball $U$ in $X$,

$$A\big|_U : U \to U,$$

and the associated Koopman operator

$$T_A : C(U) \to C(U), \quad f \mapsto f \circ A\big|_U .$$

The so constructed system is called an *affine* finite dimensional system.

Here the underlying dynamics comes from a linear mapping which has its own spectrum. The obvious question to ask is: How is the spectrum of the corresponding Koopman operator determined by the spectrum of the dynamics $A$?

**Remark 3.1.1.** The Koopman linearization of such a dynamical system $(U; A\big|_U)$ is useful to consider even if the system comes from a linear map because linearity is not an isomorphy invariant. This means that $(U; A\big|_U)$ can be isomorphic to many non-linear systems. For $(U; A\big|_U) \cong (\mathrm{K}; \varphi)$ where $(\mathrm{K}; \varphi)$ is a potentially non-linear system we have $P\sigma(T_A) = P\sigma(T_\varphi)$ since two dynamical systems are isomorphic if and only if their corresponding Koopman systems are isomorphic and the point spectrum is an isomorphy invariant. Thus, if we draw conclusions from $\sigma(A)$ on $P\sigma(T_A)$, we also obtain conclusions on $P\sigma(T_\varphi)$ (cf. [MM14]).

**Remark 3.1.2.** Take any power bounded matrix on $(\mathbb{C}^k, \|\cdot\|)$, i.e., there exists some $C \in \mathbb{R}$ such that $\sup_{n\in\mathbb{N}_0} \|A^n\| \le C$. Consider the equivalent norm $|||\cdot|||$ defined by $|||x||| := \sup_{n\in\mathbb{N}} \|A^n x\|$ for $x \in \mathbb{C}^k$. Then $A$ is a contraction with respect to the norm $|||\cdot|||$. Thus the following results hold true even for power bounded matrices.

We now compare the spectra of $A$ and $T_A$.

**Theorem 3.1.3.** *The spectrum $\sigma(A)$ of the matrix $A$ is contained in the point spectrum $P\sigma(T_A)$ of the corresponding Koopman operator.*

*Proof.* Let $\sigma(A) = \{\lambda_1, ..., \lambda_m\}$ be the spectrum of $A$ with pairwise distinct eigenvalues $\lambda_1, ..., \lambda_m$. The Jordan canonical form gives a decomposition of $X$ into $A$-invariant cyclic subspaces,

$$X = \underbrace{V_1^1 \oplus ... \oplus V_{d_1}^1}_{\text{subspaces associated with } \lambda_1} \oplus ... \oplus \underbrace{V_1^m \oplus ... \oplus V_{d_m}^m}_{\text{subspaces associated with } \lambda_m} .$$

Take $\lambda \in \sigma(A)$. After renumbering, let $V := V_1^1$ be a subspace associated with $\lambda$, hence

$$X = V \oplus V_2^1 \oplus ... \oplus V_{d_m}^m =: V \oplus W$$

where $V, W$ are $A$-invariant subspaces. This decomposition admits a basis $(x_1, ..., x_k)$ such that $(x_{d+1}, ..., x_k)$ is a basis of $W$ and $(x_1, ..., x_d)$ a basis of $V$ and such that if $d = 1$, then

$$Ax_1 = \lambda x_1$$

and, if $d > 1$, then

$$\begin{aligned} Ax_1 &= \lambda x_1, \\ Ax_2 &= \lambda x_2 + x_1, \\ &\vdots \\ Ax_d &= \lambda x_d + x_{d-1}. \end{aligned}$$

The matrix representation of $A\big|_V$ with respect to this basis is a Jordan block of size $d$, i.e., $A\big|_V \mathrel{\hat{=}} \begin{pmatrix} \lambda & 1 & & \\ & \ddots & & \\ & & \lambda & 1 \\ & & & \lambda \end{pmatrix}$.

Define a linear form $\tilde{f}$ on $X$ on this basis by

$$\tilde{f}(x_d) = 1, \quad \tilde{f}(x_i) = 0$$

for $1 \leq i \leq k$ and $i \neq d$.

Then

$$f := \tilde{f}\big|_U \tag{3.1}$$

is an eigenfunction of $T_A$ to the eigenvalue $\lambda$.
Indeed, let $x = \sum_{i=1}^k \alpha_i x_i \in U$ be arbitrary. Then $Ax \in U$ and if $d = 1$, then

$$\begin{aligned} T_A f(x) &= f(Ax) = \tilde{f}(\sum_{i=1}^k \alpha_i A x_i) = \sum_{i=1}^k \alpha_i \tilde{f}(Ax_i) \\ &= \alpha_1 \tilde{f}(\lambda x_1) + \sum_{i=2}^k \alpha_i \underbrace{\tilde{f}(Ax_i)}_{\substack{=0, \text{ since } W \\ \text{is } A\text{-invariant}}} \\ &= \alpha_1 \lambda \underbrace{\tilde{f}(x_1)}_{=1} = \alpha_1 \lambda \tilde{f}(x_1) + \lambda \sum_{i=2}^k \alpha_i \underbrace{\tilde{f}(x_i)}_{=0} \\ &= \lambda \tilde{f}(x) \\ &= \lambda f(x). \end{aligned}$$

If $d > 1$, then

$$\begin{aligned}
T_A f(x) &= f(Ax) = \tilde{f}(\sum_{i=1}^{k} \alpha_i A x_i) = \sum_{i=1}^{k} \alpha_i \tilde{f}(Ax_i) \\
&= \alpha_1 \underbrace{\tilde{f}(\lambda x_1)}_{=0} + \sum_{i=2}^{d} \alpha_i \tilde{f}(\lambda x_i + x_{i-1}) + \sum_{i=d+1}^{k} \alpha_i \underbrace{\tilde{f}(Ax_i)}_{\substack{=0, \text{ since } W \\ \text{is } A\text{-invariant}}} \\
&= \sum_{i=2}^{d} \alpha_i \big(\lambda \tilde{f}(x_i) + \tilde{f}(x_{i-1})\big) \\
&= \alpha_d \lambda \tilde{f}(x_d) + \lambda \sum_{i=1, i \neq d}^{k} \alpha_i \underbrace{\tilde{f}(x_i)}_{=0} \\
&= \lambda \tilde{f}(x) \\
&= \lambda f(x).
\end{aligned}$$

Therefore,

$$f \neq 0 \text{ and } T_A f = \lambda f.$$

□

In the following we discuss the point spectrum of the Koopman operator for special matrices.

**Theorem 3.1.4.** *If $A$ is unitary, then $P\sigma(T_A) = \langle \sigma(A) \rangle$, where $\langle \sigma(A) \rangle$ denotes the group generated by $\sigma(A)$.*

*Proof.* We first show that $\lambda_1 \cdot \ldots \cdot \lambda_m \in P\sigma(T_A)$ for $\lambda_1, \ldots, \lambda_m \in \sigma(A)$ and $m \leq k$. For $\lambda_i = \lambda_j$ for all $i, j \in \{1, \ldots, m\}$ this follows since $T_A$ is multiplicative.
Therefore let $\lambda_i \neq \lambda_j$ for some $i, j$ and $f_{\lambda_1}, \ldots, f_{\lambda_m}$ eigenfunctions associated with $\lambda_1, \ldots, \lambda_m$ such that $f_{\lambda_j}(x_j) = 1$, $f_{\lambda_j}(x_i) = 0$, for $i \neq j$ and basis vectors $x_1, \ldots, x_m$ (constructed as in the proof above, after potentially a resorting of the basis vectors). It follows that

$$T_A(f_{\lambda_1} \cdot \ldots \cdot f_{\lambda_m}) = \lambda_1 \cdot \ldots \cdot \lambda_m f_{\lambda_1} \cdot \ldots \cdot f_{\lambda_m}.$$

Moreover, $f_{\lambda_1} \cdot \ldots \cdot f_{\lambda_m} \neq 0$, since for $x = \frac{1}{m}(x_1 + \ldots + x_m) \in U$,

$$\begin{aligned}
(f_{\lambda_1} \cdot \ldots \cdot f_{\lambda_m})(x) &= f_{\lambda_1}(\tfrac{1}{m}(x_1 + \ldots + x_m)) \cdot \ldots \cdot f_{\lambda_m}(\tfrac{1}{m}(x_1 + \ldots + x_m)) \\
&= \tfrac{1}{m^m} f_{\lambda_1}(x_1) \cdot \ldots \cdot f_{\lambda_m}(x_m) = \frac{1}{m^m} \neq 0.
\end{aligned}$$

We always have $1 \in P\sigma(T_A)$. Moreover, for $\lambda \in P\sigma(T_A)$, $\lambda^{-1} = \bar{\lambda} \in P\sigma(T_A)$ since $P\sigma(T_A)$ is cyclic (confer Theorem 3.0.3 (iii)) and $\sigma(A) \subseteq \mathbb{T}$. This shows $\langle \sigma(A) \rangle \subseteq P\sigma(T_A)$.

To show the converse inclusion, consider

$$S := \{f \in C(U) : \; f \text{ eigenfunction of } T_A \text{ corresponding to } \lambda \in \langle\sigma(A)\rangle\}$$

and its linear hull

$$\mathfrak{A} := \operatorname{lin} S.$$

Clearly, $\mathfrak{A}$ is a conjugation invariant $C^*$-algebra. It separates the points of $U$, since for $x, y \in U$ with $x \neq y$, there are eigenfunctions $f_1$ and $f_2$ corresponding to $\lambda_1, \lambda_2 \in \sigma(A)$, constructed as (3.1) in the proof of Theorem 3.1.3, such that $f_1(x) \neq f_2(y)$. Hence by the Stone-Weierstraß theorem, $\mathfrak{A}$ is dense in $C(U)$ with respect to $\|\cdot\|_\infty$ on $C(U)$.

Let $\mu^{2k}$ be the $2k$-dimensional Lebesgue measure on $(U, \mathcal{B}(U))$, where $\mathcal{B}(U)$ denotes the Borel $\sigma$-algebra on $U \subseteq \mathbb{R}^{2k} \cong \mathbb{C}^k$. Since $A$ is unitary, it leaves $\mu^{2k}$ invariant, i.e. $\mu^{2k}(A^{-1}M) = \mu^{2k}(M)$ for all $M \in \mathcal{B}(U)$ (see [Bau90, Satz I.8.3]). Therefore, we consider $T_A$ as a Koopman operator on $L^2(U, \lambda^{2k})$ corresponding to the measure-preserving dynamical system $\left(U, \mathcal{B}(U), \lambda^{2k}; A\big|_U\right)$.

