Matricial approach for polynomial systems

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Novembre 17, 2009

Outline

Determinant and elimnation

A first link with optimization

Generalized companion matrices and resultants

Link with determinantal representation of rational plane curves

Sylvester matrix

Let $\mathbb{A}$ be a commutative field (and more generaly a domain), $f(x) = \sum_{i=0}^{d} a_i x^i$ and $g(x) = \sum_{j=0}^{e} b_j x^j \in \mathbb{A}[x]$ be two polynomial of degree d and e respectively (i.e. a_d and $b_e \neq 0$). We define the Sylvester matrix is defined as follows:

$$\mathrm{Syl}_{d,e}(f,g) = \begin{array}{c} 1 \\ \vdots \\ \vdots \\ \vdots \\ x^{d+e-1} \end{array} \overset{\begin{array}{cccccc} f & \cdots & x^{e-1}f & g & \cdots & x^{d-1}g \end{array}}{\begin{pmatrix} a_0 & & & b_0 & & \\ \vdots & \ddots & & \vdots & \ddots & \\ a_d & & a_0 & b_e & & b_0 \\ & \ddots & \vdots & & \ddots & \vdots \\ & & a_d & & & b_e \end{pmatrix}} .$$

Proposition

The polynomials $f(x)$ and $g(x)$ share a common in root in $\bar{\mathbb{A}}$ (and more generally in $\overline{\mathrm{Frac}(\mathbb{A})}$) if and only if $\det(\mathrm{Syl}_{d,e}(f,g)) = 0$.

Syzygetic interpretation

Let $l \in \mathbb{N}$, denote $\mathbb{A}[x]_l = \{p \in \mathbb{A}[x] | \deg(p) < l\}$. Then $\mathrm{Syl}_{d,e}(f,g)$ is the matrix associated to the following linear map:

$$\Psi_{d,e}(f,g) : \begin{cases} \mathbb{A}[x]_e \times \mathbb{A}[x]_d \longrightarrow \mathbb{A}[x]_{d+e} \\ (u,v) \longmapsto u * f + v * g \end{cases}$$

in the classical power basis of each space.

Proposition

The map $\Psi_{d,e}(f,g)$ is injective if and only if f and g are coprime.

Assume that $f = hf'$ and $g = hg'$, then $(g', f') \in \mathrm{Ker}(\Psi_{d,e}(f,g))$ (injective $\Rightarrow$coprime).
The converse is trivial: if $\mathrm{Ker}(\Psi_{d,e}(f,g)) \neq \{0\}$, then, it exist $(u,v) \in \mathbb{A}[x]_e \times \mathbb{A}[x]_d \setminus \{(0,0)\}$ such that $uf + vg = 0$. It is a non trivial Bézout relation and f and g can not be coprime (coprime $\Rightarrow$injective).

Link with elimination

Consider now f and g has generic polynomial, i.e. as polynomial in $\mathbb{Z}[a_0, \dots, a_d, b_0, \dots, b_e][x]$ and denote $I = (f, g)$. The $\mathbb{Z}[a_0, \dots, a_d, b_0, \dots, b_e] \cap I$ is a principal generated by a polynomial called resultant of f and g and denoted $\mathrm{Res}_x(f, g)$ (the index x is omitted when the context is clear).

Proposition

The elemination ideal $\mathbb{Z}[a_0, \dots, a_d, b_0, \dots, b_e] \cap I$ *is a principal ideal generated by* $\mathrm{Res}(f, g) = \det(\mathrm{Syl}_{d,e}(f, g))$.

An element of $\mathbb{Z}[a_0, \dots, a_d, b_0, \dots, b_e] \cap I$ is a element $h \in \mathbb{Z}[a_0, \dots, a_d, b_0, \dots, b_e]$ such that there exists u and v with $h = uf + vg$.

Outline

GCD and Sylvester map

Assume that $f(x)$ and $g(x)$ has $h(x)$ of degree $p > 1$ has greatest common divisor.
Every linear combination with polynomial coefficents of $f(x)$ and $g(x)$ is divisible by $h(x)$.

Lemma

$\mathrm{codim}(\mathrm{Im}(\Psi_{d,e}(f,g)) \geqslant p$.

It is simply because $1, x, \ldots, x^{p-1}$ are free in $\mathbb{A}[x]_{d+e}$ and are not in $\mathrm{Im}(\Psi_{d,e}(f,g))$.

GCD and Sylvester map

Proposition

$\dim(\mathrm{Im}(\Psi_{d,e}(f,g))) = e + f - p$ *and* $\dim(\mathrm{Ker}(\Psi_{d,e}(f,g))) = p$.

$$
\begin{array}{ccc}
0 & & \\
\downarrow & & \\
\mathrm{Ker}(\Psi_{d,e}(f,g)) & & \\
\downarrow & & \\
\mathbb{A}[x]_d \times \mathbb{A}[x]_e & \longrightarrow & \mathbb{A}[x]_d \times \mathbb{A}[x]_e / \mathrm{Ker}(\Psi_{d,e}(f,g)) \\
\downarrow & & \downarrow\approx \\
\mathbb{A}[x]_{d+e} & \longrightarrow & \mathbb{A}[x]_{d+e} / \mathrm{coker}(\Psi_{d,e}(f,g)) \\
\downarrow & & \\
\mathrm{coker}(\Psi_{d,e}(f,g)) & & \\
\downarrow & & \\
0 & &
\end{array}
$$

Geometrical interpretation

Taking $A(x) = x^d + \sum_{i=0}^{d-1} a_i x^i$ and $B(x) = x^e + \sum_{i=0}^{e} b_i x^i$ to generic polynomial in $\mathbb{Z}[a_0, \ldots, a_{d-1}, b_0, \ldots, b_{d-1}]$. We identify a monic polynomial with the vector of its coefficients exept the leading coefficient.

Each specialization of A and B gives couple (f, g) is identify to a point in $\mathbb{E}_{d,e} = \mathbb{R}^{d-1} \times \mathbb{R}^{e-1}$.

The set of couple of monic polynomials admitting a non-trivial GCD, denoted $\mathcal{G}_{d,e}$, has equation $\text{Res}_{d,e}(A, B)$.
The set of couple of polynomials having a non-trivial GCD is an algebraic subset of codimension 1 at least.

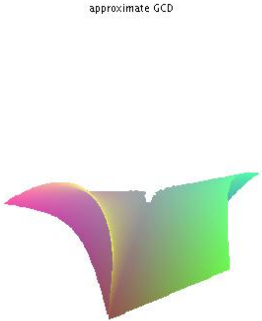
Examples

Consider $A(x) = x^2 + a_1 x + a_0$ and $B(x) = x^2 + b_1 x + b_0$. The set of couple of degree 2 monic polynomials having a GCD has equation:

$$a_0^2 - 2a_0 b_0 + b_0^2 - b_1 a_1 a_0 - b_1 a_1 b_0 + b_1^2 a_0 + b_0 a_1^2$$

Setting $b_1 = -2$, the set of couple of the form $(x^2 + a_1 x + a_0, x^2 - 2x + b_0)$ having a common gcd has equation:

$$a_0^2 + 2a_0 b_0 + b_0^2 + 2a_1 b_0 + 4a_0 + b_0 a_1^2$$



Recovering an approximate GCD

Denote $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_{e+d}$ the singular values of $\mathrm{Syl}_{d,e}(\tilde{f}, \tilde{g})$.

The value σ_i is the distance $\mathrm{Syl}_{d,e}(\tilde{f}, \tilde{g})$ to a matrix of rank i.
But the rank i matrix at distance σ_i of $\mathrm{Syl}_{d,e}(\tilde{f}, \tilde{g})$ is surely not of **Sylvester type** (bloc Toeplitz).

There exists a generally well posed problem associated (see Zhonggang Zeng 04 or -,Szanto 04 following Karmakar, Lakshman 96): Find F of degree d and G of degree e such that $\|\tilde{f} - F, \tilde{g} - G\| \leqslant \sigma_i$ and $\deg(\mathrm{GCD}(F, G)) = e + d - i$.

Outline

Companion matrices

Let $f(x)$ a degree d polynomial.

Definition

A matrix A is a companion matrix of f if its characteristic polynomial χ_A is (up to a sign) equal to f.

Example

The fröbenius matrix associated to $f(x) = \sum_{i=0}^{d} a_i x^i$ given by:

$$\begin{array}{c c} & \begin{matrix} x & x^2 & \cdots & x^d \end{matrix} \\ \begin{matrix} 1 \\ x \\ \vdots \\ x^{d-1} \end{matrix} & \begin{pmatrix} 0 & 0 & \cdots & -\frac{a_0}{a_d} \\ 1 & 0 & \cdots & -\frac{a_1}{a_d} \\ & \ddots & \ddots & \vdots \\ & & 1 & -\frac{a_{d-1}}{a_d} \end{pmatrix} \end{array}$$

is a companion matrix of $f(x)$.

Algebraic interpretation

Denote $\mathbb{B} = \mathbb{A}[x]/(f)$ the quotient algebra associated to $f(x)$. It is a vector space of dimension d. The familly $1, x, \ldots, x^{d-1}$ is a basis of $\mathbb{B}$ called the standart power basis.
Denote $\mathcal{M}_x : p \in \mathbb{B} \longrightarrow xp \in \mathbb{B}$ the multiplication by x in $\mathbb{B}$.

Proposition

Λ_f is the matrix of the multiplication by x in $\mathbb{B}$ in the standart power basis.

Let $\boldsymbol{b} = [b_1, \ldots, b_d]$ be a basis of $\mathbb{B}$ and denote $M_{x,\boldsymbol{b}}$ the matrix of $\mathcal{M}_x$ in the basis $\boldsymbol{b}$.

Proposition

For every base $\boldsymbol{b}$ of $\mathbb{B}$, the matrix $M_{x,\boldsymbol{b}}$ is a companion matrix of $f(x)$.

Funny problems

Let $f(x)$ and $g(x)$ be two polynomials of degree respectively d and e. We denote $\zeta_1, \ldots, \zeta_d$ the roots of $f(x)$.

Definition

A generalized companion of $f(x)$ and $g(x)$ is a matrix such that its eigenvalues are $g(\zeta_1), \ldots, g(\zeta_d)$.

A generalized companion matrix of $f(x)$ and x is a companion matrix of $f(x)$.

Proposition

The corank of a generalized companion matrix of $f(x)$ and $g(x)$ is the degree of $f \wedge g$.

Example of generalized companion matrix

For all $h(x) \in \mathbb{A}[x]$, we denote $N_f(h)$ the remainder of the euclidian division of h by f.

We denote $N_f(x^i g) = \sum_{j=0}^{d-1} g_{j,i} x^j$ and we consider the matrix $M_g = (g_{j,i})_{\substack{j=0\dots d-1 \\ i=0\dots d-1}}$:

$$M_g = \begin{array}{c} \\ 1 \\ \vdots \\ x^{d-1} \end{array} \begin{array}{c} \begin{array}{ccc} N_f(g) & \cdots & N_f(x^{d-1}g) \end{array} \\ \begin{pmatrix} g_{0,0} & \cdots & g_{0,d-1} \\ \vdots & \ddots & \vdots \\ g_{d-1,0} & \cdots & g_{d-1,d-1} \end{pmatrix} \end{array} .$$

Proposition

The matrix M_g is a generalized companion matrix associated to $f(x)$ and $g(x)$.

M_g is the matrix of multiplication by g in $\mathbb{B}$.

Few words on the structure of $\mathbb{B}$

Assume that $\zeta_i \neq \zeta_j$ if $i \neq j$, we denote $L_i = \frac{\prod_{j \neq i}(x - \zeta_j)}{\prod_{j \neq i}(\zeta_i - \zeta_j)}$ and

$\mathbf{1}_{\zeta_i} : p \rightarrow p(\zeta_i)$, for all $i \in \{1, \ldots, d\}$.

Proposition

The polynomials $L_1, \ldots, L_d$ form a basis of $\mathbb{B}$ called the Lagrange basis and its dual basis is $\mathbf{1}_{\zeta_1}, \ldots, \mathbf{1}_{\zeta_d}$.

In the Lagrange basis, the matrix of the multiplication by g is:

$$\begin{array}{c} \\ \mathbf{1}_{\zeta_1} \\ \vdots \\ \mathbf{1}_{\zeta_d} \end{array} \begin{array}{c} \begin{array}{ccc} gL_1 & \cdots & gL_d \end{array} \\ \left(\begin{array}{ccc} g(\zeta_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & g(\zeta_d) \end{array} \right) \end{array} .$$

This is a proof of the fact that M_g is a generalized companion matrix associated to $f(x)$ and $g(x)$.

Back to resultant and GCD

We have $\det(M_g) = (-1)^k \operatorname{Res}_x(f, g)$.
We assume that $g(\zeta_1) \neq 0, \ldots, g(\zeta_k) \neq 0$ and $g(\zeta_{k+1}) = \cdots = g(\zeta_d) = 0$.
The set $\operatorname{Ann}_{\mathbb{B}}(g) = \{h \in \mathbb{B} | h * g = 0\}$ is an ideal of $\mathbb{B}$.

Proposition

The ideal $\operatorname{Ann}_{\mathbb{B}}(g)$ *is a principal ideal generated by* $s(x) = \prod_{i=1}^{k} (x - \zeta_i)$.

We have $s(\zeta_i) g(\zeta_i) = 0$ for all $i \in \{1, \ldots, d\}$, so $s \in \operatorname{Ann}_{\mathbb{B}}(g)$.
Reciprocally, if $h \in \operatorname{Ann}_{\mathbb{B}}(g)$, we must have $h(\zeta_i) g(\zeta_i) = 0$ for all $i \in \{1, \ldots, d\}$ and so for $i \in \{1, \ldots, k\}$ and finally $s | h$.

Proposition

We have $f \wedge g = N_f(g)/s$.

Computational purposes

Shape of Sylvester matrix is $(d+e) \times (d+e)$ but structured
But if $\mathrm{Syl}_{d,e}(f,g) = U\Sigma V$ is a singular decomposition, with $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_{d+e})$, and if $\Sigma_k = \mathrm{diag}(\sigma_1, \ldots, \sigma_k, 0, \ldots, 0)$ then the truncation $U\Sigma_k V$ of $\mathrm{Syl}_{d,e}(f,g)$ is not of Sylvester type anymore !

Shape of generalized companion matrix is $\min(d,e) \times \min(d,e)$ and furthermore there is a truncation of this matrix that is a generalized companion matrix.

Bézoutian

Definition

The bézoutian polynomial of f and g is defined as
$\Theta_{f,g}(x,y) = \frac{f(x)g(y)-f(y)g(x)}{y-x} = \sum b_{i,j}x^i y^j$

Definition

The bézoutian matrix of f and g is the matrix $\mathrm{Bez}(f,g) = (b_{i,j})_{i,j}$.

Proposition

$\det(\mathrm{Bez}(f,g)) = (-1)^k \operatorname{Res}_x(f,g).$

There is a deep link between $\mathrm{Bez}(f,g)$ and M_g (Barnett formula).

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Implicitization of plane curves

Let $\Gamma : t \in \mathbb{R} \longrightarrow \begin{pmatrix} \gamma_1(t)/\delta(t) \\ \gamma_2(t)/\delta(t) \end{pmatrix}$ be a rational map and denote $\mathcal{C} = \Gamma(\mathbb{R})$.

Problem

Can we find a polynomial $P(x, y)$ such that $\bar{\mathcal{C}} = \mathcal{Z}(P)$?

Let us define $f(x, y, t) = \delta(t)x - \gamma_1(t)$ and $g(x, y, t) = \delta(t)y - \gamma_2(t)$.

Proposition

We have $P(x, y) = \mathrm{Res}_t(f, g)$ is such that $\bar{\mathcal{C}} = \mathcal{Z}(P)$.
