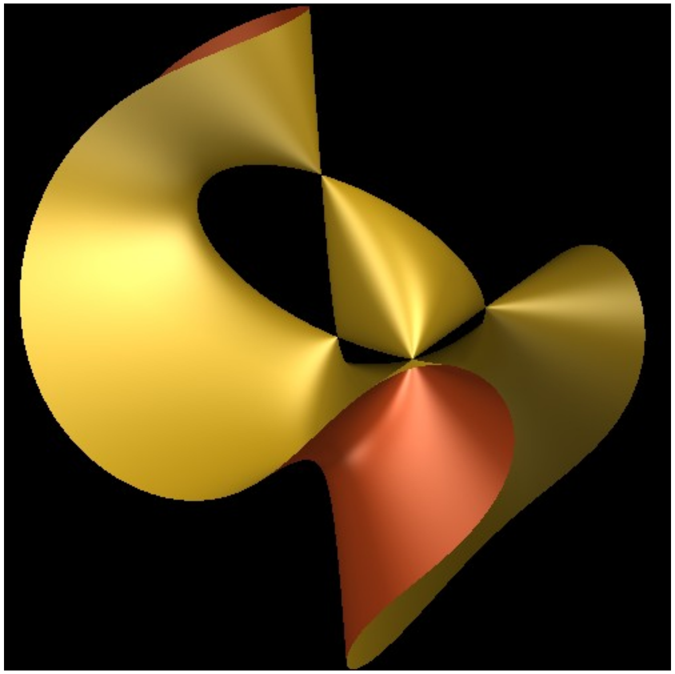
Workshop GeoLMI



LAAS-CNRS, Univ. Toulouse

19-20 November 2009

Thursday, November 19th

08:30-09:00	Welcome coffee
09:00-09:30	**Didier Henrion** Introductory comments
09:30-10:30	**Anita Buckley** Pfaffian representations of plane curves
10:30-11:00	Coffee break
11:00-12:00	**Luca Chiantini** On the pfaffian representation of general homogeneous polynomials
12:00-13:30	Lunch buffet
13:30-14:30	**Victor Vinnikov** Determinantal representations of singular curves
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15:00-16:00	**Olivier Ruatta** On LMI occuring in some real semi-algebraic geometry problems
16:00-17:00	Discussion pannel

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LMI

Linear matrix inequality

$$F(x) = F_0 + \sum_{i=1}^{n} x_i F_i \succeq 0$$

where F_i are given symmetric real matrices and constraint $\succeq 0$ means positive semidefinite (all eigenvalues real nonnegative)

Arise in control theory (Lyapunov 1890, Willems 1971, Boyd et al. 1994), combinatorial optimization, finance, structural mechanics, and many other areas

Key property = convex in x

SDP

Decision problem

$$\begin{array}{ll} \min_{x} & \sum_i c_i x_i \\ \text{s.t.} & F_0 + \sum_i x_i F_i \succeq 0 \end{array}$$

Optimization over LMIs = semidefinite programming, versatile generalization of linear (and convex quadratic) programming to the convex cone of positive semidefinite matrices

At given accuracy can be solved in polynomial time using interior-point methods (Nesterov, Nemirovski 1994)

Many public-domain solvers available

Geometry of LMI sets

How does an LMI set

$$\mathcal{F} = \{x \in \mathbb{R}^n \ : \ F(x) = F_0 + \sum_{i=1}^{n} x_i F_i \succeq 0\}$$

look like in Euclidean space ?

Build characteristic polynomial

$$\det(tI_d + F(x)) = \sum_{k=0}^{d} f_{d-k}(x) t^k$$

and the LMI set can be described as

$$\mathcal{F} = \{x \in \mathbb{R}^n \ : \ f_k(x) \geq 0, \ k = 1, 2, \ldots, d\}$$

a convex closed basic semialgebraic set

Example of 2D LMI feasible set

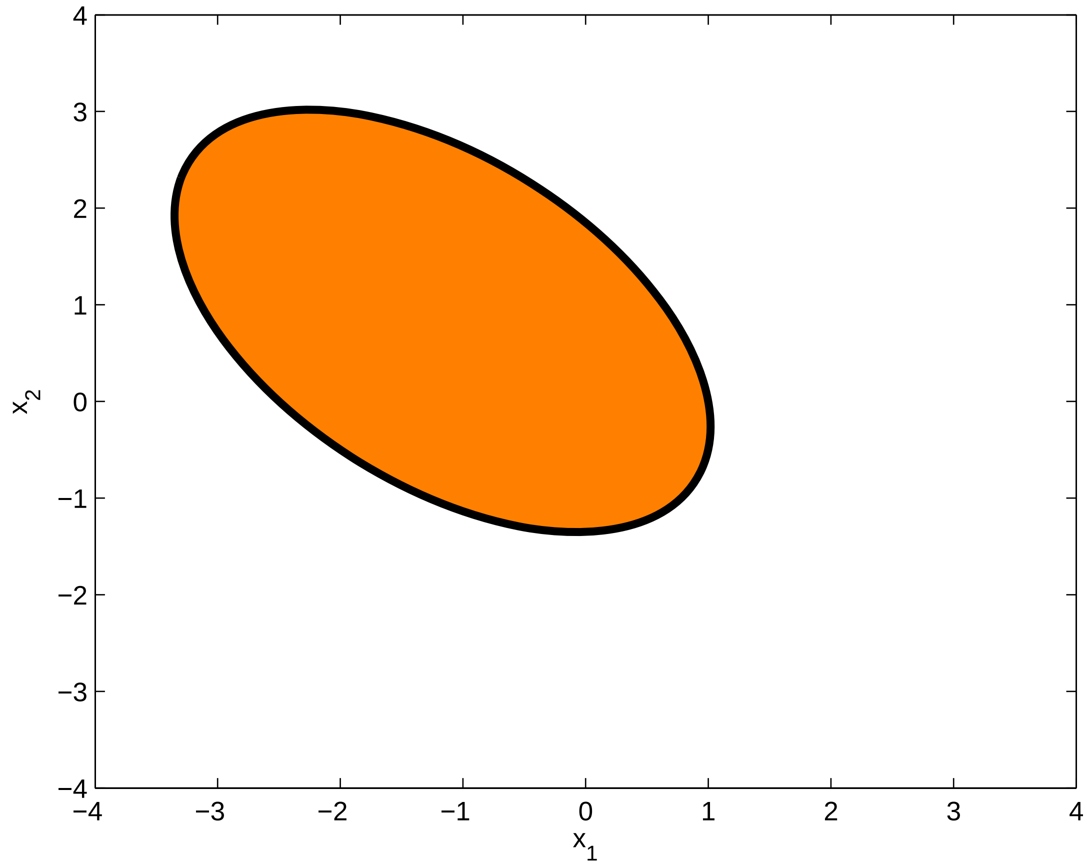
$$F(x) = \begin{bmatrix} 1 - x_1 & x_1 + x_2 & x_1 \\ x_1 + x_2 & 2 - x_2 & 0 \\ x_1 & 0 & 1 + x_2 \end{bmatrix} \succeq 0$$