Note that $C(U)$ is dense in $L^2(U, \lambda^{2k})$ with respect to the $\|\cdot\|_2$-norm and $\overline{\operatorname{lin}}^{\infty} S \subseteq \overline{\operatorname{lin}}^{2} S$. Moreover, we showed that $\operatorname{lin} S$ is dense in $C(U)$ with respect to $\|\cdot\|_\infty$. Hence,

$$\overline{\operatorname{lin}}^{2} S = L^2\left(U, \lambda^{2k}\right).$$

Since the Hilbert space $L^2(U, \lambda^{2k})$ allows the decomposition

$$L^2\left(U, \lambda^{2k}\right) = \overline{\operatorname{lin}}^{2} S \oplus S^\perp,$$

we obtain $S^\perp = \{0\}$. Assume that there is an eigenvalue $\lambda_0 \notin \langle\sigma(A)\rangle$ of $T_\varphi$ with corresponding eigenfunction $f_0 \in C(U)$. Then for any $\lambda \in \langle\sigma(A)\rangle$ with eigenfunction $f \in \mathfrak{A}$ the unitarity of $A$ implies that

$$(f_0 \,|\, f) = (T_A f_0 \,|\, T_A f) = (\lambda_0 f_0 \,|\, \lambda f) = \lambda_0 \overline{\lambda} \, (f_0 \,|\, f),$$

thus $(f_0 \,|\, f) = 0$. Hence $f_0 \in S^\perp$ which is a contradiction. Therefore, $P\sigma(T_A) = \langle\sigma(A)\rangle$. □

We can even generalize Theorem 3.1.4.

**Theorem 3.1.5.** *If $A$ is a contraction and $\sigma(A) \subseteq \mathbb{T}$, then $P\sigma(T_A) = \langle\sigma(A)\rangle$.*

*Proof.* Consider the Jordan canonical form of $A = SJS^{-1}$. Since $A$ is power bounded, every Jordan block has size one which implies that $J$ is a diagonal matrix. Since every entry on the diagonal of $J$ is unimodular, it follows that $J^{-1} = \overline{J}^{t}$, i.e., $J$ is unitary. □

**Theorem 3.1.6.** *Let* (K; $\varphi$) *be a dynamical system with* $|K| \geq 2$. *If there exists some* $x_0 \in K$ *and* $m \in \mathbb{N}$ *such that* $\varphi^m(K) = \{x_0\}$, *then* $P\sigma(T_\varphi) = \{0, 1\}$.

*Proof.* Clearly, $\{0, 1\} \subseteq P\sigma(T_\varphi)$ since $|K| \geq 2$. For the other inclusion take $\lambda \in P\sigma(T_\varphi)$ with corresponding eigenfunction $f \in C(K)$. Then

$$T_\varphi{}^m f(x) = f \circ \varphi^m(x) = \lambda^m f(x)$$

and hence $\lambda^m f(x) = f(x_0)$ for every $x \in K$. If $f(x_0) = 0$, it follows $\lambda = 0$ as $f \neq 0$. If $f(x_0) \neq 0$, then we obtain $\lambda^m f(x_0) = f(x_0)$ for $x := x_0$. This implies $\lambda^m = 1$. Since $\lambda^{m+1} f(\varphi(x_0)) = f(\varphi(x_0))$, also $\lambda^{m+1} = 1$, hence $\lambda = 1$. □

This implies the following.

**Proposition 3.1.7.** *For a nilpotent contractive matrix* $A$, $P\sigma(T_A) = \{0, 1\}$.

**Example 3.1.8.** Consider $K := \{a, b, c\}$ and the dynamics $\varphi : K \to K$ with $\varphi(a) = \varphi(b) = a$ and $\varphi(c) = b$. Then $\varphi^2(K) = \{a\}$ and the corresponding Koopman operator is the matrix $T_\varphi = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ which has spectrum $\sigma(T_\varphi) = \{0, 1\}$.

**Theorem 3.1.9.** *If there is* $\lambda \in \sigma(A)$ *with* $|\lambda| < 1$, *then* $0 \in P\sigma(T_A)$.

*Proof.* If $0 \in \sigma(A)$, the statement is true according to Theorem 3.1.3. So let $\lambda \neq 0$ for all $\lambda \in \sigma(A)$, hence $A$ is injective. We first show that $A\big|_U : U \to U$ is not surjective. Let $\lambda \in \sigma(A)$ with $0 < |\lambda| < 1$, and $x$ an associated eigenvector. Assume that $x \in rg(A\big|_U)$. Then there is $y \in U$ such that $Ay = x$. By the linearity of $A$, we have $A(\lambda y) = \lambda x = Ax$, and therefore $A(\lambda y - x) = 0$. The injectivity of $A$ implies $y = \frac{x}{\lambda}$, hence $\|y\| = \frac{1}{|\lambda|}\|x\| > 1$. This contradicts $y \in U$, so $x \notin rg(A\big|_U)$. Therefore $A\big|_U$ is not surjective and so $T_A$ is not injective 2.2.18. Hence there exists $0 \neq f \in C(U)$ with $T_A f = 0$, i.e., $0 \in P\sigma(T_A)$.

□

In the following we assume that $A$ has an eigenvalue in the interior of the unit disk. Then all interior points of $D$ belong to $P\sigma(T_A)$. We begin with the one-dimensional case.

**Theorem 3.1.10.** *Consider the linear mapping* $A : \mathbb{C} \to \mathbb{C}$, $x \mapsto \alpha x$ *for some* $\alpha \in \mathbb{C}$. *If* $0 < |\alpha| < 1$, *then* $D^\circ = \{\lambda \in \mathbb{C} : |\lambda| < 1\} \subseteq P\sigma(T_A)$.

*Proof.* Let $\lambda \in D^\circ$. As seen in Theorem 3.1.9, we have $0 \in P\sigma(T_A)$. Thus take $\lambda \neq 0$ and construct an eigenfunction $f$ corresponding to $\lambda$ as follows:
Construct $f$ piecewise on the circular rings $U_{n,n+1} := \{x \in D : |\alpha^{n+1}| \leq |x| \leq |\alpha^n|\}$ and write $f_{n,n+1}$ for a function on $U_{n,n+1}$, $n \in \mathbb{N}_0$.
Start on the outmost circular ring $U_{0,1}$ and define $f_{0,1}$ first on the circles $\mathbb{T}$ and $\alpha\mathbb{T}$ as

$$f_{0,1}(z) := z$$
$$f_{0,1}(\alpha z) := \lambda z$$

for $z \in \mathbb{T}$. Then extend $f_{0,1}$ continuously to the ring $U_{0,1}$.
Now define $f_{n,n+1}$ for $x \in U_{n,n+1}$, $n \geq 1$, as

$$f_{n,n+1}(x) := \lambda^n f_{0,1}\left(\alpha^{-n}x\right)$$

and

$$f(0) := 0.$$

A simple calculation shows that $f$ is indeed continuous on $U$ and satisfies $f(1) = 1$. Moreover, for any $n \geq 0$ and $x \in U_{n,n+1}$ we have

$$\begin{aligned} T_A f(x) &= f(Ax) = f(\underbrace{\alpha x}_{\in U_{n+1,n+2}}) = f_{n+1,n+2}(\alpha x) \\ &= \lambda^{n+1} f_{0,1}\left(\alpha^{-(n+1)}\alpha x\right) = \lambda \cdot \lambda^n f_{0,1}\left(\alpha^{-n}x\right) \\ &= \lambda f_{n,n+1}(x) \\ &= \lambda f(x). \end{aligned}$$

Therefore, $f$ is an eigenfunction corresponding to $\lambda$, thus $\lambda \in P\sigma(T_A)$.

□

**Remark 3.1.11.** Clearly $\dim \mathrm{Eig}(\lambda, T_A) = \infty$ for every $0 < |\lambda| < 1$.

**Example 3.1.12.** We give an explicit example for an eigenfunction corresponding to $0 < |\lambda| < 1$.
First note that for every $x_0 \in U_{0,1} = \{x \in D : |\alpha| \leq |x| \leq |1|\}$ there is some $z = e^{i\varphi_z} \in \mathbb{T}$ such that $x_0$ lies on the continuous path $\alpha(t)e^{i\varphi_z(t)}$ from $z$ to $\alpha z$, where $\alpha(t) := 1 + (|\alpha| - 1)t$ and $\varphi_z(t) := \varphi_z + \varphi_\alpha t$ for $0 \leq t \leq 1$, $0 \leq \varphi_z, \varphi_\alpha, \varphi_{x_0} \leq 2\pi$. This is shown by solving $|x_0|e^{i\varphi_{x_0}} = \alpha(s)e^{\varphi_z(s)}$, hence by solving the linear system

$$\begin{aligned} |x_0| &= 1 + (|\alpha| - 1)s \\ \varphi_{x_0} &= \varphi_z + \varphi_\alpha s. \end{aligned}$$

We obtain the unique solution

$$s = \frac{|x_0| - 1}{|\alpha| - 1} \tag{3.2}$$

$$\varphi_z = \varphi_\alpha \frac{|x_0| - 1}{|\alpha| - 1} - \varphi_{x_0}. \tag{3.3}$$

By extending this to the unit ball $U$, we obtain for any $x \in U$ a unique representation $x = \alpha^n \alpha(s) e^{i\varphi_z(s)}$, where $n \geq 0$ is the unique number such that $x \in U_{n,n+1}$. Even this $n$ can be given explicitely. Since $|\alpha^{n+1}| \leq |x| \leq \alpha^n$, $(n+1)\ln|\alpha| \leq \ln|x| \leq n \ln|\alpha|$, thus

$$n = \lfloor \frac{\ln|x|}{\ln|\alpha|} \rfloor. \tag{3.4}$$

By means of this representation define

$$f : U \to \mathbb{C},\ x \mapsto \lambda^n \left(1 + (\lambda - 1)s\right) z$$

and

$$f(0) := 0.$$

Then $f$ is as in the proof of Theorem 3.1.10 with $f_{0,1}(\alpha(s)e^{i\varphi_z(s)}) = (1 + (\lambda - 1)s)\, z$. Using (3.2), (3.3) and (3.4), we obtain

$$f : U \to \mathbb{C},\ x \mapsto \lambda^{\lfloor \frac{\ln|x|}{\ln|\alpha|} \rfloor} \left(1 + (\lambda - 1)\frac{|x| - 1}{|\alpha| - 1}\right) e^{i\left(\varphi_\alpha \frac{|x|-1}{|\alpha|-1} - \varphi_x\right)}.$$

Next we show that the result in Theorem 3.1.10 can be extended to the space $\mathbb{C}^k$.

