On the pfaffian representation of general homogeneous polynomials

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COMPLETE INTERSECTIONS ON GENERAL HYPERSURFACES.
On the pfaffian representation of general homogeneous polynomials

**PROBLEM:** Find a representation of general forms of degree $d$, by means of simpler forms of smaller degree.

The interest in the problem relies also in its geometric counterpart.

**EXAMPLE:** write a general form in 3 variables, of degree $d$, as a determinant of a matrix of forms.
EXAMPLE: write a general form in 3 variables, of degree d, as a determinant of a matrix of forms.

\[ F_d = \det \begin{pmatrix} F' & G' \\ L & M \end{pmatrix} \]

degrees \[ \begin{pmatrix} d-1 & d-1 \\ 1 & 1 \end{pmatrix} \]

\[ F_d = LF' - MG' \]

L, M linear forms

\[ F_d \in \langle L, M \rangle \]

plane curve
EXAMPLE: write a general form in 3 variables, of degree $d$, as a determinant of a matrix of forms.

$$F_d = \det \begin{pmatrix} F' & G' \\ L & M \end{pmatrix}$$

degrees \[ \begin{pmatrix} d-2 & d-2 \\ 2 & 2 \end{pmatrix} \]

$F_d = LF' - MG'$

$L, M$ quadratic

$F_d \in \langle L, M \rangle$

4 general points
\[ F_d = \det \begin{pmatrix} F' & G' \\ L & M \end{pmatrix} \]

degrees \[ \begin{pmatrix} d - 3 & d - 3 \\ 3 & 3 \end{pmatrix} \]

\[ F_d = LF' - MG' \]

L,M cubics

\[ F_d \in \langle L, M \rangle \]

The 9 points are NOT general

(9 general points lie only in 1 cubic)

QUESTION: are there 9 points, on a general curve of degree d, which are c.i. of two cubics?
Representation of forms \quad \leftrightarrow \quad \text{Existence of certain structures (subvarieties) on a general hypersurface}

Geometric problem \quad \rightarrow \quad \text{Understand the geometry of subvarieties of a general hypersurface}

\quad \rightarrow \quad \text{Find which varieties are there inside a general hypersurface}

\text{e.g.}
\text{Find which sets of points one can find inside a general plane curve}

\text{EXAMPLE: write a general form in 3 variables, of degree } d, \text{ as a determinant of a matrix of forms.}
THEOREM  For any choice of $a, b < d$, a general plane curve of degree $d$ contains a set of points which is complete intersection of type $a, b$. 

Idea for the proof  

$F_d = LF' - MG'$

$\deg(L) = a, \deg(M) = b$

---

**Projective space parameterizing curves of degree $d$**

**$F = \text{GENERAL FORM}$**

Forms splitting in a product $LF'$, with $\deg(L) = a$

Forms splitting in a product $MG'$, with $\deg(M) = b$

*image of $\mathbb{P}(a) \times \mathbb{P}(d - a)$ under Segre’s map + projection*
**THEOREM**  For any choice of $a, b < d$, a general plane curve of degree $d$ contains a set of points which is complete intersection of type $a, b$.

**Terracini's Lemma**  \( \text{tg space to the join} = \text{join of \text{tg spaces}} \)

**QUESTION:**  for generic forms $L, M, F', G'$ is \( \langle L, M, F', G' \rangle_d = \text{Ring}_d \) ?

**YES!**  \( \text{Lefschetz Hard Theorem} \) (Stanley)
Generalization to $\mathbb{P}^3$

FORGET IT!

$F_d$ cannot belong to $= \langle L, F' \rangle_d$, $\deg(L), \deg(F') < d$, because all of its curves are complete intersection of $F_d$

not a 2x2 determinant

$F_d = LF' - MG'$

large degree

NOETHER – LEFSCHETZ principle

Larger determinants?

Same problem
**Problem:** find a decomposition of type $F_d = AL + BM + CN$ with $\deg(L), \deg(M), \deg(N) < d$

for a general form of degree $d$ in 4 variables.

$$F_d \in < L, M, N >$$

e.g.

$$\deg(L) = \deg(M) = \deg(N) = 1$$

done

On a general surface of degree $> 3$, there are “few curves”,
but many sets of points!
**Problem:** find a decomposition of type \( F_d = AL + BM + CN \) with \( \deg(L), \deg(M), \deg(N) < d \) for a general form of degree \( d \) in 4 variables.

\[ \uparrow \]

Which complete intersection sets of points are there on a general surface of degree \( d \gg 0 \) in \( \mathbb{P}^3 \)?

**assume** \( \deg(L) \leq \deg(M) \leq \deg(N) \)

**DEFINITION** We say that a triple \( (\deg(L), \deg(M), \deg(N)) \) is **asymptotic** if for all \( d > d_0 \) a general surface of degree \( d \) contains a complete intersection set of points of type \( (\deg(L), \deg(M), \deg(N)) \)

**Example** \( (1,1,1) \) is asymptotic
Problem: find a decomposition of type $F_d = AL + BM + CN$ with $\deg(L), \deg(M), \deg(N) < d$ for a general form of degree $d$ in 4 variables.

Which complete intersection sets of points are there on a general surface of degree $d \gg 0$ in $\mathbb{P}^3$?

assume $\deg(L) \leq \deg(M) \leq \deg(N)$

Theorem (Carlini, ---, Geramita)
If $\deg(L) \leq 4$, the triple $(\deg(L), \deg(M), \deg(N))$ is asymptotic
If $\deg(L) > 6$, the triple $(\deg(L), \deg(M), \deg(N))$ is not asymptotic
If $5 \leq \deg(L) \leq 6$, the asymptotic triples are:

$\deg(L) = 5 \quad \deg(M) < 12 \quad \deg(L) = 5 \quad \deg(M) = 12 \quad \deg(N) = 12$

$\deg(L) = 6 \quad \deg(M) < 8 \quad \deg(L) = 6 \quad \deg(M) = 8 \quad \deg(N) = 8, 9$
**Problem:** is q DOMINANT?

**BASIC CONSTRUCTION**

$I = \{(F,Z): Z \subseteq F\}$

Complete intersections sets of points

$\mathbb{P}(d) \quad q$ \hspace{2cm} $F \bullet$

known from the Hilbert function of $Z$

$q$ \hspace{2cm} $p$ \hspace{2cm} $\bullet Z$

general fiber of $p = \mathbb{P}(H^0(I_Z(d)))$

if $\dim(I) < \dim(\mathbb{P}(d))$, return: NO

if $\dim(I) \geq \dim(\mathbb{P}(d))$, compute $tg$ spaces
I = \{(F,Z): \ Z \subseteq F\}

Problem: is q DOMINANT?

GENERALIZING the BASIC CONSTRUCTION

\(\mathbb{P}(d)\)

\(F \bullet\)

\(Z \bullet\)

\(T = \text{type of variety} = \text{irreducible component of some Hilbert scheme}\)

\[\dim(I) = \dim(T) + (\dim(\mathbb{P}(d)) - H_Z(d))\]

NB: if T parameterizes objects of dim > 0, no chance of ASYMPTOTIC positive answer

Problem: is q DOMINANT? DOMINANT \(\rightarrow\) \(\dim(T) \geq H_Z(d)\)
Back to the representation of forms $F$ having a c.i. set of points $Z$ of type $a,b,c$

Indeed $(LA+MB+NC)^2$ is the determinant of a skew-symmetric matrix

$F = (LA + MB + NC)$ is a 4x4 Pfaffian
Back to the representation of forms

\[ 0 \to \mathcal{O}(-a - b - c) \to \mathcal{O}(-a - c) \xrightarrow{M} \mathcal{O}(-b) \to \mathcal{I}_Z \to 0 \]

\[ \mathcal{O}(-b - c) \oplus \mathcal{O}(-c) \]

\[ \mathcal{O}(-d) = \mathcal{O}(-d) \]

**N.B.** In \( \mathbb{P}^3 \) a representation is always possible, for \( a = b = c = 1 \): every surface contains one point!

\[ \text{deg}(M) = \begin{pmatrix} 0 & d-1 & d-1 & d-1 \\ d-1 & 1 & 1 & 1 \\ d-1 & 1 & 1 & 1 \\ d-1 & 1 & 1 & 1 \end{pmatrix} \]

**N.B.** In \( \mathbb{P}^n \), \( n \geq 4 \), false in high degree.
GENERAL PFAFFIANS

\[ Z = \text{arithmatically Gorenstein 0-dimensional set} \subseteq F \]

\[
\begin{array}{ccccccc}
0 & \to & \mathcal{P}_0 & \to & \mathcal{P}_1 & \xrightarrow{M} & \mathcal{P}_2 & \to & \mathcal{I}_Z & \to & 0 \\
\end{array}
\]

\[ \mathcal{O}(-d) = \mathcal{O}(-d) \]

\[ \to F \]

\[ \mathcal{P}_0 = \text{line bundle} \quad \mathcal{P}_1 \quad \mathcal{P}_2 = \text{free} \]

\[ F = \text{pfaffian of a matrix} \quad N = \begin{pmatrix}
0 \\
\begin{pmatrix} n^T \\
M
\end{pmatrix}
\end{pmatrix} \]

\[ n = \text{vector of forms of degree} < d \]

Pfaffian representations of \( F \) \iff \text{Find arithmatically Gorenstein 0-dimensional sets} \subseteq F
SITUATION FOR PFAFFIAN SURFACES IN $\mathbb{P}^3$

A general surface $F$ is the pfaffian of a skew-symmetric matrix when:

- **Homogeneous pfaffians** ($n, M$ homogeneous of the same degree)
  - degree 1 (linear pfaffians)
    \[ \text{deg}(F) \leq 15 \quad (\text{Adler – Beauville - Schreyer}) \]
  - degree 2 (quadratic pfaffians)
    \[ \text{deg}(F) \leq 15 \quad (\text{Faenzi}) \]
  - higher degree
    \[ \text{deg}(F) \leq 8 \quad (---, \text{Faenzi}) \]

$F = \text{pfaffian of a matrix} \quad N = \begin{pmatrix} 0 & n \\ n^T & M \end{pmatrix} \quad \text{t}$

Never asymptotic
SITUATION FOR PFAFFIAN SURFACES IN $\mathbb{P}^3$

A general surface $F$ is the pfaffian of a skew-symmetric matrix when:

- **Quasi-homogeneous pfaffians**
  - $(n, M)$ homogeneous of different degrees
  - $F = \text{pfaffian of a matrix } N = \begin{pmatrix} n^T & M \\ n & M \end{pmatrix}$
  - $\deg(F) \leq 8$
  - $M$ matrix of forms of degrees $b > 1$
  - $M$ matrix of linear forms

  \[
  \begin{array}{c|c}
  \text{Non asymptotic} & \text{Some is asymptotic (depends on } n) \\
  \end{array}
  \]

  \[
  \begin{array}{c|c}
  \text{Quasi-homogeneous pfaffians (depends on } n) \\
  \end{array}
  \]

  \[
  \begin{array}{c|c}
  \text{---, Faenzi} & \text{---, Faenzi} \\
  \end{array}
  \]
SITUATION FOR PFAFFIAN SURFACES IN $\mathbb{P}^3$

- