Self-Adjoint Determinantal Representations of Smooth Cubic Surfaces

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2. Cubic Surfaces
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   - Determinantal Representations
   - Twisted Cubic Curves on a Cubic Surface
   - Self-Adjoint Determinantal Representations
   - Definite Determinantal Representations
3. Conclusion

Notation

- F algebraically closed field
- $R = F[x_0, x_1, \ldots, x_n]$ polynomial ring
- $p(\mathbf{x}) = p(x_0, x_1, \ldots, x_n)$ a homogeneous polynomial (a form) of degree $d \geq 2$
- S the corresponding hypersurface in $\mathbb{P}^n = \mathbb{P}^n(F)$, $n \geq 2$
- $\mathcal{O} = \mathcal{O}_{\mathbb{P}^n}$ the sheaf of regular functions

Definition

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Given matrices $A_i \in M_d(F)$ then $A(\mathbf{x}) = \sum_{i=0}^{n} x_i A_i$ is a (linear) determinantal representation of p (or S) if

$$\det A(\mathbf{x}) = c \cdot p(\mathbf{x})$$

for some nonzero $c \in F$.

More generally, $A(\mathbf{x})$ a $e \times e$ matrix of forms of degree h is a determinantal representation of p (or S) if

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Symmetric and self-adjoint determinantal representations

- A determinantal representation
$$A(\mathbf{x}) = \sum_{j=0}^{n} x_j A_j \text{ is symmetric if } A_j = A_j^T \text{ for all } j.$$
- If $F = \mathbb{C}$ then a determinantal representation
$$A(\mathbf{x}) = \sum_{j=0}^{n} x_j A_j \text{ is self-adjoint if } A_j = A_j^* \text{ for all } j.$$
- A self-adjoint determinantal representations is definite if there exist $a_j \in \mathbb{R}$ such that $\sum_{j=0}^{n} a_j A_j$ is positive definite.

An easy example: Quadrics

- p a quadratic form of rank ≤ 4, then p is equivalent to one of the following: x^2, xy, $x^2 - yz$, $xt - yz$.

  These all have determinantal representations:

$$x^2 = \begin{vmatrix} x & 0 \\ 0 & x \end{vmatrix}, \quad xy = \begin{vmatrix} x & 0 \\ 0 & y \end{vmatrix},$$

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Further Examples: Cubic Curves

- An irreducible cubic curve is equivalent to a curve in Weierstraß form

$$p(x,y,z) = y^2 z - x(x - \theta_1 z)(x - \theta_2 z) = 0.$$

It has a determinantal representation

$$p(x,y,z) = \begin{vmatrix} -x & y & 0 \\ 0 & x - \theta_1 z & y \\ z & 0 & x - \theta_2 z \end{vmatrix}.$$

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Curves

- Hesse 1844 proved that every planar cubic curve has three essentially different symmetric determinantal representations

Theorem (Dixon, 1901)

Every curve $p(x, y, z) = 0$ has a determinantal representation:

$$p(x, y, z) = c \cdot \det(xA + yB + zC).$$

Furthermore, one can choose A, B, C symmetric.

General Hypersurfaces

Theorem (Dickson, 1920)

A generic homogeneous polynomial in $n+1$ variables of degree d has a determinantal representation if and only if

- $n = 2$ *(curves)*
- $n = 3$ *and* $d = 2, 3$ *(surfaces)*
- $n = 4$ *and* $d = 2$ *(threefolds)*

In the case $n = d = 3$ only generically, in the other cases all. Only the cubic surfaces equivalent to $x^2t + xz^2 + y^3 = 0$ have no determinantal representation. [Brundu & Logar, 1998]

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'Curious Facts'

From the theory of generalized Clifford algebras or maximal Cohen-Macaulay modules; Backelin, Herzog, Sanders, 1986:

Theorem

For every polynomial p there exists a $k \in \mathbb{N}$ such that p^k has a determinantal representation.