**Theorem 3.1.13.** *Consider $X := (\mathbb{C}^k; \|\cdot\|)$ for some norm $\|\cdot\|$ and a contractive linear mapping $A \in \mathcal{L}(X)$. If there is $\alpha \in \sigma(A)$ with $0 < |\alpha| < 1$, then*

$$D^\circ = \{\lambda \in \mathbb{C} : |\lambda| < 1\} \subseteq P\sigma(T_A).$$

*Proof.* Let $(x_1, ..., x_k)$ be a basis of $\mathbb{C}^k$ such that for $\alpha \in \sigma(A)$ with $0 < |\alpha| < 1$ we have

$$\begin{aligned} Ax_1 &= \alpha x_1 \\ Ax_2 &= \alpha x_2 + x_1 \\ &\vdots \\ Ax_d &= \alpha x_d + x_{d-1}, \end{aligned}$$

i.e., the matrix representation of $(x_1, ..., x_d)$ is a Jordan block associated to $\alpha$ (cf. Jordan canonical form in the proof of Theorem 3.1.3). Without loss of generality assume $\|x_i\| \leq 1$ for $i = 1, .., k$.
Then $0 \in P\sigma(T_A)$ according to Theorem 3.1.9. Thus take $0 \neq \lambda \in D^\circ$ and show $\lambda \in P\sigma(T_A)$. To do so we construct an eigenfunction $F$ corresponding to $\lambda$ by an analogous procedure as in the one-dimensional case.
Define $F_{n,n+1}$ on $U_{n,n+1} := \{x \in U : x = \sum_{i=1}^k \mu_i x_i \text{ with } |\alpha^{n+1}| \leq |\mu_d| \leq |\alpha^n|\}$.
For $x \in U$ with $x = \sum_{i=1}^k \mu_i x_i$ and $\mu_d \in \mathbb{T}$, define

$$F_{0,1}(x) := \mu_d$$

and

$$F_{0,1}(\alpha x) := \lambda \mu_d.$$

Now extend $F_{0,1}$ continuously to $U_{0,1}$ such that $F_{0,1}(x) = F_{0,1}(\mu_d x_d)$, hence $F_{0,1}$ is independent of the coefficients $\mu_1, \ldots, \mu_{d-1}, \mu_{d+1}, \ldots, \mu_k$.
For $x \in U_{n,n+1}$, $n \geq 1$, define

$$F_{n,n+1}(x) := \lambda^n F_{0,1}\left(\alpha^{-n} x\right)$$

and for $x \in U_0 := \{x \in U : x = \sum_{i=1}^k \mu_i x_i \text{ with } \mu_d = 0\}$

$$F(x) := 0.$$

Then $F \neq 0$ is continuous which can be calculated as in Theorem 3.1.10.
Moreover, for $k \geq 1$ and $x = \sum_{i=1}^k \mu_i x_i \in U_{n,n+1}$,

$$\begin{aligned}
T_A F(x) &= F(Ax) = F\left(A \sum_{i=1}^k \mu_i x_i\right) \\
&= F\underbrace{\left(\alpha\mu_1 x_1 + \sum_{i=2}^d \alpha \mu_i x_i + \mu_{i-1} x_{i-1} + \sum_{i=d+1}^k \overbrace{A\mu_i x_i}^{\notin \operatorname{lin}\{x_d\}}\right)}_{\in U_{n+1,n+2}} \\
&= F_{n+1,n+2}\left(\alpha \mu_d x_d\right) \\
&= \lambda^{n+1} F_{0,1}\left(\alpha^{-(n+1)} \alpha \mu_d x_d\right) \\
&= \lambda \cdot \lambda^n F_{0,1}\left(\alpha^{-n} \mu_d x_d\right) \\
&= \lambda F_{n,n+1}\left(\mu_d x_d\right) \\
&= \lambda F_{n,n+1}\left(\sum_{i=1}^k \mu_i x_i\right) \\
&= \lambda F_{n,n+1}(x) \\
&= \lambda F(x).
\end{aligned}$$

Therefore, $F$ is an eigenfunction corresponding to $\lambda$, thus $\lambda \in P\sigma(T_A)$. $\square$

From the results above we obtain the following.

**Theorem 3.1.14.** *Consider* $X := (\mathbb{C}^k; \|\cdot\|)$ *for some norm* $\|\cdot\|$ *and a contractive linear mapping* $A \in \mathcal{L}(X)$.

*(i) If there is* $\lambda \in \sigma(A)$ *such that* $0 < |\lambda| < 1$ *and* $\sigma(A) \cap \mathbb{T} \neq \varnothing$, *then*

$$D^\circ \cup \langle \sigma(A) \cap \mathbb{T} \rangle \subseteq P\sigma(T_A).$$

*(ii) If there is* $\lambda \in \sigma(A)$ *such that* $0 < |\lambda| < 1$ *and* $\sigma(A) \cap \mathbb{T} = \varnothing$, *then*

$$P\sigma(T_A) = D^\circ \cup \{1\}.$$

*Proof.* It remains to show $P\sigma(T_A) \subseteq D^\circ \cup \{1\}$ in $(ii)$. This shall be proved in 4.1.9. □

It remains open, whether the converse inclusion in 3.1.14 $(i)$ holds true.

## 3.2 Partitioning of the state space

We consider a dynamical system $(K;\varphi)$ with corresponding Koopman system $(C(K);T_\varphi)$ and show that the point spectrum of the Koopman operator allows us to draw conclusions on the structure of $K$ and the dynamics $\varphi$. In this section, $\zeta_k$ denotes a primitive $k^{th}$ root of unity. We first show that the dimension of the fixed space dominates the dimension of eigenspaces corresponding to roots of unity. To do so, we prove the following more general lemma. I thank Pavel Zorin for providing the proof.

**Lemma 3.2.1.** *Let $K$ be a compact space and $f_1,...,f_n \in C(K)$ linearly independent. Denote $M := \operatorname{lin}\{f_1,...,f_n\}$ and $M^k := \operatorname{lin}\{g^k : g \in M\}$ for some $k \in \mathbb{N}$. Then*

$$\dim M \leq \dim M^k.$$

*Proof.* Our aim is to find $n$ linearly independent functions in $M^k$. For this, we first show that there exists some $K' \subseteq K$ with $|K'| = n$ such that $\dim M\big|_{K'} = n$, where $M\big|_{K'} := \{f\big|_{K'} : f \in M\}$.
We give a proof by induction over $n$. Clearly, the assertion is true for $n = 1$.
For $M = \operatorname{lin}\{f_1,...,f_n\}$, we know by induction hypothesis that there is $K'_{n-1} = \{x_1,...,x_{n-1}\}$ such that $\dim \operatorname{lin}\{f_1,...,f_{n-1}\}\big|_{K'_{n-1}} = n-1$. Define

$$\tilde{f}_n := f_n - \sum_{i=1}^{n-1} \alpha_i f_i \in M$$

where the choice of $\alpha_1,...,\alpha_{n-1}$ is such that $\tilde{f}_n(x_i) = 0$ for $i = 1,...,n-1$. This is possible since

$$\begin{pmatrix} f_1(x_1) & \dots & f_{n-1}(x_1) \\ \vdots & & \vdots \\ f_1(x_{n-1}) & \dots & f_{n-1}(x_{n-1}) \end{pmatrix}$$

is invertible by induction hypothesis and hence

$$\begin{pmatrix} f_n(x_1) \\ \vdots \\ f_n(x_{n-1}) \end{pmatrix} = \begin{pmatrix} f_1(x_1) & \dots & f_{n-1}(x_1) \\ \vdots & & \vdots \\ f_1(x_{n-1}) & \dots & f_{n-1}(x_{n-1}) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{n-1} \end{pmatrix}$$

has a solution.
Now choose $x_n \in K$ such that $\tilde{f}_n(x_n) \neq 0$. Then for $K' := \{x_1,...,x_n\}$ we have

$$\dim \operatorname{lin}\{f_1,...,f_{n-1},\tilde{f}_n\}\big|_{K'} = \dim M\big|_{K'} = n.$$

This means that $M\big|_{K'} = C(K') \cong \mathbb{C}^n$ and hence $\mathbb{1}_{\{x_j\}} \in M\big|_{K'}$ for $j = 1,...,n$, i.e., there is $\lambda_{ij}$, $i,j = 1,...,n$, such that $\mathbb{1}_{\{x_j\}} = \sum_{i=1}^n \lambda_{ij} f_i\big|_{K'}$.

Clearly, $\mathbb{1}_{\{x_1\}}, ..., \mathbb{1}_{\{x_n\}}$ are linearly independent. Define

$$f_{x_j} := \sum_{i=1}^{n} \lambda_{ij} f_i \in M$$

for every $j = 1, ..., n$. Then $f_{\{x_1\}}, ..., f_{\{x_n\}}$ are linearly independent since $f_{x_j}\big|_{K'} = \mathbb{1}_{x_j}$. Since $\mathbb{1}_{x_j}^k = \mathbb{1}_{x_j}$, also $f_{x_1}^k, ..., f_{x_n}^k$ are linearly independent for every $k \in \mathbb{N}$. Hence we have constructed $n$ linearly independent functions in $M^k$, thus

$$\dim M \leq \dim M^k.$$

□

**Remark 3.2.2.** Note that in the proof of Lemma 3.2.1, we have shown that for every $m \in \mathbb{N}$ with $\dim M \geq m$, also $\dim M^k \geq m$ for $k \in \mathbb{N}$. This means that Lemma 3.2.1 does not only hold true for $M$ finite dimensional but can be generalized to $M$ infinite dimensional. The same is the case for the following lemma, which follows from Lemma 3.2.1.

