Applied Koopmanism

MFO mini-workshop organized by
Didier Henrion, Igor Mezić and Mihai Putinar

February 8-12, 2016

1 Workshop focus and goal

There has been a substantial surge of results utilizing operator-theoretic approach in study of dynamical and control systems [10, 5, 20, 8]. There is a need of integration of such efforts into a comprehensive theory. In this context, the workshop will focus on the interplay of ergodic theory, operator theory, geometric dynamical systems and convex optimization methods in pure and applied context to provide a pathway to a more efficient interchange between the fields and definition of potential coupling theorems.

There are two classes of operators that are of particular interest in the context of the workshop: Koopman-type operators [5] and Perron-Frobenius-type operators [10, 7]. Koopman (or composition) operator is a linear infinite-dimensional operator that can be defined for any nonlinear dynamical system. The linear operator retains the full information of the nonlinear state-space dynamics. The formalism based on Koopman operator representation holds promise for extension of dynamical systems methods to systems in high-dimensional spaces as well as hybrid systems, with a mix of smooth and discontinuous dynamics. Recently, Koopman operator properties have been intensely studied, and applications pursued in fields as diverse as fluid mechanics and power grid dynamics. Perron-Frobenius operator is also a linear operator, and, when defined in an appropriate function space, the adjoint to the Koopman operator. Physically, the Perron-Frobenius operator is useful in studying propagation of dynamical systems' densities. It has shown major promise for applications such as Lagrangian properties of fluid flows and control and optimization of dynamical systems. We now describe several specific topics of interest for discussion at the workshop.

One of the topics that indicates how merging of techniques from optimization and ergodic theory can be useful is the development of dedicated *convex optimization techniques* for the numerical study of dynamical systems. More specifically, we are interested in tailoring the moment-SOS hierarchies of semidefinite programming (SDP) – originally developed for polynomial optimization – to obtain relevant information on the support of invariant measures for dynamical systems with semialgebraic dynamics and constraints. Invariant measures have been studied extensively in dynamical systems theory [10] and Markov decision processes [9] and it is now recognized that key properties of a dynamical system can be assessed by considering only a few moments of a measure transported along the system flow [1]. The constructive proof of the ergodic partition theorem [17, 4] provides characterization of ergodic sets, which are the smallest invariant sets that ensure measurability of partition. Ergodic sets are the supports of ergodic measures, that can in turn be studied via their moments, or their Fourier coefficients in the periodic case. Here too, the key idea consists in observing the action of invariant measures on a countable number of observables, or test functions, see e.g. [5] or [16]. Even more recently, invariant measures and weak Kolmogorov-Arnold-Moser (KAM) theory have been used to study geometrical properties of the joint spectral radius (JSR) of a set of linear operators [19, 6].

Hierarchies of finite-dimensional convex optimization problems have been introduced in the early 2000s to solve numerically non-convex optimization problems with semialgebraic data, with convergence guarantees [11]. The overall startegy consists of building a family of semidefinite programming (SDP) problems [2] of increasing size, with primal SDP problems relaxing the original polynomial optimization problem in the space of truncated moments of a measure supported on the globally optimal solutions, and dual SDP problems certifying bounds on the global optimum with specific polynomial sum-of-squares (SOS) representations. In the context of polynomial optimization, this is called the moment-SOS hierarchy [13] or sometimes Lasserre's hierarchy [20], and this relies on fundamental results of convex algebraic geometry, see [21] or [3]. The approach has been extended in [12] to optimal control problems on ordinary differential equations (ODEs), and more recently, to construct families of semialgebraic approximations of the support of measures transported along the flow of controlled ODEs [8].

Another topic of interest is the discussion of the relationship between *geometric properties of dynamical systems and spectral properties of the associated operators*. In fact, the hallmark of the work on the operator-theoretic approach in the last two decades is the linkage between geometrical properties of dynamical systems - whose study has been advocated and strongly developed by Poincaré and followers - with the geometrical properties of the level sets of Koopman eigenfunctions [17, 14, 15]. The operator-theoretic approach has been shown capable of detecting objects of key importance in geometric study, such as invariant sets, but doing so globally, as opposed to locally as in the geometric approach. It also provides an opportunity for study of high-dimensional evolution equations in terms of dynamical systems concepts [18, 22] via a spectral decomposition, and links with associated numerical methods for such evolution equations [23].

We believe that the workshop is a timely and unique opportunity to bring together experts in operator theory, convex optimization, dynamical systems, and systems control, and let them focus on topics that combine their fields principally aimed at the *construction of tailored moment-SOS hierarchies for studying the geometry of the support of invariant measures arising in dynamical systems.*

2 Organizers

2.1 Didier Henrion

D. Henrion is a CNRS Researcher at LAAS in Toulouse, France. He is also a Professor at the Faculty of Electrical Engineering of the Czech Technical University in Prague, Czech Republic. He is interested in polynomial optimization for systems control, focusing on the development of constructive tools for addressing mathematical problems arising from systems control theory. See `homepages.laas.fr/henrion`

2.2 Igor Mezić

I. Mezić is a Professor at the Department of Mechanical Engineering of the University of California at Santa Barbara, USA. He is interested in reformulation of dynamical systems theory utilizing spectral properties of the Koopman operator and its relationship with geometrical properties of dynamical systems in high dimensions, and under uncertainty. See `www.engr.ucsb.edu/~mgroup/joomla`

2.3 Mihai Putinar

M. Putinar is a Professor at the Department of Mathematics of the University of California at Santa Barbara, USA, at the Nanyang Technological University, Singapore, and at Newcastle University, UK. He is an expert in operator theory, functional analysis and real algebraic geometry. See `mihaiputinar.com`

3 Confirmed participants

1. **Amir Ali Ahmadi**, Oper. Res. Financ. Engr., Princeton Univ., USA, systems control and optimization
2. **Marko Budišić**, Math., Univ. Wisconsin-Madison, USA, ergodic theory and dynamical systems
3. **Stéphane Gaubert**, Math., INRIA and Ecole Polytechnique, Paris, France, systems control and optimization
4. **Roxana Hess**, LAAS-CNRS, Univ. Toulouse, France, real algebraic geometry and optimization
5. **Oliver Junge**, Math., Tech. Univ. Munich, Germany, dynamical systems, control and optimization
6. **Raphaël Jungers**, Math., Univ. Catholique Louvain, Belgium, systems control and optimization
7. **Milan Korda**, Elec. Engr., EPFL, Switzerland, systems control and optimization
8. **Kari Valentina Küster**, Math., Univ. Tübingen, Germany, operator theory and ergodic theory
9. **Yuri Latushkin**, Math., Univ. Missouri, USA, dynamical systems and control theory
10. **Alexandre Mauroy**, Elec. Engr. Comput. Sci., Univ. Liège, Belgium, dynamical systems, control, and operator theory
11. **Ryan Mohr**, Mech. Eng., Univ. California Santa Barbara, USA, dynamical systems and operator theory

12. **Ian Morris**, Math., Univ. Surrey, UK, ergodic theory and dynamical systems
13. **Rainer Nagel**, Math., Univ. Tübingen, Germany, operator theory and ergodic theory
14. **Clancy Rowley**, Comp. App. Math. Princeton Univ., USA, dynamical systems and control theory

4 Mathematics subject classification

37M25 Dynamical systems and ergodic theory - Computational methods for ergodic theory (approximation of invariant measures, computation of Lyapunov exponents, entropy)

90C22 Operations research, mathematical programming - Semidefinite programming

93B28 Systems theory; control - Operator-theoretic methods

References

[1] D. Barkley, I. G. Kevrekidis, A. M. Stuart. The moment map: nonlinear dynamics of density evolution via a few moments. SIAM J. Appl. Dyn. Syst., 5(3):403-434, 2006.

[2] A. Ben-Tal, A. Nemirovski. Lectures on modern convex optimization. SIAM, Philadelphia, 2001.

[3] G. Blekherman, P. A. Parrilo, R. R. Thomas (Editors). Semidefinite optimization and convex algebraic geometry. SIAM, Philadelphia, 2013.

[4] M. Budišić. Ergodic quotients in analysis of dynamical systems. PhD thesis, Univ. California Santa Barbara, 2012.

[5] M. Budišić, R. Mohr, I. Mezić. Applied Koopmanism. Chaos 22:047510, 2012.

[6] V. Calvez, P. Gabriel, S. Gaubert. Non-linear eigenvalue problems arising from growth maximization of positive linear dynamical systems. `arXiv:1404.1868`, April 2014.

[7] M. Dellnitz, O. Junge. On the approximation of complicated dynamical behavior, SIAM J. Numer. Anal., 36 (2), 1999.

[8] D. Henrion, M. Korda. Convex computation of the region of attraction of polynomial control systems. IEEE Trans. Autom. Control, 59(2):297-312, 2014.

[9] O. Hernández-Lerma, J. B. Lasserre. Markov chains and invariant probabilities. Birkhäuser, Basel, 2003.

[10] A. Lasota, M. C. Mackey. Probabilistic properties of deterministic systems. Cambridge Univ. Press, Cambridge, UK, 1985.

[11] J. B. Lasserre. Global optimization with polynomials and the problem of moments. SIAM J. Optim. 11(3):796–817, 2001.

[12] J. B. Lasserre, D. Henrion, C. Prieur, E. Trélat. Nonlinear optimal control via occupation measures and LMI relaxations. SIAM J. Control Optim. 47(4):1643-1666, 2008.

[13] J. B. Lasserre. Moments, positive polynomials and their applications. Imperial College Press, London, UK, 2010.

[14] A. Mauroy, I. Mezić. On the use of Fourier averages to compute the global isochrons of (quasi)periodic dynamics, Chaos, 22(3):033112, 2012.

[15] A. Mauroy, I. Mezić, J. Moehlis. Isostables, isochrons, and Koopman spectrum for the action-angle reduction of stable fixed point dynamics, Physica D, 261:19-30, 2013.

[16] A. Mauroy, I. Mezić. Global stability analysis using the eigenfunctions of the Koopman operator `arXiv:1408.1379`, August 2014.

[17] I. Mezić. On the geometrical and statistical properties of dynamical systems: theory and applications. PhD Thesis, California Inst. Tech., 1994.

[18] I. Mezić. Spectral properties of dynamical systems, model reduction and decompositions. Nonlinear Dyn. 41:309-325, 2005.

[19] I. D. Morris. Mather sets for sequences of matrices and applications to the study of joint spectral radii. Proc. London Math. Soc., 107(3):121-150, 2013.

[20] J. Nie. Optimality conditions and finite convergence of Lasserre's hierarchy. Math. Prog., Ser. A, 146(1-2):97-121, 2014.

[21] M. Putinar, S. Sullivant (Editors). Emerging applications of algebraic geometry. The IMA Volumes in Mathematics and its Applications, vol. 149, Springer, Berlin, 2009.

[22] C.W. Rowley, I. Mezić, S. Bagheri, P. Schlatter, D.S. Henningson. Spectral analysis of nonlinear flows, J. Fluid Mech. 641:115127, 2009.

[23] P. J. Schmid. Dynamic mode decomposition of numerical and experimental data. J. Fluid Mech. 656:5-28, 2010.