System of 3 polynomial inequalities $f_i(x) \geq 0$

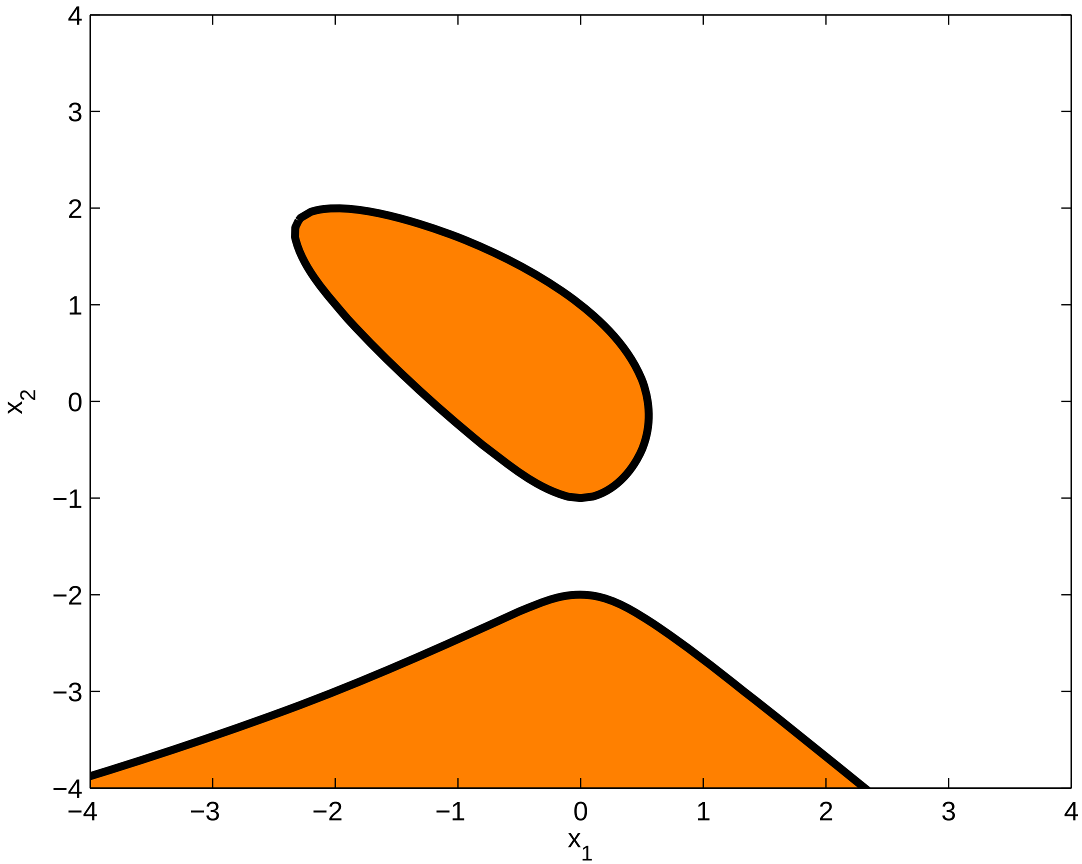
The first one is

$$f_1(x) = \text{trace}\, F(x) = 4 - x_1 \geq 0$$

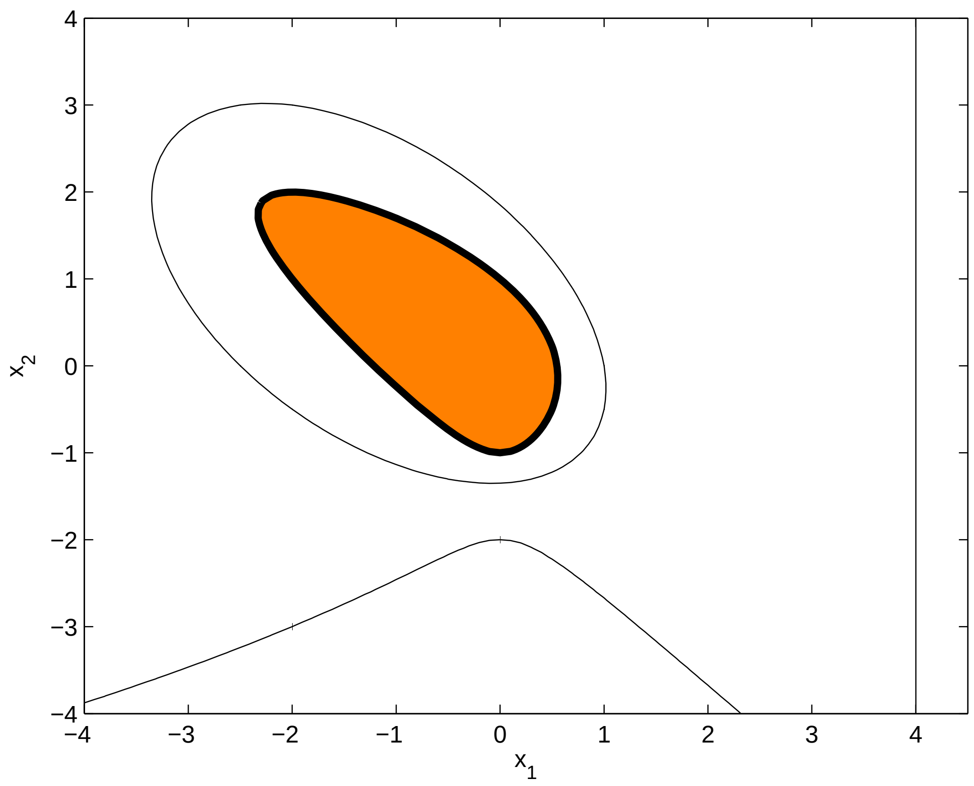
$$f_2(x) = 5 - 3x_1 + x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2 \geq 0$$



$$f_3(x) = \det F(x) = 2 - 2x_1 + x_2 - 3x_1^2 - 3x_1x_2 - 2x_2^2 - x_1x_2^2 - x_2^3 \geq 0$$