**Lemma 3.2.3.** *If* $\{1, \zeta_k, ..., \zeta_k^{k-1}\} \subseteq P\sigma(T_\varphi)$ *and* $\dim \mathrm{Fix}(T_\varphi) < \infty$, *then*

$$\dim \mathrm{Eig}(\zeta_k^l, T_\varphi) \leq \dim \mathrm{Fix}(T_\varphi)$$

*for every* $l = 1, ..., k-1$.

*Proof.* Let $f \in \mathrm{Eig}(\zeta_k^l, T_\varphi)$ for any $1 \leq l \leq k-1$. Then

$$T_\varphi f^k = (T_\varphi f)^k = \left(\zeta_k^l f\right)^k = \zeta_k^{l \cdot k} f^k = f^k.$$

Thus,

$$\{f^k : f \in \mathrm{Eig}(\zeta_k^l, T_\varphi)\} \subseteq \mathrm{Fix}(T_\varphi).$$

By Lemma 3.2.1 it follows $\dim \mathrm{Fix}(T_\varphi) \geq \dim \mathrm{Eig}(\zeta_k^l, T_\varphi)$. □

Based on this, we now work out a partitioning of the state space $K$ and describe the behavior of the dynamics $\varphi$ on the components in the case of a one-dimensional fixed space.

**Theorem 3.2.4.** *Let* $(K;\varphi)$ *be a dynamical system with corresponding Koopman system* $(C(K); T_\varphi)$. *Let* $\{1, \zeta_k, \zeta_k^2, ..., \zeta_k^{k-1}\} \subseteq P\sigma(T_\varphi)$ *and* $\dim \mathrm{Fix}(T_\varphi) = 1$. *Then* $K$ *is a union of* $k$ *pairwise disjoint, open and closed sets* $K_1, ..., K_k$ *and*

$$\varphi(K_1) \subseteq K_2,\ \varphi(K_2) \subseteq K_3, ..., \ \varphi(K_k) \subseteq K_1.$$

*Proof.* Let $f$ be an eigenfunction corresponding to $\zeta_k$, hence $f_l := f^l$ is an eigenfunction to $\zeta_k^l$, and $\mathrm{Eig}(\zeta_k^l) = \mathrm{lin}\{f_l\}$ since $\dim \mathrm{Eig}(\zeta_k^l) \le \dim \mathrm{Fix}(T_\varphi) = 1$ for every $l = 0, ..., k-1$ according to Lemma 3.2.3.
Consider the closed subspace

$$\mathfrak{A} := \mathrm{lin}\{f_1, ..., f_k\},$$

which is a $C^*$-algebra.
Indeed, $(\mathfrak{A}, +, \|\cdot\|_\infty)$ is a Banach space by definition and a Banach algebra since for arbitrary $g := \sum_{i=1}^k \alpha_i f_i$ and $h := \sum_{j=1}^k \beta_j f_j \in \mathfrak{A}$, we have $g \cdot h = \sum_{i=1}^k \sum_{j=1}^k \alpha_i \beta_j f^{i+j} \in \mathfrak{A}$. Moreover, $\bar{g} = \sum_{i=1}^k \bar{\alpha_i} \bar{f_i} \in \mathfrak{A}$ since $\bar{f}^i$ is an eigenfunction associated with $\overline{\zeta_k}^i = \zeta_k^{-i} \in P\sigma(T_\varphi)$. Because of $\dim \mathrm{Fix}(T_\varphi) = 1$, it follows that $f^k = c \cdot \mathbb{1}$ for some $c \in \mathbb{C}$, hence $\mathbb{1} \in \mathfrak{A}$.

Thus, by the theorem of Gelfand-Naimark [EFHN15, Theorem 4.23] there exists a compact space $L$ such that

$$\mathfrak{A} \cong C(L).$$

Because of $\dim \mathfrak{A} = k$, it follows that

$$L \cong \{1, 2, ..., k\}.$$

Since $\mathfrak{A} \subseteq C(K)$, there is an injective algebra homomorphism $J : C(L) \to C(K)$. By [EFHN15, Theorem 4.13], $J$ is a Koopman operator, that is $J = J_\Phi$ for a surjective continuous mapping $\Phi : K \to L$.
The continuity of $\Phi$ implies that each $\Phi^{-1}(\{i\})$ is both open and closed in $K$. Since $\Phi^{-1}(\{i\}) \cap \Phi^{-1}(\{j\}) = \emptyset$ for every $i \neq j$, $K$ can be written as a disjoint union of open and closed sets, i.e.,

$$K = \dot{\bigcup_{i \in \{1,...,k\}}} K_i,$$

for $K_i := \Phi^{-1}(\{i\})$.

It remains to show the cyclic behavior of $\varphi$ on $K_1, ..., K_k$.
Consider the restriction $T_\varphi\big|_{C(L)}$ which is an algebra homomorphism on $C(L)$, hence

$$T_\varphi\big|_{C(L)} = T_\psi$$

is a Koopman operator for a continuous mapping $\psi : L \to L$ [EFHN15, Theorem 4.13].
Moreover, $T_\psi : \mathfrak{A} \to \mathfrak{A}$ is surjective since for any $g := \sum_{i=1}^k \alpha_i f_i \in \mathfrak{A}$ we have $T_\psi f = g$ for $f := \sum_{i=1}^k \frac{\alpha_i}{\zeta_k^i} f_i$. It follows that $\psi$ is injective, hence bijective as $L$ is finite, that is, $\psi : L \to L$ is a permutation.

As $L$ is discrete, the Koopman operator $T_\psi$ is a permutation matrix (cf. Example 2.2.16). Assume that $\psi$ has more than one cycle, i.e., $T_\psi = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}$ with $P_1$ and $P_2$ permutation matrices. Then $\begin{pmatrix} \mathbb{1} \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ \mathbb{1} \end{pmatrix}$ are non-constant fixed points of $T_\psi$ which

contradicts $\dim \mathrm{Fix}(T_\varphi) = 1$.
So $\psi$ is a cyclic permutation without fixed points.

Without loss of generality, assume $\psi(i) = i+1$ for $i = 1,...,k-1$ and $\psi(k) = 1$. Consider the following diagram

$$\begin{tikzcd} C(L) \arrow[r, "T_\psi"] \arrow[d, hook, "J"'] & C(L) \arrow[d, hook, "J"] \\ C(K) \arrow[r, "T_\varphi"'] & C(K) \end{tikzcd}$$

which is commutative since $J$ is an embedding and $T_\psi = T_\varphi\big|_{C(L)}$.
By [EFHN15, Lemma 4.14, Cor. 4.15] this implies that also

$$\begin{tikzcd} L \arrow[r, "\psi"] & L \\ K \arrow[u, two heads, "\Phi"] \arrow[r, "\varphi"'] & K \arrow[u, two heads, "\Phi"'] \end{tikzcd}$$

is commutative. From $\Phi \circ \varphi = \psi \circ \Phi$, it follows that

$$\Phi(\varphi(K_i)) = \psi(\Phi(K_i)) = \psi(\{i\}) = \begin{cases} i+1 & \text{for } i = 1,...,k-1 \\ 1 & \text{for } i = k \end{cases}$$

and therefore

$$\varphi(K_i) \subseteq \Phi^{-1}(\{i+1\}) = K_{i+1}$$

for every $i = 1,...,k-1$ and $\varphi(K_k) \subseteq K_1$. □

We give two examples illustrating this assertion.

**Example 3.2.5.** Let $K_1,...,K_k$ be $k$ copies of $\{z, \zeta_{k+1}z, ..., \zeta_{k+1}^k z\}$ for some $z \in \mathbb{T}$, $k \geq 1$, and $K := \dot{\bigcup}_{i=1}^k K_i$. Define

$$\varphi : K \to K,\ z_i \mapsto \zeta_k z_{i+1},$$

where $z_{i+1} \in K_{i+1 \bmod k}$ is a copy of $z_i \in K_i$.
Then $\{1, \zeta_k, ..., \zeta_k^{k-1}\} \subseteq P\sigma(T_\varphi)$, because for $f \in C(K)$ with $f\big|_{K_i} = \zeta_k^{(i-1)\cdot l} \cdot \mathbb{1}$,

$$T_\varphi f(\zeta_{k+1}^j z_i) = f\left(\zeta_{k+1}^{j+1} z_{i+1}\right) = \zeta_k^{il} \cdot \mathbb{1} = \zeta_k f\left(\zeta_{k+1}^j z_i\right)$$

for all $i = 1, ..., k$ and $j = 1, ..., k+1$ and $f \neq 0$. Thus, $f$ is an eigenfunction corresponding to $\zeta_k^l$.
Now we show that $(K;\varphi)$ is transitive, i.e. there is some $x \in K$ such that $\mathrm{orb}(x) = K$. Choose $x = z_1$. Then for any $\zeta_{k+1}^j z_i \in K$ for $n := i - 1 + (i-1-j)k$,

$$\begin{aligned}
\varphi^n(z_1) &= \zeta_{k+1}^n z_{n-1 \mod k} \\
&= \zeta_{k+1}^{i-1+(i-1-j)k} z_{i+(i-1-j)k \mod k} \\
&= \zeta_{k+1}^{(k+1)(i-1-j)-(i-1-j)+i-1} z_i \\
&= \zeta_{k+1}^j z_i,
\end{aligned}$$

hence $\zeta_{k+1}^j z_i \in \mathrm{orb}(z_1)$ for all $j = 0, ..., k$, $i = 1, ..., k$. Therefore, $(K;\varphi)$ is transitive which implies $\dim \mathrm{Fix}(T_\varphi) = 1$.
Hence the conditions for Theorem 3.2.4 are fulfilled. Indeed, $K = \dot\bigcup_{i=1}^k K_i$ is a union of disjoint open and closed sets and also $\varphi(K_1) \subseteq K_2, ..., \varphi(K_k) \subseteq K_1$.