From the theory of linear matrix inequalities and control theory:

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Characterization of equivalence 1

- For determinantal representation A denote by $M(A)$ the cokernel of the multiplication map

$$A : \mathcal{O}(-h)^d \to \mathcal{O}^d$$

Theorem (Cook and Thomas, 1979)

Two determinantal representations A and B are equivalent if and only if the corresponding sheaves $M(A)$ and $M(B)$ are isomorphic.

Characterization of equivalence 2

- Dually, denote by ϵ_A the kernel of the map

$$A(\alpha) : F^d \rightarrow F^d$$

  evaluated at each $\alpha \in \mathbb{P}^n$
- ϵ_A is a 'line bundle' supported on S

Theorem (Vinnikov, 1989, Beauville 2000, Dolgachev)

Two determinantal representations A and B of a smooth hypersurface S are equivalent if and only if the corresponding line bundles ϵ_A and ϵ_B are isomorphic.

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Parametrization for Smooth Curves

Theorem (Vinnikov, 1989, Beauville)

Nonequivalent determinantal representations of a smooth irreducible curve are parametrized by points on the Jacobian variety of the curve that are not on the exceptional subvariety.

Example: Irreducible cubic curves

Theorem

Nonequivalent determinantal representations of a cubic curve C given by $y^2z = x^3 + pxz^2 + qz^3$ are parametrised by the affine points $(s, t, 1)$ on C:

$$\begin{vmatrix} x + \frac{t}{2}z & y + sz & \left(p + \frac{3}{4}t^2\right) z \\ 0 & x - tz & y - sz \\ -z & 0 & x + \frac{t}{2}z \end{vmatrix},$$

where $s^2 = t^3 + pt + q$.

Other Results

- Every smooth irreducible real curve has a self-adjoint representation. Parametrization known. (Vinnikov, 1993)
- It has a definite representation if and only if its real part forms a maximal nest of ovals. (Vinnikov, 1993)
- There is a connection between the theory of hyperbolic polynomials and definite determinantal representations.

Existence of Lines

Theorem (Cayley, Salmon, 1849)

On every smooth cubic surfaces S there are exactly 27 lines.

Theorem (Schläfli, 1858)

On S there are two sixtuples of skew lines

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{pmatrix} \quad (1)$$

such that a_i and b_j meet if and only if $i \neq j$.

The configuration (1) is caled a double-six.

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such that a_i and b_j meet if and only if $i \neq j$.

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27 lines

- On S there are 36 double-sixes and 72 sixtuples of skew lines.
- A sixtuple of skew lines uniquely determines a double-six, a half of a double-six uniquely determines a double-six.

Blow-up at Six Points and 27 lines

- Given 6 point in $\mathbb{P}^2$ in general position, then the blow-up at these points gives a cubic surface. Lines on S: 6 exceptional lines, 15 lines through pairs of points, and the last 6 lines are obtained from conics through 5 of the points. The exceptional lines and lines 'from conics' form a double-six.
- Given a double-six, let $\pi_{ij} =< b_i, a_j >$ be the plane spanned by a_i and b_j for $i \neq j$. It contains another line on S, call it c_{ij}. We have $c_{ij} = \pi_{ij} \cap \pi_{ji}$.
- There are 45 tritangent planes. Each line is contained in 5 tritangent planes.

Steinner's Triederpaar

Planes $\pi_{12}, \pi_{23}, \pi_{31};\ \pi_{13}, \pi_{32}, \pi_{21}$ form a triederpaar.
The planes intersect in 9 lines on S and these lines together with one more point uniquely determine S:

$$\pi_{12}\pi_{23}\pi_{31} + \lambda\pi_{13}\pi_{32}\pi_{21} = 0$$

is an equation for S. Assume $\lambda = 1$.

Determinantal Representation

Then

$$\Re(\mathbf{x}) = \begin{pmatrix} 0 & \pi_{12} & \pi_{13} \\ \pi_{21} & 0 & \pi_{23} \\ \pi_{31} & \pi_{32} & 0 \end{pmatrix}$$

is a determinantal representation of S. The i-th column in $\Re(\mathbf{x})$ gives equations for a_i, the j-th row gives equations for b_j:

$$\Re(\mathbf{x}) = \begin{array}{c} \begin{pmatrix} 0 & \pi_{12} & \pi_{13} \\ \pi_{21} & 0 & \pi_{23} \\ \pi_{31} & \pi_{32} & 0 \end{pmatrix} \\ \begin{matrix} \downarrow & \quad\downarrow & \quad\downarrow \\ a_1 & \quad a_2 & \quad a_3 \end{matrix} \end{array} \begin{matrix} \rightarrow & b_1 \\ \rightarrow & b_2 \\ \rightarrow & b_3 \end{matrix}$$

These six lines form a half of a double-six and they uniquely determine it.

Determinantal Representation and Six Skew Lines

How do we see the double-six as a blow-up of six points?
Define a 3×4 linear matrix L in variables z_0, z_1, z_2 by

$$A(\mathbf{x}) \cdot \begin{pmatrix} z_0 \\ z_1 \\ z_2 \end{pmatrix} = L(z_0, z_1, z_2) \cdot \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

There are exactly six points $P_k = \begin{pmatrix} \alpha_k \\ \beta_k \\ \gamma_k \end{pmatrix}$ in $\mathbb{P}^2$ such that L has rank 2. The zero locus of $A(\mathbf{x})(P_i) = 0$, $i = 1, \ldots, 6$ are exactly the six skew lines a_i of the blow-up on S.

This gives a bijective correspondence between a double-six and representations $A(\mathbf{x})$ and $A(\mathbf{x})^T$.

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The Adjoint Matrix

Let $\widetilde{A}(\mathbf{x}) = \operatorname{adj} A(\mathbf{x})$ be the adjoint of $A(\mathbf{x})$. Then

$$A(\mathbf{x})\,\widetilde{A}(\mathbf{x}) = p(\mathbf{x})I,$$

where $p(\mathbf{x}) = 0$ is an equation for S.
The zero locus of each column of $\widetilde{A}(\mathbf{x})$ defines a twisted cubic curve on S. All columns together define a twodimensional linear system of twisted cubics on S. There are 72 such linear systems.

Theorem (Classical?, Beauville, 2000)

A smooth cubic surface allows exactly 72 nonequivalent determinantal representations. There is one-to-one correspondence between:

- *equivalence classes of determinantal representations of S,*
- *sixes of skew lines on S,*
- *linear systems of twisted cubic curves on S.*

Example: Fermat's Surface

Fermat surface $x^3 + y^3 + z^3 + t^3 = 0$.
Consider determinantal representation

$$M = \begin{pmatrix} 0 & x+y & z+t \\ \omega z + t & 0 & x + \omega y \\ \omega x + y & z + \omega t & 0 \end{pmatrix}.$$

Find the double-six corresponding to M and M^T:

$$M \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} v & v & w & w \\ w & \omega w & \omega u & u \\ \omega u & u & v1 & \omega v \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$$

The matrix on the right hand side has rank 2 at points

$$(1,0,0),\ (0,1,0),\ (0,0,1),\ (1,1,1),\ (1,\omega,\omega^2),\ (1,\omega^2,\omega)$$

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Self-Adjoint Determinantal Representations

Proposition

A real smooth cubic surface S has a self-adjoint determinantal representation if and only if there exists a double-six

$$\begin{pmatrix} a_1 & \dots & a_6 \\ b_1 & \dots & b_6 \end{pmatrix}$$

and three distinct indices i, j, k such that

$$\begin{array}{lll} a_i = \overline{b_1}, & a_j = \overline{b_2}, & a_k = \overline{b_3}, \\ b_i = \overline{a_1}, & b_j = \overline{a_2}, & b_k = \overline{a_3} \end{array} .$$

Proof 1

- Let $U = U^*$. It is equivalent to

$$\mathfrak{R} = XUY, \quad X, Y \in \mathrm{GL}_3(\mathbb{C})$$

.

- Then $\mathfrak{R}$ and $\mathfrak{R}^*$ are equivalent since

$$\mathfrak{R}^* = Y^*UX^* = Y^*X^{-1}\mathfrak{R}Y^{-1}X^*.$$

They correspond to the same six skew lines a_i of a double-six $\binom{a_1 \ldots a_6}{b_1 \ldots b_6}$.
- Then

Proof 2

$$\mathfrak{R} = \begin{pmatrix} 0 & \pi_{12} & \pi_{13} \\ \pi_{21} & 0 & \pi_{23} \\ \pi_{31} & \pi_{32} & 0 \end{pmatrix} \begin{matrix} \rightarrow & b_1 \\ \rightarrow & b_2 \\ \rightarrow & b_3 \end{matrix}$$

$$\begin{matrix} \downarrow & \downarrow & \downarrow \\ a_1 & a_2 & a_3 \end{matrix}$$

and

$$\mathfrak{R}^* = \begin{pmatrix} 0 & \overline{\pi_{21}} & \overline{\pi_{31}} \\ \overline{\pi_{12}} & 0 & \overline{\pi_{32}} \\ \overline{\pi_{13}} & \overline{\pi_{23}} & 0 \end{pmatrix} \begin{matrix} \rightarrow & b_i = \overline{a_1} \\ \rightarrow & b_j = \overline{a_2} \\ \rightarrow & b_k = \overline{a_3} \end{matrix}$$

$$\begin{matrix} \downarrow & \downarrow & \downarrow \\ a_i & a_j & a_k \\ \| & \| & \| \\ \overline{b_1} & \overline{b_2} & \overline{b_3} \end{matrix}$$

Lines on a Real Cubic Surface

There are three types of lines on a real cubic surface:

- real line: $l = \bar{l}$,
- conjugate of the 1st kind: $l \cap \bar{l}$ is exactly one real point,
- conjugate of the 2nd kind: $l \cap \bar{l} = \emptyset$.

Theorem (Cayley, Segre)

A smooth real cubic surface can only be one of the 5 types :

Type	Number of lines:		
	Real	1st kind	2nd kind
F_1	27	0	0
F_2	15	0	12
F_3	7	4	16
F_4	3	12	12
F_5	3	24	0

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Types of Conjugate Double-Sixes on Real Cubic Surfaces

A double-six satisfies the condition of the proposition if and only if its 2 sixtuplets are mutually conjugate. There are four types of such double-sixes:

- the I-st kind: $\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ \overline{a_1} & \overline{a_2} & \overline{a_3} & \overline{a_4} & \overline{a_5} & \overline{a_6} \end{pmatrix}$,
- the II-nd kind: $\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ \overline{a_2} & \overline{a_1} & \overline{a_3} & \overline{a_4} & \overline{a_5} & \overline{a_6} \end{pmatrix}$,
- the III-rd kind: $\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ \overline{a_2} & \overline{a_1} & \overline{a_4} & \overline{a_3} & \overline{a_5} & \overline{a_6} \end{pmatrix}$,
- the IV-th kind: $\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ \overline{a_2} & \overline{a_1} & \overline{a_4} & \overline{a_3} & \overline{a_6} & \overline{a_5} \end{pmatrix}$.

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Self-Adjoint Determinantal Representations

Theorem

The number of nonequivalent self-adjoint determinantal representations of smooth real cubic surfaces:

Type	Number_{kind} of double-sixes	Number of self-adjoint representations
F_1	0	0
F_2	1_I	2
F_3	2_{II}	4
F_4	3_{III}	6
F_5	12_{IV}	24

Example: Fermat's Surface 2

Fermat's surface has 3 real lines and is of Segre type F_4. It has 6 nonequivalent self-adjoint determinantal representations. E.g.

$$N = \begin{pmatrix} x+y & z+\omega t & 0 \\ z+\omega^2 t & 0 & x+\omega y \\ 0 & x+\omega^2 y & z+t \end{pmatrix}.$$

The M before is not equivalent to a self-adjoint determinantal representations.

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Definite Determinantal Representations

A self-adjoint determinantal representation $U = z_0 U_0 + z_1 U_1 + z_2 U_2 + z_3 U_3$ of S is called *definite* if there exist $c_0, c_1, c_2, c_3 \in \mathbb{R}$ such that

$$c_0 U_0 + c_1 U_1 + c_2 U_2 + c_3 U_3 > 0$$

and is *indefinite* otherwise.
A nonzero vector $h \in \mathbb{C}^3$ is *self-orthogonal vector* of U if

$$h^* U_i h = 0, \quad \text{for all } i = 0, 1, 2, 3.$$

If a self-adjoint determinantal representation has a self-orthogonal vector, then it is indefinite.

The Existence of Definite Determinantal Representations 1

Theorem

Every self-adjoint determinantal representation of a surface of types F_2, F_3 and F_4 has a self-orthogonal vector and so it is indefinite.

To prove the theorem we show that each self-adjoint representation of a surface of types F_2, F_3 and F_4 has a self-orthogonal vector.

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The Existence of Definite Determinantal Representations 2

Theorem

Every real cubic surface of type F_5 has up to equivalence 16 definite determinantal representations (among the 24 nonequivalent self-adjoint representations).

The Existence of Definite Determinantal Representations 3

The proof of the theorem depends on the geometry of a surface of type F_5. The real part of a surface of types F_1, F_2, F_3 and F_4 has one connected component and of a surface of type F_5 has two connected components, one of which is ovoidal. Now the geometry of tritangent planes and of so-called parabolic planes is used to determine the existence of a positive definite quadratic form in the system of quadratic forms

$$\mathcal{Q}(\mathbf{x}) = \mathbf{v}^* U(\mathbf{x})\mathbf{v}.$$

We have an example of an indefinite determinantal representations of a surface of type F_5 without self-orthogonal vector.

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The idea of the proof

There exists a point $(\eta, \xi, 1) \in \mathbb{P}^2$ such that the zero locus of

$$\mathfrak{R}^t \begin{pmatrix} \eta \\ \xi \\ 1 \end{pmatrix} = 0$$

is $\overline{a_3}$. Then

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \eta & \xi & 1 \end{pmatrix} \mathfrak{R} = \begin{pmatrix} \pi_{21} & 0 & \pi_{23} \\ 0 & \pi_{12} & \pi_{13} \\ \xi\pi_{21} + \pi_{31} & \eta\pi_{12} + \pi_{32} & \eta\pi_{13} + \xi\pi_{23} \\ \downarrow & \downarrow & \downarrow \\ a_1 & a_2 & a_3 \end{pmatrix}$$

is self-adjoint.

The idea of the proof

View the system of quadratic forms:

$$\mathcal{Q} = (\overline{\alpha}, \overline{\beta}, \overline{\gamma}) \begin{pmatrix} \pi_{21} & 0 & \pi_{23} \\ 0 & \pi_{12} & \pi_{13} \\ \overline{\pi_{23}} & \overline{\pi_{13}} & \pi_{43} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} =$$

$$\frac{1}{\pi_{21}}(\alpha\pi_{21} + \gamma\pi_{23})(\overline{\alpha}\pi_{21} + \overline{\gamma\pi_{23}}) + \frac{1}{\pi_{12}}(\beta\pi_{12} + \gamma\pi_{13})(\overline{\beta}\pi_{12} + \overline{\gamma\pi_{13}}) +$$

$$\frac{1}{\pi_{12}\pi_{21}}(\pi_{12}\pi_{21}\pi_{43} - \pi_{12}\pi_{23}\overline{\pi_{23}} - \pi_{21}\pi_{13}\overline{\pi_{13}})\gamma\overline{\gamma}.$$

$\mathcal{Q}$ contains a definite quadratic form if and only if there is a real point $P \in \mathbb{P}^3$ such that $\pi_{12}(P), \pi_{21}(P)$ and $F(P)$ are all positive or all negative.

Interesting Questions

Theorem (Beauville, 2000)

A smooth quartic surface has a determinantal representation if and only if it contains a nonhypereliptic curve of genus 3 *embedded in* $\mathbb{P}^3$ *by a linear system of degree* 6.

- Consider quartic surfaces and parametrize all determinantal representations, if they exist.
- What real quartic surfaces admit a self-adjoint determinantal representation?

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