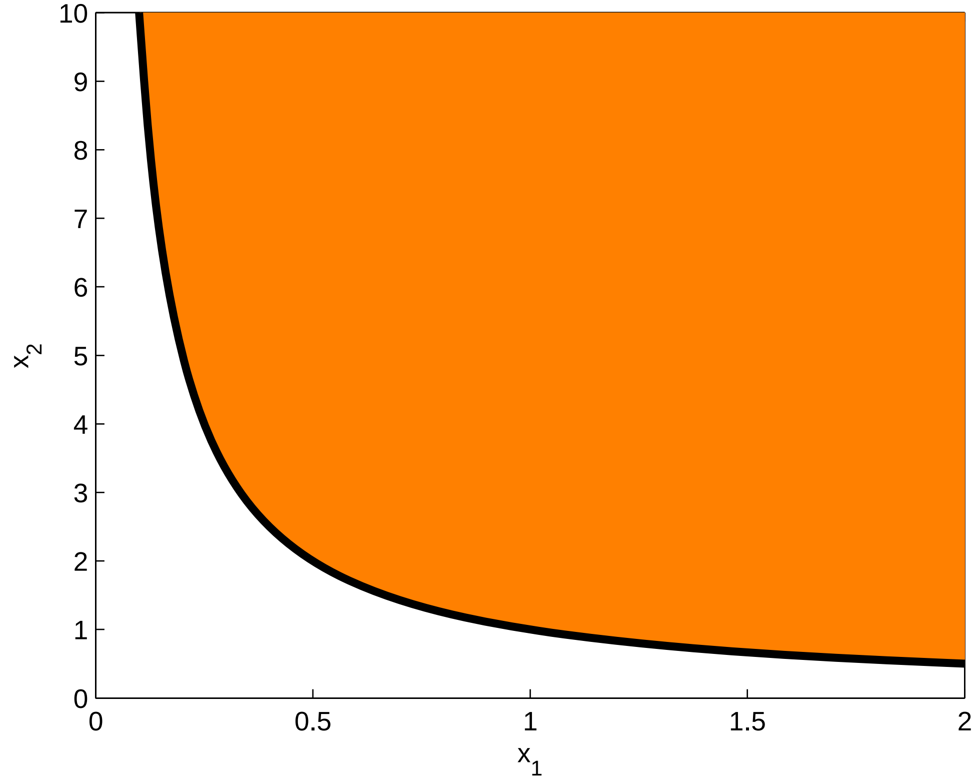
LMI set = intersection of level-sets $f_k(x) \geq 0$, $k = 1, 2, 3$



Boundary of LMI region shaped by determinant

Other polys only isolate convex connected component

LMI set or not ?



$x_1 x_2 \geq 1$ and $x_1 \geq 0$

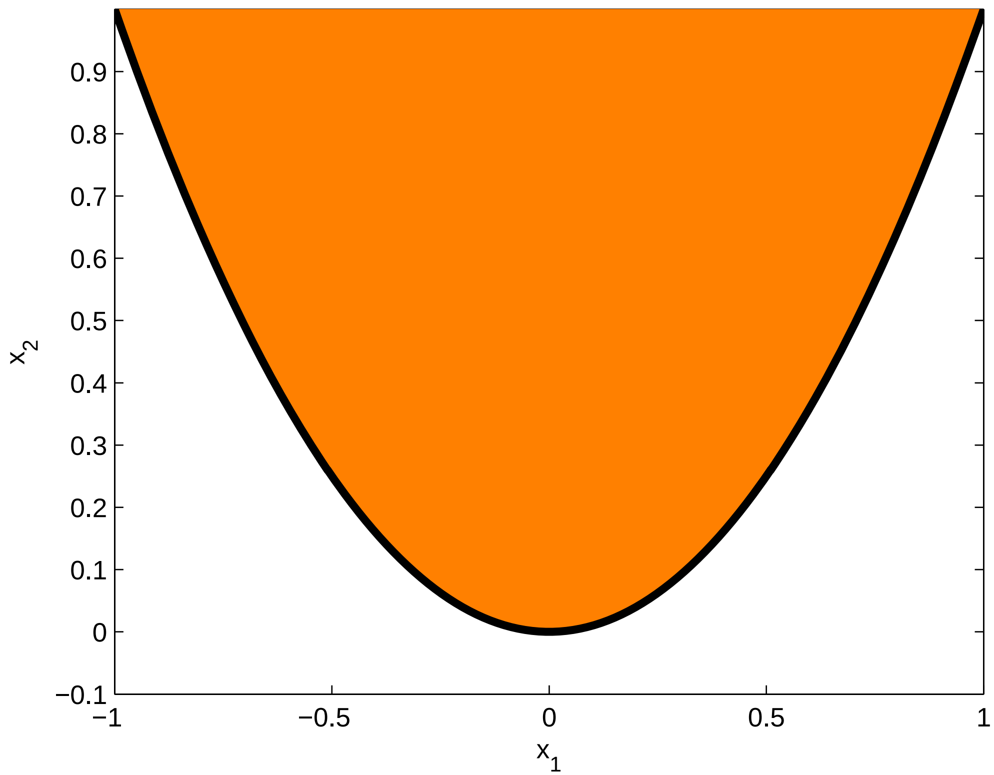
LMI

$$x_1 x_2 \geq 1 \text{ and } x_1 \geq 0$$

$$\Longleftrightarrow$$

$$\begin{bmatrix} x_1 & 1 \\ 1 & x_2 \end{bmatrix} \succeq 0$$

LMI set or not ?



$$x_2 \geq x_1^2$$

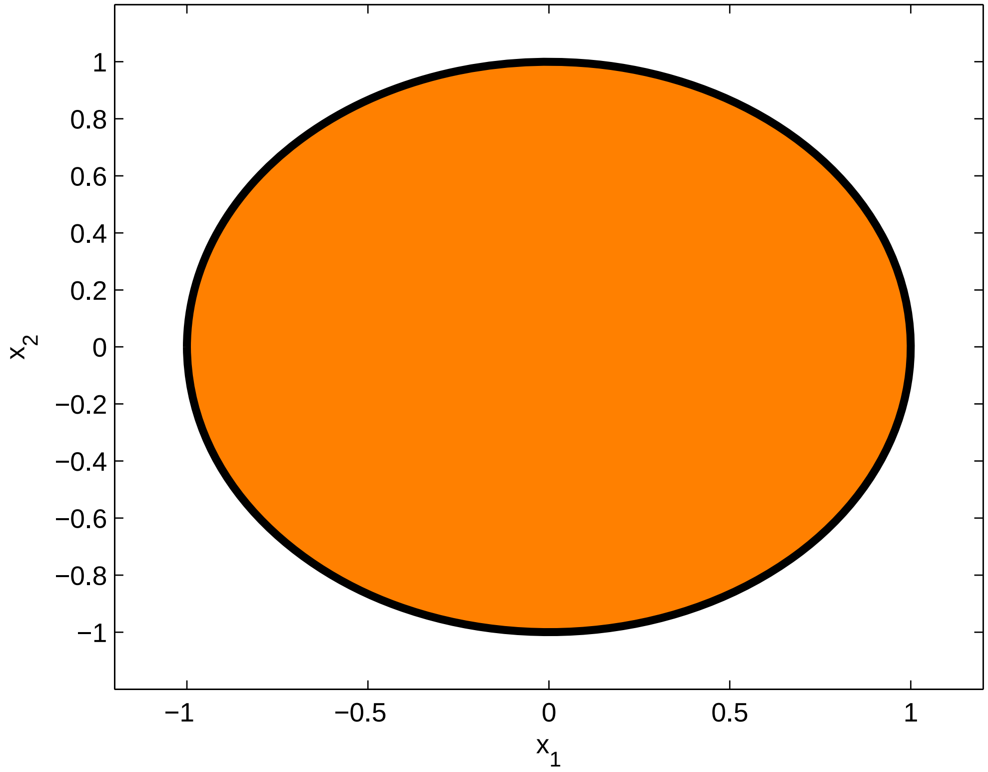
LMI

$$x_2 \geq x_1^2$$

$$\Longleftrightarrow$$

$$\begin{bmatrix} 1 & x_1 \\ x_1 & x_2 \end{bmatrix} \succeq 0$$

LMI set or not ?