**Example 3.2.6.** Let $K_1, ..., K_k$ be $k$ copies of $\mathbb{T}$, $K := \dot\bigcup_{i=1}^k K_i$ and

$$\varphi : K \to K,\ z_i \mapsto \alpha z_{i+1}$$

an irrational torus rotation for $z_i \in K_{i \mod k}$, $i = 1, ..., k$. Clearly, there are no non-trivial $\varphi$-invariant sets, so $(K;\varphi)$ is minimal. This implies $\dim \mathrm{Fix}(T_\varphi) = 1$. Moreover, $\{1, \zeta_k, ..., \zeta_k^{k-1}\} \subseteq P\sigma(T_\varphi)$ since for $f \in C(K)$ with $f\big|_{\mathbb{T}_i} \equiv \zeta_k^{il}$,

$$T_\varphi f(z_i) = f(\varphi(z_i)) = f(\alpha z_{i+1}) = \zeta_k^{(i+1)l} = \zeta_k^l \zeta_k^{il} = \zeta_k^l f(z_i),$$

for every $z_i \in \mathbb{T}_i$, $i = 1, ..., k$.
Hence $(K;\varphi)$ satisfies the assumptions of Theorem 3.2.4. As we see in the definition, $K$ is a disjoint union of open and closed sets and $\varphi(K_i) \subseteq K_{i+1 \mod k}$.

In the following assertion we show that for a finite dimensional fixed space a similar partitioning can be obtained. In general, the periodic property of $\varphi$ gets lost.

**Theorem 3.2.7.** *Let $(K,\varphi)$ be a dynamical system with corresponding Koopman system $(C(K), T_\varphi)$. Let $\{1, \zeta_k, \zeta_k^2, ..., \zeta_k^{k-1}\} \subseteq P\sigma(T_\varphi)$ and $\dim \mathrm{Fix}(T_\varphi) < \infty$.*
*Then $K$ is a union of $d$ pairwise disjoint, open and closed sets $K_1, ..., K_d$, where $d := \sum_{l=1}^k d_l$ with $d_l := \dim \mathrm{Eig}(\zeta_k^l, T_\varphi)$ for every $l = 1, ..., k$.*

*Proof.* As $\dim \mathrm{Fix}(T_\varphi) < \infty$, also $\dim \mathrm{Eig}(\zeta_k^l, T_\varphi) < \infty$ for every $l = 1, ..., k$ (see Lemma 3.2.3). Consider the subspace

$$\mathfrak{A} := \mathrm{lin} \bigcup_{i=1}^k \mathrm{Eig}(\zeta_k^i, T_\varphi)$$

which is a $C^*$-algebra.
Indeed, for $\{f_1, ..., f_d\}$ basis of $\mathfrak{A}$, where $f_i \in \operatorname{Eig}(\zeta_k^{l_i})$ for $i = 1, ...d$, for any $g := \sum_{i=1}^d \alpha_i f_i$ and $h := \sum_{j=1}^d \beta_j f_j \in \mathfrak{A}$ also $g \cdot h = \sum_{i=1}^d \sum_{j=1}^d \alpha_i \beta_j f_i f_j \in \mathfrak{A}$, since $f_i \cdot f_j \in \operatorname{Eig}(\zeta_k^{l_i+l_j}, T_\varphi)$. Moreover, $\bar{g} = \sum_{i=1}^d \bar{\alpha_i} \bar{f_i} \in \mathfrak{A}$ since $\bar{f_i}$ is an eigenfunction associated with $\overline{\zeta_k^{l_i}} \in P\sigma(T_\varphi)$. Since $\mathbb{1} \in \operatorname{Eig}(1, T_\varphi)$, $\mathbb{1} \in \mathfrak{A}$.
Continue analogously as in the proof of Theorem 3.2.4 with $k = d$.

□

That the periodic behavior of $\varphi$ is in general not preserved if $\dim \operatorname{Fix}(T_\varphi) \neq 1$ is pointed out by the next two examples.

**Example 3.2.8.** Consider a space $K$ consisting of elements, say, $a, b, c$ and $\varphi : K \to K$ with

$$\varphi(a) := b,\ \varphi(b) := a,\ \varphi(c) := c.$$

Then $C(K) \cong \mathbb{C}^3$ and

$$T_\varphi : \mathbb{C}^3 \to \mathbb{C}^3,\ (x, y, z) \mapsto (y, x, z)\,.$$

We have $\{1, -1\} \subseteq P\sigma(T_\varphi)$ with

$$\operatorname{Eig}(-1, T_\varphi) = \operatorname{lin}\{(-1, 1, 0)\}$$

and

$$\operatorname{Fix}(T_\varphi) = \operatorname{lin}\{(0, 0, 1)\,, (1, 1, 1)\},$$

hence $\dim \operatorname{Fix}(T_\varphi) = 2$.
Since $K$ is discrete, there is a disjoint splitting of K in three open and closed sets, but since $z$ is a fixed point, $\varphi$ is not minimal, hence a periodic behavior of $\varphi$ on the components is not possible.

**Example 3.2.9.** Let $K_1, ..., K_k$ be $k$ copies of $\mathbb{T}$ and $K := \dot{\bigcup}_{i=1}^k K_i$.
Define for $\alpha \in \mathbb{T}$ with $\alpha^n \neq 1$ for $n = 1, 2, ..$ and $z_i \in K_i$, $i = 1, .., k$,

$$\varphi : K \to K,\ z_i \mapsto \begin{cases} \alpha z_{i+1 \mod (k-1)} & \text{for } i = 1, ..., k-1 \\ \alpha z_i & \text{for } i = k \end{cases}.$$

Then $\{1, \zeta_k, ..., \zeta_k^{k-1}\} \subseteq P\sigma(T_\varphi)$ since for $f \in C(K)$ with $f\big|_{K_i} \equiv \zeta_k^{il}$, $i \neq k$, and $f\big|_{K_k} = 0$ we have $f \in \operatorname{Eig}(\zeta_k^l, T_\varphi)$.

The following example shows that the condition $\dim \mathrm{Eig}(\zeta_k^l) < \infty$ for $l = 1, ..., k$ in Theorem 3.2.7 is necessary.

**Example 3.2.10.** Consider the rotation system $(\mathbb{T}; \zeta_k)$,

$$\varphi : \mathbb{T} \to \mathbb{T},\ z \mapsto \zeta_k z$$

and corresponding Koopman system $(C(\mathbb{T}); T_\varphi)$ with $T_\varphi f(z) = f(\zeta_k z)$ for every $z \in \mathbb{T}$. The fixed space is $\mathrm{Fix}(T_\varphi) = \{f \in C(\mathbb{T}) : f \text{ is } \zeta_k\text{-periodic}\}$, thus $\dim \mathrm{Fix}(T_\varphi) = \infty$. Obviously, there is no non-trivial decomposition of $\mathbb{T}$ in disjoint open and closed sets.

By a similar example we show that Theorem 3.2.7 is in general only valid for a subset of the form $\{1, \zeta_k, \zeta_k^2, ..., \zeta_k^{k-1}\} \subseteq P\sigma(T_\varphi)$.

**Example 3.2.11.** Let $(\mathbb{T}; \alpha)$ be an irrational torus rotation, i.e., $\alpha \in \mathbb{T}$ is not a root of unity. Then $\{1, \alpha, \alpha^2, ..., \alpha^k\} \subseteq P\sigma(T_\varphi)$ for all $k \geq 0$. We have $\mathrm{Eig}(\alpha^l, T_\varphi) = \mathrm{lin}\{p_l\}$ for the polynomials $p_l : \mathbb{T} \to \mathbb{T}$, $p_l(z) = z^l$ for $l = 0, ..., k$. Thus, all eigenspaces are one-dimensional, but there is no such decomposition of $\mathbb{T}$ as in Theorem 3.2.7.

# 4 The Jacobs-de-Leeuw-Glicksberg splitting of a Koopman operator

In Chapter 1 we pointed out that the advantage of working with the Koopman system is that we can use the well-developed linear theory. In this chapter we apply an operator theoretic result – the Jacobs-de-Leeuw-Glicksberg splitting – to the Koopman system and draw conclusions on the underlying dynamics. Before doing so we give the general theory.

**Definition 4.0.1.** Let $T \in \mathcal{L}(X)$ for a Banach space $(X, \|\cdot\|)$. Then there exists a *Jacobs-de-Leeuw-Glicksberg splitting* if

$$X = X_s \oplus X_r$$

with *stable part*

$$X_s := \{x \in X : \exists\, (n_i)_{i \in \mathbb{N}} \subseteq \mathbb{N} \text{ with density}^1 1 \text{ such that } T^{n_i} x \overset{i\to\infty}{\longrightarrow} 0 \text{ weakly}\}$$

and *reversible part*

$$X_r := \overline{\operatorname{lin}}\{x \in X : \exists\, \lambda \in \mathbb{T} \text{ such that } Tx = \lambda x\}.$$

**Remark 4.0.2.** Clearly, $X_s$ and $X_r$ are $T$-invariant.

The question now is, when the Jacobs-de-Leeuw-Glicksberg decomposition exists.

**Theorem 4.0.3.** *Let $T \in \mathcal{L}(X)$ for a Banach space $(X, \|\cdot\|)$. If $T$ has relatively weakly compact orbits, then there exists a projection $Q \in \mathcal{L}(X)$ which commutes with $T$ and induces the Jacobs-de-Leeuw-Glicksberg decomposition with*

$$X_s = \ker(Q)$$

*and*

$$X_r = \operatorname{Im}(Q).$$

[1] The *density* $d(M)$ *of a set* $M \subseteq \mathbb{N}$ is defined as $d(M) := \lim_{n\to\infty} \frac{|M \cap [1,n]|}{n}$ if the limit exists.

*Proof.* We refer to [Eis10, Thm. I.1.11] and [Eis10, Thm. II.4.1] for the proof. □

**Notation 4.0.4.** If the Jacobs-de-Leeuw-Glicksberg splitting exists for a Banach space $(X, \|\cdot\|)$, then for $x \in X$ we denote $x = x_s + x_r$ where $x_s \in X_s$ and $x_r \in X_r$.

**Remark 4.0.5.** The splitting gives insight into the asymptotic behavior of the powers $T^n$ of the operator $T$. The terminology *reversible* part for $X_r$ comes from the identity

$$X_r = \{x \in X : y \in \overline{\{T^n x : n \in \mathbb{N}\}}^{\sigma} \Rightarrow x \in \overline{\{T^n y : n \in \mathbb{N}\}}^{\sigma}\}$$

(cf. [EFHN15, Thm. 16.24]) in case that $T$ has relatively weakly compact orbits, which describes a way of returning to each point. Clearly, $X_s$ is called the *stable* part since „$T^n x \overset{n\to\infty}{\longrightarrow} 0$" in some sence for $x \in X_s$.

We now give some examples for which a Jacobs-de-Leeuw-Glicksberg splitting exists.

**Example 4.0.6.** Let $T \in \mathcal{L}(X)$ for a Banach space $(X, \|\cdot\|)$. If $T$ has relatively strongly compact orbits, then

$$X = X_s \oplus X_r$$

where

$$X_s = \{x \in X : \|T^n x\| \overset{n\to\infty}{\longrightarrow} 0\}$$

and

$$X_r = \overline{\operatorname{lin}}\{x \in X : \exists\, \lambda \in \mathbb{T} \text{ such that } Tx = \lambda x\}.$$

*Proof.* We follow [Eis10, Thm. I.1.16 and II.2.4]. Clearly, if $T$ has relatively strongly compact orbits, it has relatively weakly compact orbits and hence $T$ has a Jacobs-de-Leeuw-Glicksberg splitting.
Assume that $\sup_{n\in\mathbb{N}_0} \|T^n\| = \infty$. By the principle of uniform boundedness there exists some $x \in X$ such that $\sup_{n\in\mathbb{N}_0} \|T^n x\| = \infty$ which implies that for every $m \in \mathbb{N}_0$ there is some $j_m \in \mathbb{N}_0$ such that $\|T^{j_m} x\| > m$. Then $(T^{j_m} x)_{m\in\mathbb{N}_0}$ has no convergent subsequence which contradicts the relatively strong compactness of $\{T^n x : n \in \mathbb{N}_0\}$. Therefore $M := \sup_{n\in\mathbb{N}} \|T^n\| < \infty$.
To show that $X_s = \{x \in X : \|T^n x\| \overset{n\to\infty}{\longrightarrow} 0\}$, we first note that if $\lim_{i\to\infty} \|T^{n_i} x\| = 0$ for a subsequence $(n_i)_{i\in\mathbb{N}}$ of $\mathbb{N}$, then $\lim_{n\to\infty} \|T^n x\| = 0$, since for any $\varepsilon > 0$ and $k > 0$ such that $\|T^{n_k} x\| \le \varepsilon$, we have for $n \ge n_k$ that

$$\|T^n x\| \le \|T^{n-n_k}\| \|T^{n_k} x\| \le M\varepsilon.$$

Now take $x \in X_s$. Then there is a subsequence $(n_i)_{i\in\mathbb{N}}$ of $\mathbb{N}$ with density 1 such that $T^{n_i} x \overset{i\to\infty}{\longrightarrow} 0$ weakly. The relatively strong compactness of the orbit $\{T^n x : n \in \mathbb{N}_0\}$ implies $\lim_{k\to\infty} \|T^{n_{i_k}} x\| = 0$ for a subsequence $(n_{i_k})_{k\in\mathbb{N}}$ of $(n_i)_{i\in\mathbb{N}}$. Hence $\lim_{n\to\infty} \|T^n x\| = 0$.
Conversely, if $T^n x$ converges strongly to 0, it clearly converges weakly to 0. □

In fact, for a power bounded operator we have equivalence in Example 4.0.6. For a not power bounded operator this is in general not equivalent. A counterexample can be found in [EN00, Ex. V.2.12].

**Remark 4.0.7.** If $T \in \mathcal{L}(X)$ is power bounded, then the existence of the Jacobs-de-Leeuw-Glicksberg splitting

$$X = X_s \oplus X_r$$

in the strong sense (see Example 4.0.6) implies that $T$ has relatively strongly compact orbits.

*Proof.* It is sufficient to show that the orbits of a dense subset of $X$ are relatively strongly compact (see e.g. [EFHN15, Ex. 16.10]).
Clearly, the orbit of any $x \in X_s$ is relatively strongly compact, thus it remains to show the assertion for any $x \in \operatorname{lin}\{x \in X : \exists\, \lambda \in \mathbb{T} \text{ such that } Tx = \lambda x\}$.
Take $x \in \operatorname{lin}\{x \in X : \text{there exists some } \lambda \in \mathbb{T} \text{ such that } Tx = \lambda x\}$ and $(n_i)_{i\in\mathbb{N}_0} \subseteq \mathbb{N}_0$. If $Tx = \lambda x$ for some $\lambda \in \mathbb{T}$, then $(T^{n_i}x)_{i\in\mathbb{N}_0} = (\lambda^{n_i}x)_{i\in\mathbb{N}_0}$ has clearly a convergent subsequence, since for $\lambda = \zeta_k$, where $\zeta_k$ is a root of unity, it takes a finite number of values and for $\lambda$ not a root of unity, the set of the powers of $\lambda$ is dense in $\mathbb{T}$. By a diagonal argument we obtain that also an arbitrary sequence $(y_i)_{i\in\mathbb{N}_0} \subseteq \overline{\{T^n x : n \in \mathbb{N}_0\}}$ has a strong cluster point. By the preceding considerations we now see that the orbit of any $x \in \operatorname{lin}\{x \in X : \exists\, \lambda \in \mathbb{T} \text{ such that } Tx = \lambda x\}$ is relatively strongly compact. □

**Remark 4.0.8.** Remark 4.0.7 reveals that for Koopman operators Example 4.0.6 is an equivalence statement.

**Example 4.0.9.** Let $X$ be a reflexive Banach space and $T \in \mathcal{L}(X)$ powerbounded. Then $T$ has relatively weakly compact orbits and hence $X = X_r \oplus X_s$.

*Proof.* Since $X$ is reflexive, it follows by the theorem of Banach Alaoglu that the unit ball $U$ in $X$ is weakly compact.
The boundedness of $(T^n)_{n\in\mathbb{N}_0}$ implies that $\|T^n x\| \le C_x$ for every $x \in X$ and a constant $C_x$. Hence $\{T^n x : n \in \mathbb{N}_0\} \subseteq C_x \cdot U$ and also $\overline{\{T^n x : n \in \mathbb{N}_0\}}^{\sigma} \subseteq C_x \cdot U$. The weak compactness of $U$ implies that also $\overline{\{T^n x : n \in \mathbb{N}_0\}}^{\sigma}$ is weakly compact for every $x \in X$. Therefore $X = X_r \oplus X_s$ holds. □

**Remark 4.0.10.** Example 4.0.9 implies that for powerbounded dynamics on finite dimensional Banach spaces the Jacobs-de-Leeuw-Glicksberg splitting is possible.

A consequence of relatively weakly compact orbits is the following.

**Proposition 4.0.11.** *Let $X$ be a Banach space and $T \in \mathcal{L}(X)$ with relatively weakly compact orbits. Then $T$ is mean ergodic.*

*Proof.* If $\mathcal{S}x := \overline{\{T^n x : n \in \mathbb{N}_0\}}$ is weakly compact, then by the Kreĭn-Šmulian theorem also the closed convex hull $\overline{\operatorname{co}}\, \mathcal{S}x$ is weakly compact.
This implies for $A_n x := \frac{1}{n}\sum_{i=0}^{n-1} T^i x \in \overline{\operatorname{co}}\, \mathcal{S}x$ that $\{A_n x : n \in \mathbb{N}_0\}$ has a weak cluster point for every $x \in X$. Since $T$ has relatively weakly compact orbits, it is powerbounded (see [Eis10, Lemma I.1.6]). It follows that $T$ is mean ergodic (see [EFHN15, Thm. 8.20]). □

## 4.1 The Jacobs-de-Leeuw-Glicksberg splitting of a Koopman operator

We now turn to the Jacobs-de-Leeuw-Glicksberg decomposition in case of Koopman operators. Example 4.0.9 gives the following result for measure-preserving dynamical systems (see Remark 2.1.2).

**Proposition 4.1.1.** *Let* $(\Omega, \Sigma, \mu; \varphi)$ *be a measure-preserving dynamical system with corresponding Koopman system* $(L^p[\Omega, \mu]; T_\varphi)$ *for* $1 \le p < \infty$. *Then*

$$L^p[\Omega, \mu] = L^p[\Omega, \mu]_r \oplus L^p[\Omega, \mu]_s.$$

*Proof.* Since $L^p[\Omega, \mu]$ is reflexive for $1 < p < \infty$, we only have to consider the case $p = 1$. In this case the Jacobs-de-Leeuw-Glicksberg spitting follows because $T_\varphi$ is a Dunford-Schwarz operator on $L^1[\Omega, \mu]$ and $\mu$ is a finite measure. We refer to [EFHN15, Thm. 8.24.] for further details. □

Is there a similar result for Koopman systems with observable space $C(K)$? The following shows that this case is more complex since it depends on the dynamics whether the orbits of $T_\varphi$ are relatively weakly compact.

**Proposition 4.1.2.** *Let* $(\mathrm{K}; \varphi)$ *be a dynamical system with corresponding Koopman system* $(C(K); T_\varphi)$. *If the Jacobs-de-Leeuw-Glicksberg splitting exists, then* $T_\varphi$ *is mean ergodic.*

*Proof.* We first show the convergence of the Cesàro means $A_n$ on $X_s$.
Take $f \in X_s$. By definition there is a sequence $(n_i)_{i \in \mathbb{N}_0}$ with density 1 such that $\langle T^{n_i} f, \mu \rangle \xrightarrow{i \to \infty} 0$ for every $\mu \in C(K)'$. Then

$$\langle A_n f, \mu \rangle = \langle \frac{1}{n} \sum_{i=0}^{n-1} T_\varphi{}^i f, \mu \rangle = \frac{1}{n} \sum_{i=0}^{n-1} \langle T_\varphi{}^i f, \mu \rangle \xrightarrow{i \to \infty} 0$$

by the Koopman-von Neumann lemma (see e.g. [Eis10, Lemma II.4.2 S.61]), hence $T_\varphi$ is weakly mean ergodic on $X_s$. By [EFHN15, Thm. 8.20] $T_\varphi$ then is also strongly mean ergodic on $X_s$.
Now let $f \in X_r$ such that $T_\varphi f = \lambda f$ for some $\lambda \in \mathbb{T}$. Analogously to Example 2.2.21, we obtain if $\lambda$ is a $k^{th}$ root of unity that

$$A_n f \xrightarrow[n \to \infty]{\|\cdot\|} \tfrac{1}{k} \left( \sum_{i=0}^{k-1} T_\varphi{}^i f \right).$$

If $\lambda$ is not a root of unity it follows that

$$\|A_n f\| = \sup_{s\in K} |\tfrac{1}{n}\sum_{i=0}^{n-1} {T_\varphi}^i f(s))| = \sup_{s\in K} |\tfrac{1}{n}\sum_{i=0}^{n-1} \lambda^i f(s)| = \sup_{s\in K} |\tfrac{1}{n}\frac{1-\lambda^n}{1-\lambda} f(s)| \overset{n\to\infty}{\longrightarrow} 0.$$

Clearly, this implies that also $A_n g$ is convergent for $g \in \mathrm{lin}\{f \in X :$ there exists some $\lambda \in \mathbb{C}$, $|\lambda| = 1$, such that $Tf = \lambda f\}$. Since all $A_n$ are contractive, they are also convergent on the closure, thus on $X_r$ (see [Sch66, III.4.5]). □

**Example 4.1.3.** Proposition 4.1.2 reveals that there is no Jacobs-de-Leeuw-Glicksberg decomposition of a Koopman system $(C(K); T_\varphi)$ if $T_\varphi$ is not mean ergodic.
As an example we choose the dynamics

$$\varphi : [0,1] \to [0,1],\ x \mapsto x^2$$

and show that $T_\varphi$ is not mean ergodic on $C([0,1])$. By [EFHN15, Thm. 8.20] it is sufficient to show that $\mathrm{Fix}(T_\varphi)$ does not separate $\mathrm{Fix}({T_\varphi}')$. Clearly, $\mathrm{Fix}(T_\varphi) = \langle \mathbb{1} \rangle$ (cf. Example 2.3.17).
Now consider the Dirac measures $\delta_0$ and $\delta_1 \in C([0,1])'$. Then $\delta_0, \delta_1 \in \mathrm{Fix}({T_\varphi}')$ since

$${T_\varphi}' \delta_0(f) = \delta_0(T_\varphi f) = T_\varphi f(0) = f(0) = \delta_0(f)$$

and analogously

$${T_\varphi}' \delta_1(f) = \delta_1(f)$$

for every $f \in C([0,1])$. Clearly, $\delta_0 \neq \delta_1$ and for any $f = c \cdot \mathbb{1} \in \mathrm{Fix}(T_\varphi)$ we have $\delta_0(f) = c = \delta_1(f)$, hence $\mathrm{Fix}(T_\varphi)$ does not separate $\mathrm{Fix}({T_\varphi}')$.

It is still open how to characterize weakly compact orbits of $T_\varphi$ by means of the underlying dynamics whereas the case of strongly compact orbits of $T_\varphi$ is well understood: We recall from Lemma 2.3.22 that for a topological dynamical system $(\mathrm{K}; \varphi)$ with corresponding Koopman system $(C(K), T_\varphi)$, the orbits of $T_\varphi$ are relatively strongly compact if and only if $\{\varphi^n : n \in \mathbb{N}_0\}$ is equicontinuous.
This immediately gives the following.

**Proposition 4.1.4.** *Let* $(\mathrm{K}; \varphi)$ *be a topological dynamical system and* $(C(K); T_\varphi)$ *the corresponding Koopman system. If* $\{\varphi^n : n \in \mathbb{N}_0\}$ *is equicontinuous, then the Jacobs-de-Leeuw-Glicksberg splitting is possible in the strong sense (confer Example 4.0.6).*

We give an example which shows that relatively weakly compact orbits and relatively strongly compact orbits do not coincide on $C(K)$.

**Example 4.1.5.** Consider the one-point compactification $K := \mathbb{Z} \cup \{\infty\}$ with dynamics

$$\varphi : K \to K,\, x \mapsto \begin{cases} x+1 & \text{if } x \in \mathbb{Z} \\ \infty & \text{if } x = \infty \end{cases}$$

and corresponding Koopman system $(C(K); T_\varphi)$ where

$$C(K) = \{(x_z)_{z\in\mathbb{Z}} \in \ell^\infty : \lim_{z\to\infty} x_{-z} = \lim_{z\to\infty} x_z\}$$

and

$$T_\varphi : C(K) \to C(K),\, (x_z)_{z\in\mathbb{Z}} \mapsto (x_{z+1})_{z\in\mathbb{Z}}.$$

Then $T_\varphi$ has relatively weakly compact orbits by [DS58, Cor. IV.6.4], since clearly for any $x \in C(K)$ the orbit $\{{T_\varphi}^n x : n \in \mathbb{N}_0\}$ is bounded and ${T_\varphi}^n x = (x_{z+n})_{z\in\mathbb{Z}}$ is pointwise convergent, i.e., $(x_{z+n})_{n\in\mathbb{N}_0}$ converges for $n \to \infty$.

Not all orbits of $T_\varphi$ are relatively strongly compact.
Take e.g. $x := (...,0,0,1,0,0,...) \in C(K)$. Then for any $n, m \in \mathbb{N}_0$ we have

$$\| {T_\varphi}^n x - {T_\varphi}^m x\| = \sup_{z\in\mathbb{Z}} |x_{z+n} - x_{z+m}| = 1$$

for $n \neq m$, thus there is no subsequence $(n_i)_{i\in\mathbb{N}_0}$ of $\mathbb{N}_0$ such that $(T^{n_i}x)_{i\in\mathbb{N}_0}$ is convergent in $\|\cdot\|$. Hence $\{{T_\varphi}^n x : n \in \mathbb{N}_0\}$ is not relatively strongly compact.

We summarize the previous considerations to the following.

**Remark 4.1.6.** Let $(\mathrm{K};\varphi)$ be a dynamical system with corresponding Koopman system $(C(K); T_\varphi)$. Consider the following conditions.

(i) $T_\varphi$ has relatively strongly compact orbits.

(ii) $T_\varphi$ has relatively weakly compact orbits.

(iii) The Jacobs-de-Leeuw-Glicksberg decomposition exists.

(iv) $T_\varphi$ is mean ergodic.

Clearly, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) but, presumably, the converse implications are false. Example 4.1.5 shows that (ii) does not imply (i). It would be interesting to find counterexamples for the other implications.

We now turn to the Jacobs-de-Leeuw-Glicksberg splitting in case of affine dynamical systems considered in Section 3.1.

**Example 4.1.7.** Consider $X := (\mathbb{C}^k, \|\cdot\|)$ for some norm $\|\cdot\|$ and a contractive matrix $A \in \mathcal{L}(X)$. Then the set of iterates of the affine dynamical system $\left(U; A\big|_U\right)$ is equicontinuous since for any $x \in U$, $\varepsilon > 0$, $y \in \mathbb{C}^k$ with $\|x - y\| < \delta := \varepsilon$, we have

$$\|A^n x - A^n y\| \leq \|A^n\| \|x - y\| \leq \delta = \varepsilon.$$

By Proposition 4.1.4 there is a Jacobs-de-Leeuw-Glicksberg decomposition of $(C(U); T_A)$. Clearly, $A$ has relatively compact orbits, thus there is a Jacobs-de-Leeuw-Glicksberg decomposition also for $(X; A)$.
An open question is how these decompositions are related to each other. We give some remarks.

**Remark 4.1.8.** (i) Let $\|A\| = 1$, $A^n \neq 0$ for all $n \in \mathbb{N}$ and $(A^n)_{n \in \mathbb{N}}$ convergent, i.e., there is $P \in \mathbb{C}^{k \times k}$ such that $\|A^n - P\| \overset{n\to\infty}{\longrightarrow} 0$. Then $\sigma(A) \cap \mathbb{T} = 1$ and there is a Jacobs-de-Leeuw-Glicksberg splitting $\mathbb{C}^k = X_s^A \oplus X_r^A$.
Clearly, $X_s^A = \ker(P)$ which implies $X_r^A = \operatorname{Im}(P) = \operatorname{Fix}(A)$ since $P$ is a projection onto $\operatorname{Fix}(A)$.
For the Jacobs-de-Leeuw-Glicksberg splitting $C(U) = X_s^{T_A} \oplus X_r^{T_A}$ we obtain

$$\begin{aligned} X_s^{T_A} &= \{f \in C(U) : f \circ A\big|_U^n \xrightarrow[n\to\infty]{\|\cdot\|} 0\} \\ &= \{f \in C(U) : f \circ P\big|_U \equiv 0\} \\ &= I_{\operatorname{Im}(P|_U)} \end{aligned}$$

where $I_{\operatorname{Im}(P|_U)}$ denotes the closed ideal of functions which vanish on $\operatorname{Im}(P\big|_U)$.

(ii) Let $\|A\| = 1$ and $A$ periodic, i.e., there is $m \in \mathbb{N}$ such that $A^m = A$. Moreover, for $\lambda \in \sigma(A)$, either $\lambda^{m-1} = 1$ or $\lambda = 0$.
Then there is a Jacobs-de-Leeuw-Glicksberg splitting $\mathbb{C}^k = X_s^A \oplus X_r^A$.
Clearly, $X_s^A = \ker(A)$ which implies $X_r^A = \operatorname{Im}(A)$. Since for $f \in C(U)$ we have $f \circ A\big|_U^n = f \circ A\big|_U^{n \bmod m} \xrightarrow{\|\cdot\|} 0$ for $n \to \infty$ if and only if $f \circ A\big|_U^n \equiv 0$ for all $n = 1, \ldots, m-1$, we obtain

$$\begin{aligned} X_s^{T_A} &= \{f \in C(U) : f \circ A\big|_U \equiv 0\} \\ &= I_{\operatorname{Im}(A|_U)}. \end{aligned}$$

Which conclusions can we draw from the Jacobs-de-Leeuw-Glicksberg splitting? The following answers Theorem 3.1.14 (ii) which we left open in Chapter 3.

**Example 4.1.9.** Let $X := (\mathbb{C}^k, \|\cdot\|)$ for some norm $\|\cdot\|$, $A \in \mathcal{L}(X)$ such that $\|A\| < 1$ and $(C(U); T_A)$ the Koopman system of the affine dynamical system $\big(U; A\big|_U\big)$.
By Example 4.1.7 there is a Jacobs-de-Leeuw-Glicksberg decomposition of $(C(U); T_A)$. Since $A^n \xrightarrow{\|\cdot\|} 0$ for $n \to \infty$, we obtain

$$C(U)_s = \{f \in C(U) : T_A^n f \xrightarrow{\|\cdot\|} 0 \text{ for } n \to \infty\} = \{f \in C(U) : f(0) = 0\} = I_{\{0\}}.$$

This implies

$$C(U)_r = \{f \in C(U) : \text{ there is } \lambda \in \mathbb{T} \text{ such that } T_A f = \lambda f\} = \langle \mathbb{1}_U \rangle = \mathrm{Fix}(T_A)$$

and therefore $P\sigma(T_A) = D^\circ \cup \{1\}$.

Further investigation should be done on how the Jacobs-de-Leeuw-Glicksberg decomposition describes the longterm behavior of the Koopman operator and of the dynamics. We give one observation which follows directly from Example 4.0.6

**Proposition 4.1.10.** *Let* $(\mathrm{K}; \varphi)$ *be a topological dynamical system with corresponding Koopman system* $(C(K); T_\varphi)$ *where* $T_\varphi$ *has relatively strongly compact orbits. Then* $\| {T_\varphi}^n f - {T_\varphi}^n f_r\| \longrightarrow 0$ *for* $n \longrightarrow \infty$.

## 4.2 Conclusions from the reversible part in case of a Koopman operator

Even for a Koopman system that does not allow the Jacobs-de-Leeuw-Glicksberg splitting we can define the reversible part $X_r$ and draw conclusions on the structure of the underlying dynamical system.

**Proposition 4.2.1.** *Let* $(\mathrm{K};\varphi)$ *be a topological dynamical system with corresponding Koopman system* $(C(K);T_\varphi)$*. Define*

$$X_r := \overline{\mathrm{lin}}\{f \in X : \exists\,\lambda \in \mathbb{C},\ |\lambda| = 1,\ \textit{such that } Tf = \lambda f\}.$$

*Then there is a compact space* $L$ *such that* $X_r \cong C(L)$*.*
*If* $(\mathrm{K};\varphi)$ *is minimal, then* $L = G$ *for a compact monothetic group* $G$ *and*

$$T_\varphi\Big|_{X_r} = T_\psi$$

*for a group rotation* $\psi : G \to G$, $\psi(g) = g_0 g$ *for some generating* $g_0 \in G$*.*

*Proof.* By the Gelfand-Naimark theorem (see Theorem 2.2.9) there is an isometric isomorphism $\Phi : C(L) \to X_r$ for some compact space $L$, hence $X_r \cong C(L)$.
To show the second assertion, define

$$T := \Phi^{-1} \circ T_\varphi \circ \Phi : C(L) \to C(L)$$

which is a $C^*$−algebra homomorphism and well-defined since $X_r$ is $T_\varphi$-invariant. Thus by Theorem 2.2.17 there is some continuous map $\psi : L \to L$ such that $T = T_\psi$.
We show that the system $(L;\psi)$ is minimal and $T_\psi$ has discrete spectrum, i.e.,

$$C(L) = \overline{\mathrm{lin}}\{f \in X : \exists\,\lambda \in \mathbb{C},\ |\lambda| = 1,\ \text{such that } T_\psi f = \lambda f\}$$

and apply the Halmos-von Neumann theorem (see e.g. [DNP87, Thm. VIII.2]) which gives the second assertion.
Clearly, $J := \mathrm{id}_{C(K)} \circ \Phi : C(L) \to C(K)$ is an embedding, hence by Theorem 2.2.17 there is some surjective continuous map $\theta : K \to L$ such that $J = T_\theta$. The diagram

$$\begin{tikzcd} C(L) \arrow[r, "T_\psi"] \arrow[d, hook, "T_\theta"'] & C(L) \arrow[d, hook, "T_\theta"] \\ C(K) \arrow[r, "T_\varphi"'] & C(K) \end{tikzcd}$$

is commutative, hence

$$\begin{tikzcd} L \arrow[r, "\psi"] & L \\ K \arrow[u, two heads, "\theta"] \arrow[r, "\varphi"'] & K \arrow[u, two heads, "\theta"'] \end{tikzcd}$$

is also commutative, thus $\psi \circ \theta = \theta \circ \varphi$.
Now let $M \subseteq L$ be closed and $\psi$-invariant, i.e., $\psi(M) \subseteq M$. Then $\theta(\varphi(\theta^{-1}(M))) \subseteq M$ and hence $\varphi(\theta^{-1}(M)) \subseteq \theta^{-1}(M)$. The minimality of $(\mathrm{K};\varphi)$ implies $\theta^{-1}(M) = K$ or $\theta^{-1}(M) = \varnothing$, thus $M = \theta(K) = L$ or $M = \theta(\varnothing) = \varnothing$ which shows the minimality of $(L;\psi)$.
Take $f \in C(L)$. Then there is some $g = \sum_{i=0}^{\infty} g_i \in X_r$ where $T_\varphi g_i = \lambda_i g_i$ for $\lambda_i \in \mathbb{T}$, $i \in \mathbb{N}_0$, such that $f = \Phi^{-1}(g) = \sum_{i=0}^{\infty} \underbrace{\Phi^{-1}(g_i)}_{:=f_i}$. Clearly, $T_\psi f = \sum_{i=0}^{\infty} \lambda_i f_i$ since $\Phi$ is an isomorphism.
This implies that $C(L)$ has discrete spectrum. □

**Remark 4.2.2.** Note that in general there is no projection $Q \in \mathcal{L}(X)$ such that $X_r = \mathrm{Im}(Q)$.

**Proposition 4.2.3.** *Let* $(\mathrm{K};\varphi)$ *be a topological dynamical system with corresponding Koopman system* $(C(K);T_\varphi)$. *Then there is a decomposition of the state space as*

$$K = \dot{\bigcup}_{s\in L} K_s,$$

*where* $K_s := \psi^{-1}(s)$ *for a surjective continuous mapping* $\psi : K \to L$ *and* $X_r \cong C(L)$. *Moreover, for all* $s \in L$ *exists some* $t_s \in L$ *such that*

$$\varphi(K_s) \subseteq K_{t_s}$$

*and* $K_{t_{s_1}} \neq K_{t_{s_2}}$ *for* $s_1 \neq s_2$.

*Proof.* We basically repeat the proof of Theorem 3.2.4. Since $X_r \subseteq C(K)$, there is an injective algebra homomorphism $J : C(L) \to C(K)$. By Theorem 2.2.17 there is a surjective continuous mapping $\psi : K \to L$ such that $J = T_\psi$. Clearly, $K = \dot{\bigcup}_{s\in L} K_s$ for $K_s := \psi^{-1}(s)$.
Consider the restriction $T_\varphi : X_r \to X_r$ which is an algebra isomorphism. Hence there exists a continuous mapping $\theta : L \to L$ such that $T_\varphi\big|_{X_r}$ can be represented as $T_\theta$. Since $T_\theta$ is an isomorphism, $\theta$ is a homeomorphism. Analogously as in Theorem 3.2.4, we obtain $\theta \circ \psi = \psi \circ \varphi$ and hence $\psi(\varphi(K_s)) = \theta(\psi(K_s)) = \theta(\{s\}) = \{t\}$ for $t := \theta(s)$. Therefore, $\varphi(K_s) \subseteq \psi^{-1}(t) = K_t$. From the injectivity of $\theta$ follows $K_{t_{s_1}} \neq K_{t_{s_2}}$ for $s_1 \neq s_2$. □

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# Zusammenfassung in deutscher Sprache

Diese Arbeit beschäftigt sich mit der Koopman-Linearisierung von dynamischen Systemen, die eine bislang weniger gebräuchliche Art der Linearisierung darstellt. Der Vorteil an dieser ist, dass sie im Gegensatz zu herkömmlichen Linearisierungen global ist und in gewisser Weise die vollständige Information über das zugrundeliegende dynamische System erhält.

Nach einer Einleitung und Motivation als Kapitel 1 wird in Kapitel 2 zunächst ein Überblick über topologisch dynamische Systeme und relevante Eigenschaften – beispielsweise Periodizität, Transitivität und Rekurrenz – von diesen gegeben. Anschließend werden Koopmansysteme bestehend aus dem Observablenraum $C(K)$ der stetigen Funktionen auf einem Kompaktum $K$ und dem zu einer Dynamik zugehörigen Koopmanoperator eingeführt. Nach Einführung des jeweiligen Isomorphiebegriffes von dynamischen Systemen und Koopmansystemen und der Feststellung, dass dynamische Systeme genau dann isomorph sind, wenn die zugehörigen Koopmansysteme isomorph sind, werden in einer Art „Wörterbuch“ die Entsprechungen von Eigenschaften eines dynamischen Systems im zugehörigen Koopmansystem und umgekehrt dargestellt.

Kapitel 3 beschäftigt sich mit spektralen Eigenschaften von Koopmanoperatoren. Nach einer generellen Übersicht über diese werden Koopmansysteme von affinen endlichdimensionalen Dynamiken betrachtet, die von einer $k \times k$-Matrix herrühren. Es wird untersucht, wie sich spektrale Eigenschaften dieser Matrizen in den zugehörigen Koopmansystemen widerspiegeln. Umgekehrt wird anschließend untersucht, wie man von spektralen Eigenschaften eines Koopmansystems auf die unterliegende Dynamik schließen kann, wobei Partitionen des Zustandsraumes erhalten werden.

Kapitel 4 greift als Anwendung eines Theorems aus der Operatorentheorie auf Koopmanoperatoren die Jacobs-de-Leeuw-Glicksberg-Zerlegung auf, die den Grundraum des Operators in einen stabilen und einen reversiblen Teil zerlegt. Zunächst erfolgt eine Darstellung der allgemeinen Theorie welche dann auf Koopmansysteme übertragen wird. Es werden Untersuchungen angestellt, wie sich die Existenz der Zerlegung mithilfe des dynamischen Systems charakterisieren lässt und welche Schlussfolgerungen man aus der Existenz der Jacobs-de-Leeuw-Glicksberg-Zerlegung ziehen kann. Auch hier werden wieder endlichdimensionale lineare Dynamiken und ihre zugehörigen Koopmansysteme betrachtet. Eine Reihe von offenen Fragen wird formuliert.