$$x_1^2 + x_2^2 \leq 1$$

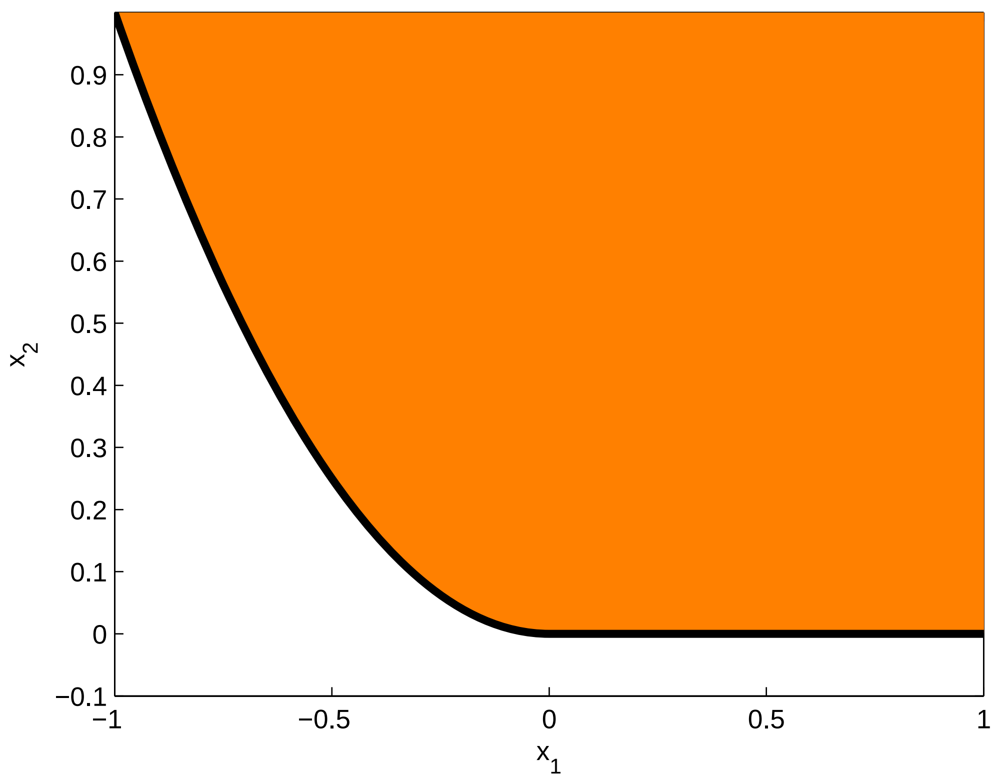
LMI

$$x_1^2 + x_2^2 \leq 1$$

$$\Longleftrightarrow$$

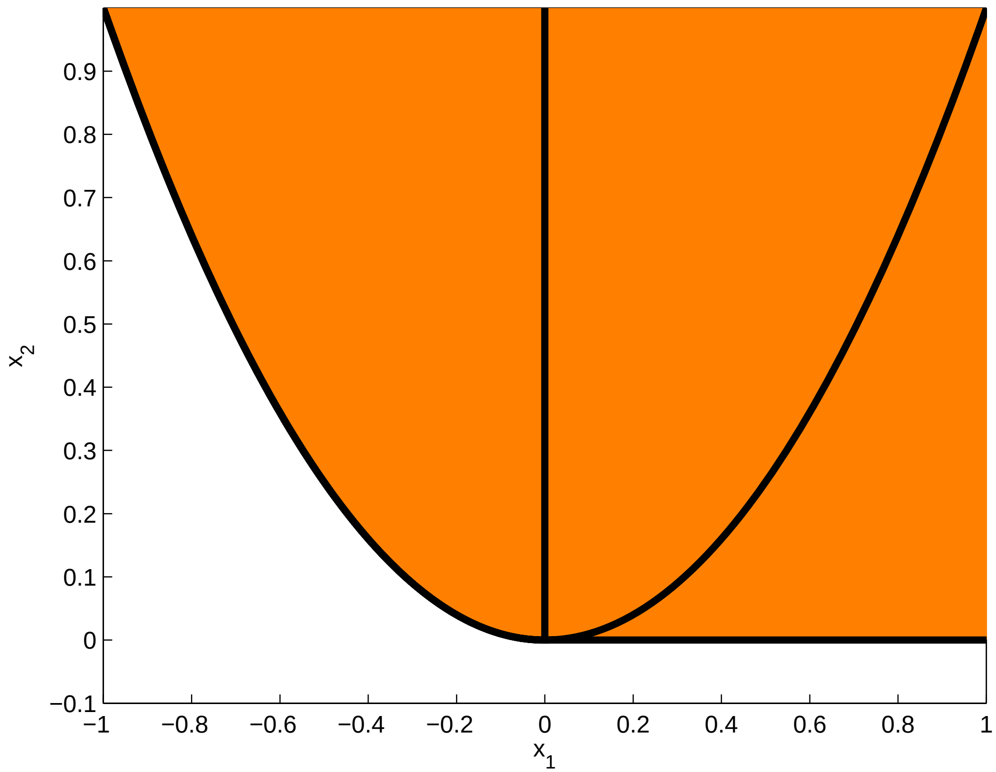
$$\begin{bmatrix} 1 + x_1 & x_2 \\ x_2 & 1 - x_1 \end{bmatrix} \succeq 0$$

LMI set or not ?



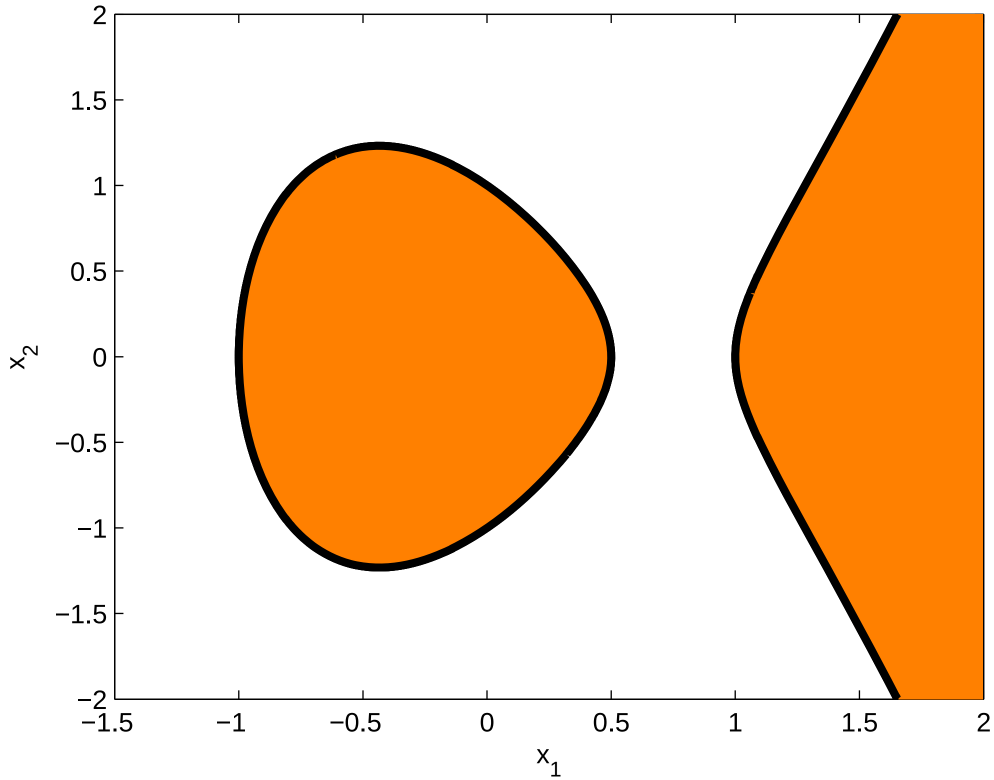
$$\{x \in \mathbb{R}^2 \, : \, t^4 + 2x_1 t^2 + x_2 \geq 0, \, \forall t \in \mathbb{R}\}$$

NOT LMI: not basic semialgebraic



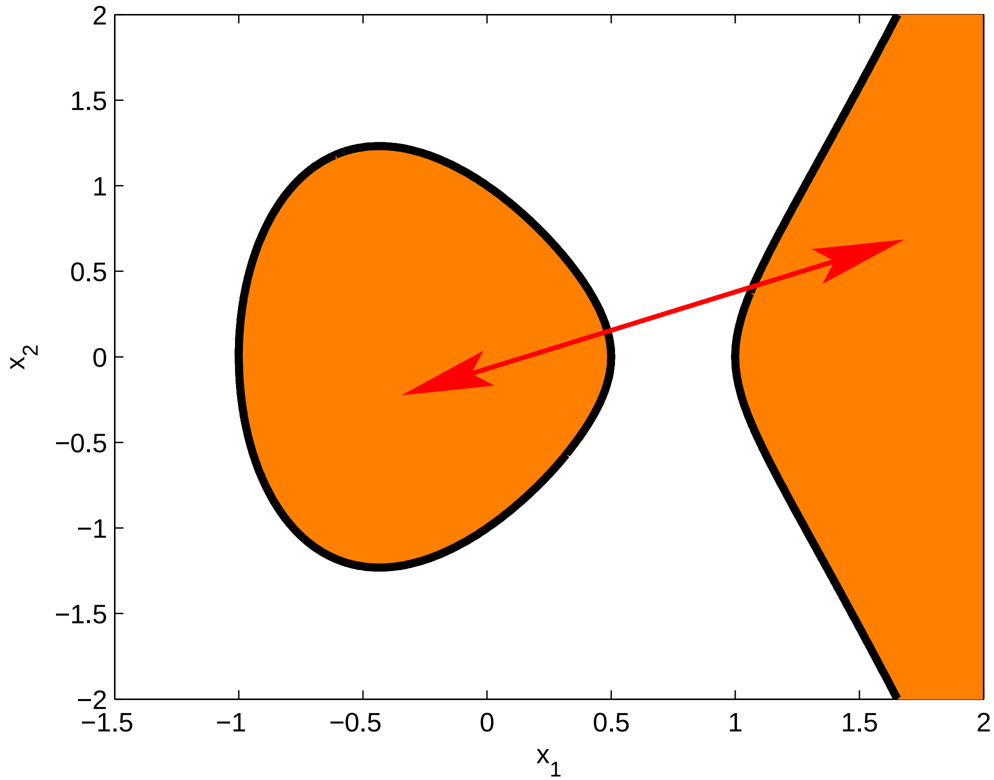
$$x_2 \geq x_1^2 \text{ or } x_1, x_2 \geq 0$$

LMI set or not ?



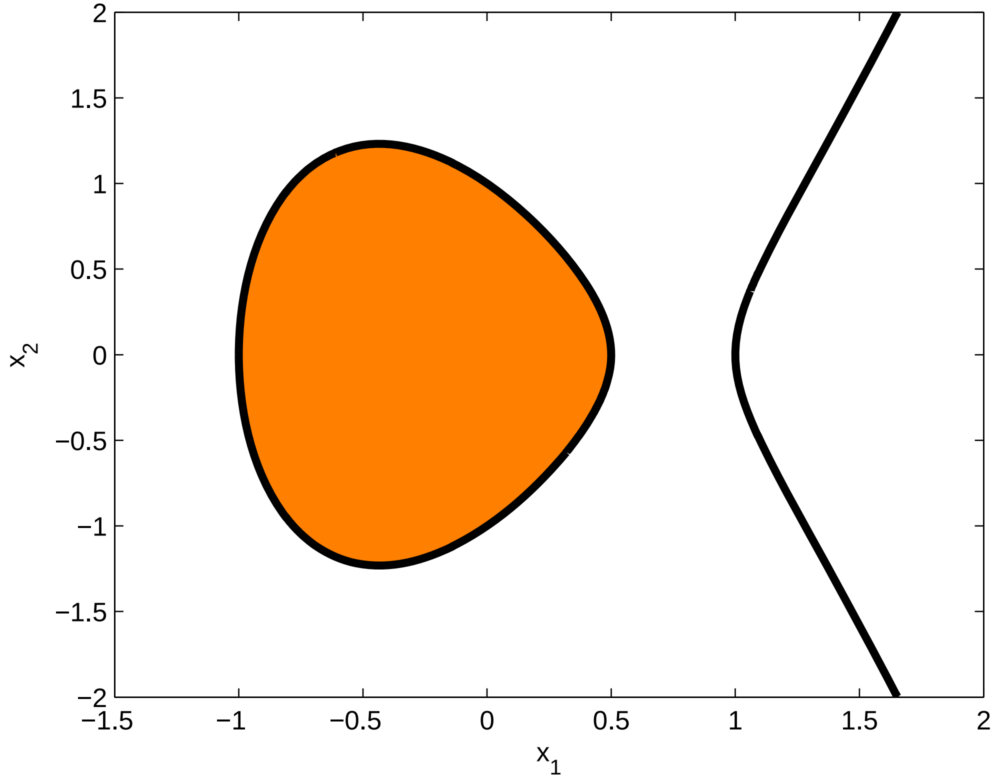
$$1 - 2x_1 - x_1^2 - x_2^2 + 2x_1^3 \geq 0$$

NOT LMI: not connected



$$1 - 2x_1 - x_1^2 - x_2^2 + 2x_1^3 \geq 0$$

LMI set or not ?



$$1 - 2x_1 - x_1^2 - x_2^2 + 2x_1^3 \geq 0 \text{ and } x_1 \leq \frac{1}{2}$$

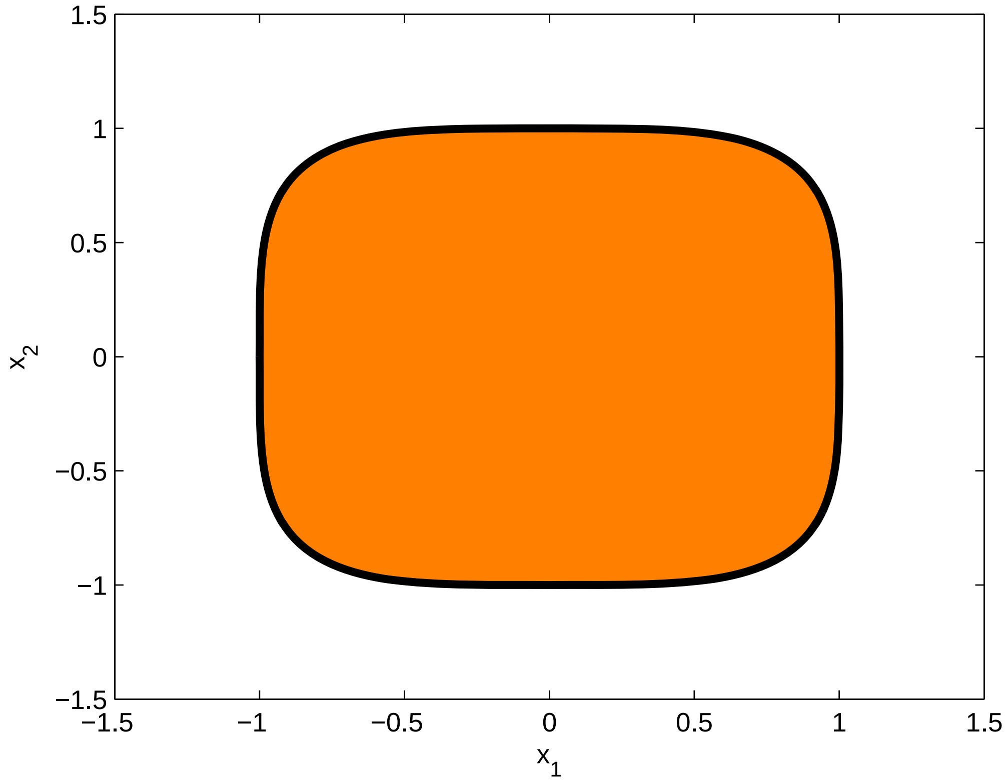
LMI

$$1 - 2x_1 - x_1^2 - x_2^2 + 2x_1^3 \geq 0 \text{ and } x_1 \leq \tfrac{1}{2}$$

$$\Longleftrightarrow$$

$$\begin{bmatrix} 1 & x_1 & 0 \\ x_1 & 1 & x_2 \\ 0 & x_2 & 1 - 2x_1 \end{bmatrix} \succeq 0$$

LMI set or not ?



$$x_1^4 + x_2^4 \leq 1$$

NOT LMI

but projection of an LMI

$$\begin{bmatrix} 1+u_1 & u_2 & & & & \\ u_2 & 1-u_1 & & & & \\ & & 1 & x_1 & & \\ & & x_1 & u_1 & & \\ & & & & 1 & x_2 \\ & & & & x_2 & u_2 \end{bmatrix} \succeq 0$$

with two liftings u_1 and u_2

Key modeling questions

Which convex closed basic semialgebraic sets are LMI sets ?

Which convex closed semialgebraic sets are lifted LMI sets ?

Symmetric linear determinantal representation

Consider the non-empty semialgebraic set

$$\mathcal{F} = \{x \in \mathbb{R}^n \,:\, f(x) \geq 0\}$$

where $f(x)$ is a given polynomial of degree d

Without loss of generality, assume that we are given a point e (typically the origin) satisfying $f(e) = 1$

Since the boundary of an LMI set is shaped by a determinant, can we find symmetric real (or hermitian complex) matrices F_i such that

$$F(x) = F_0 + \sum_{i=1}^{n} x_i F_i, \quad \det F(x) = f(x)$$

Definite determinantal representation = LMI

Once we have $\det F(x) = f(x)$, we would like to know whether

$$\begin{aligned}\mathcal{F} &= \text{closure}\,\{x \in \mathbb{R}^n \,:\, \det F(x) > 0\} \ni e \\ &= \{x \in \mathbb{R}^n \,:\, F(x) \succeq 0\}\end{aligned}$$

Since $f(e) = 1$, it holds $e \in \text{int}\,\mathcal{F}$ and $F(e) \succ 0$
so the representation must be definite for $\mathcal{F}$ to be LMI

Under which conditions on f can we find such a definite representation ?

Define the algebraic curve

$$\mathcal{C} = \{x \in \mathbb{R}^n \,:\, f(x) = 0\}$$

containing the boundary of $\mathcal{F}$

Rigid convexity

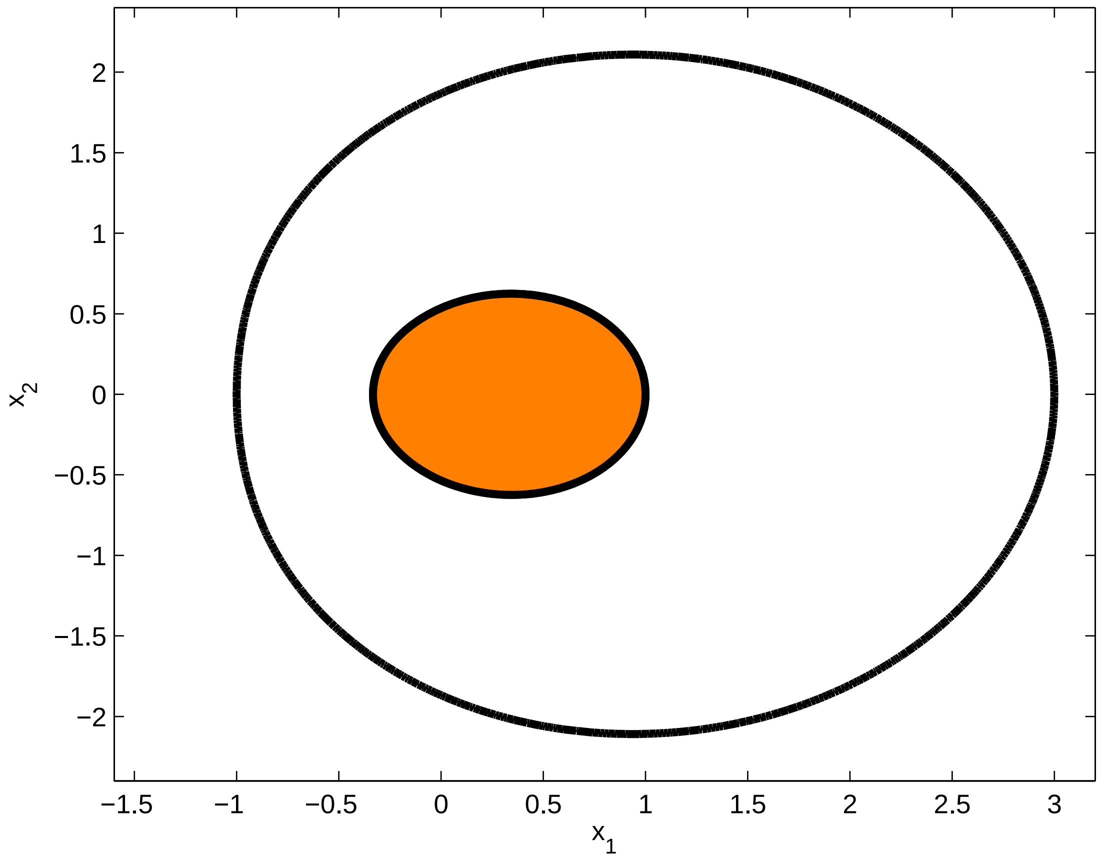
Helton and Vinnikov (2002) described a necessary and sufficient condition for $\mathcal{F}$ to be LMI when $n = 2$

Any line passing through an interior point of $\mathcal{F}$ must intersect $\mathcal{C}$ exactly d times (counting multiplicities and points at infinity)

Rigid convexity implies convexity

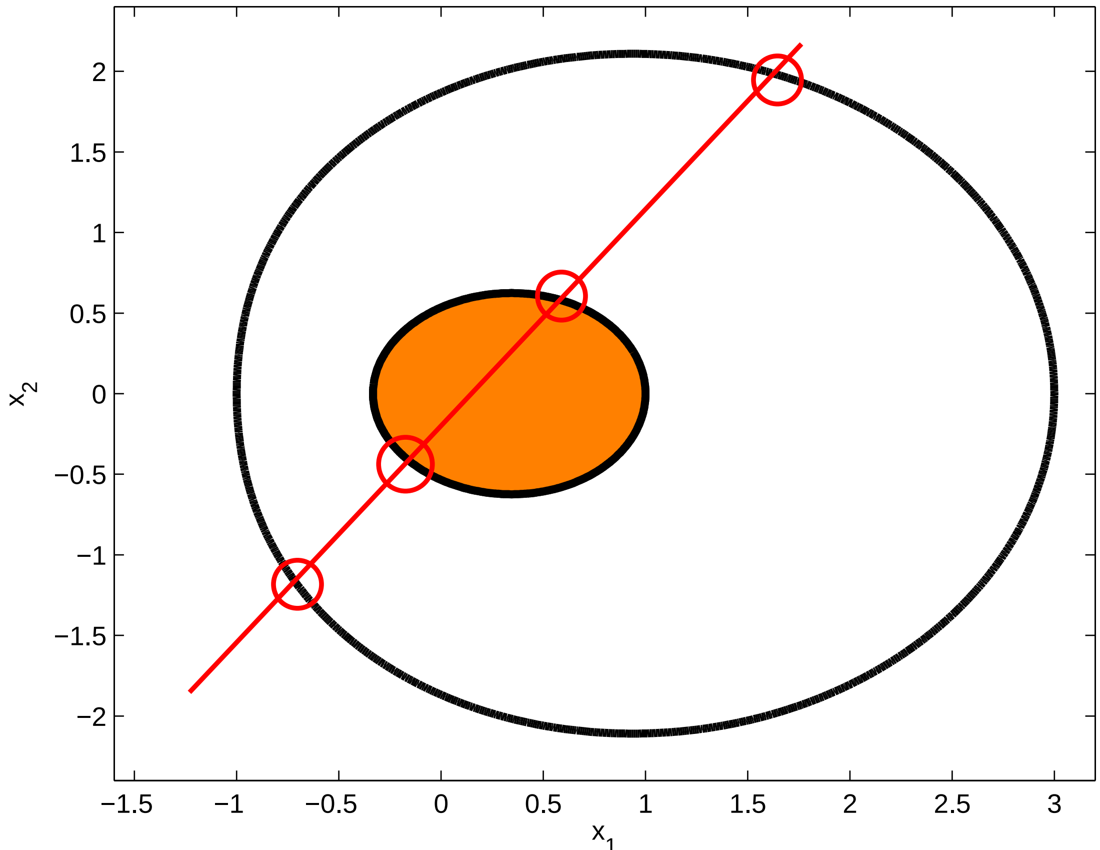
Connections with polynomial hyperbolicity (PDEs)

Cartesian ovals



$$(3((x_1+1)^2+x_2^2+1)-10(x_1+1))^2-10((x_1+1)^2+x_2^2+1)+12(x_1+1)+1 \geq 0$$

Cartesian ovals



Inner oval is rigidly convex hence LMI

Constructive methods

Checking rigid convexity amounts to checking positive semidefiniteness of the Hermite matrix of polynomial $p(x)$

Given $f(x)$ and e, once we know that the set

$$\mathcal{F} = \{x \in \mathbb{R}^n \;:\; f(x) \geq 0\} \ni e$$

is rigidly convex, how can we systematically build symmetric or hermitian matrices F_i such that

$$\mathcal{F} = \{x \in \mathbb{R}^n \;:\; F(x) = F_0 + \sum_{i=1}^{n} x_i F_i \succeq 0\}$$

and so $f(x) = \det F(x)$? When/how can we enforce $F_0 = I$?

Vinnikov's construction (1993)

In principle, matrices F_i can be built from polynomial $f(x)$ using complex Riemann surface theory

Pencil $F(x)$ can be obtained via the construction of a basis for a complete linear system of the algebraic curve $f(x) = 0$

Determinantal representations are characterized by line bundles parametrized in the Jacobian variety of the curve

Coefficients of F_i can be obtained via the period matrix of the curve and transcendental theta functions evaluated by numerical integration

No numerical implementation attempted so far..

Rational curves

An algebraic plane curve

$$\{x \in \mathbb{R}^2 \, : \, f(x) = 0\}$$

of genus zero, that is, with a maximal number of singularities, admits a rational parametrization

$$x_1(u) = \frac{f_1(u)}{f_0(u)}, \quad x_2(u) = \frac{f_2(u)}{f_0(u)}$$

with $f_i(u)$ real polys of real indeterminate u

Degrees of f_i do not exceed degree of f

Coeffs of f_i chosen in (typically small) algebraic extension of the coefficient field of f

Bézoutian

The resultant of the two polys

$$\begin{aligned} g_1(x_1, u) &= f_0(u) - f_1(u)x_1 \\ g_2(x_2, u) &= f_0(u) - f_2(u)x_2 \end{aligned}$$

vanishes whenever they have a common root as polynomials in u (variable to be eliminated)

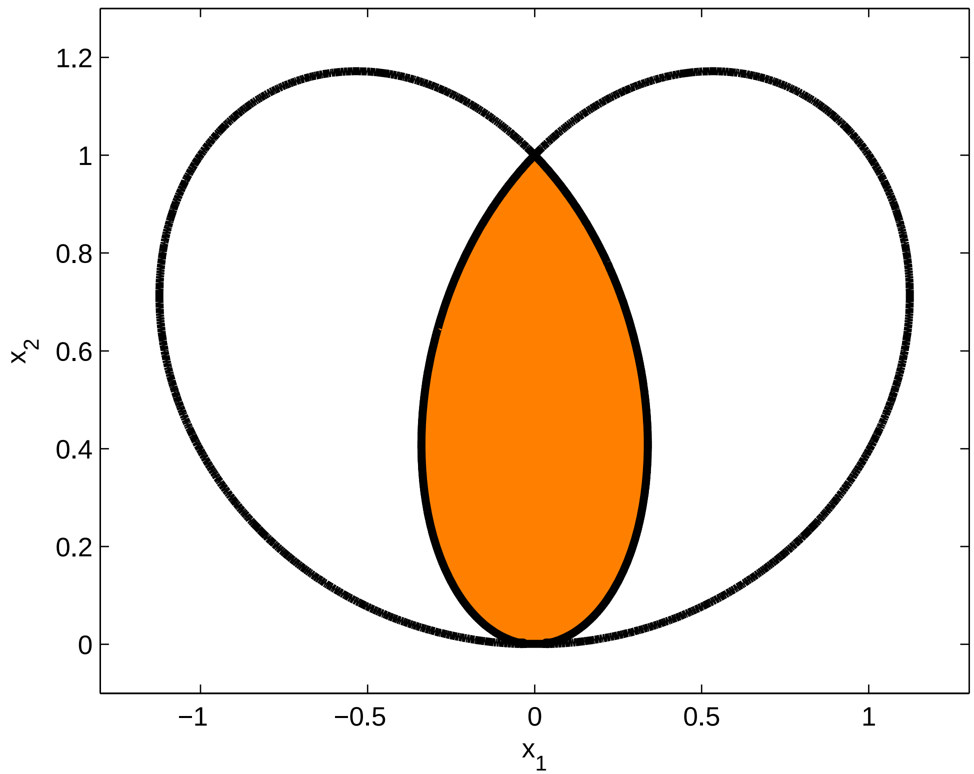
It is equal to the determinant of the Bézoutian matrix arising in the quadratic form

$$\frac{g_1(u)g_2(v) - g_1(v)g_2(u)}{u - v} = \sum_{a,b} [B]_{a,b} u^a v^b$$

By multilinearity, $F(x) = B(x)$ is symmetric and linear in x and

$$\det F(x) = f(x)$$

Capricorn curve



$$f(x) = x_1^2(x_1^2 + x_2^2) - 2(x_1^2 + x_2^2 - x_2)^2$$

Capricorn LMI

$$F(x) = \begin{bmatrix} 1960 - 868x - 1924y & -952 - 940x + 740y & -168 + 180x + 180y & 56 - 4x - 52y \\ -952 - 940x + 740y & 776 + 540x + 476y & -8 - 36x - 84y & -72 + 20x + 52y \\ -168 + 180x + 180y & -8 - 36x - 84y & 40 + 60x + 92y & 8 + 20x - 28y \\ 56 - 4x - 52y & -72 + 20x + 52y & 8 + 20x - 28y & 8 - 4x - 4y \end{bmatrix}$$

Definite around $e = (0, 1/2)$ hence LMI

LMI surfaces

Cayley cubic

$$\frac{1}{u_0}+\frac{1}{u_1}+\frac{1}{u_2}+\frac{1}{u_3}=0$$
$$u_0u_1u_2+u_0u_1u_3+u_0u_2u_3+u_1u_2u_3=0$$

Under involutary linear mapping

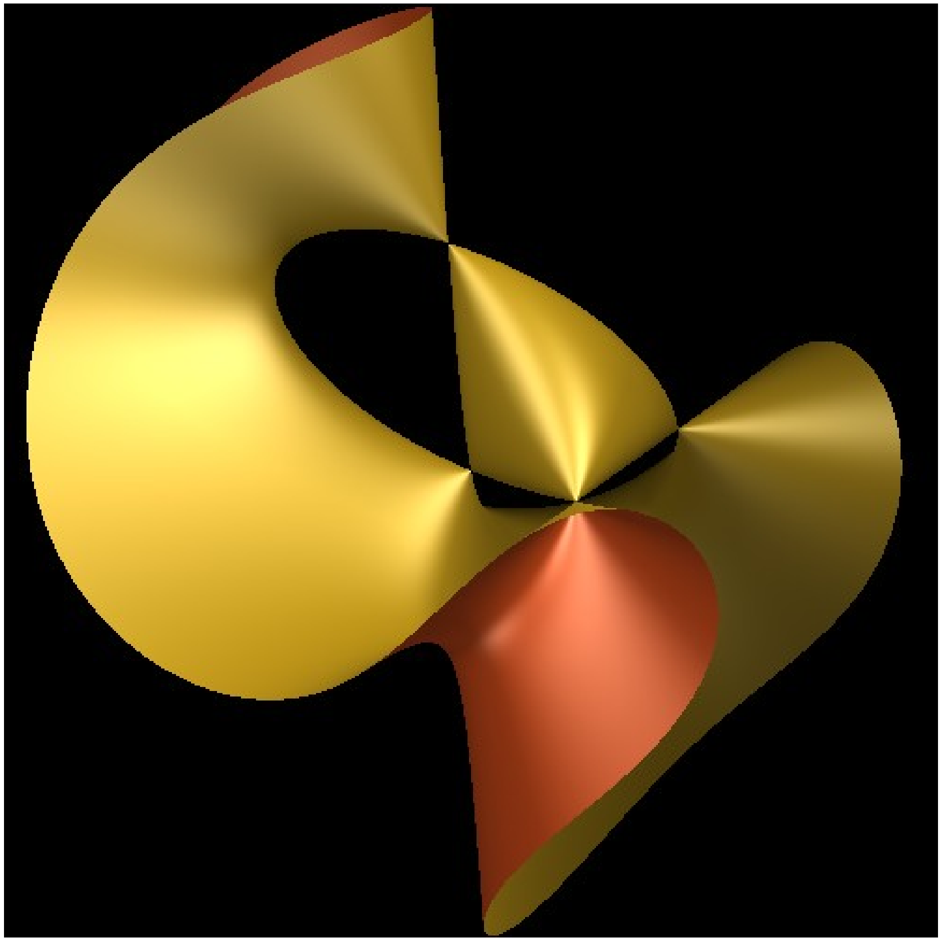
$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

the dehomogenized ($x_0=1$) algebraic equation becomes

$$1-x_1^2-x_2^2-x_3^2-2x_1x_2x_3=\det\begin{bmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & x_3 \\ x_2 & x_3 & 1 \end{bmatrix}$$

the 3x3 moment matrix of the MAXCUT LMI relaxation

Cayley cubic with 4 vertices



Research problems

- LMI modeling of smooth quartics
- higher degree algebraic plane curves
- (lifted) LMI modelling of singular curves
- LMI modeling in space ($n = 3$)
- applications in optimisation

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Acknowledgements

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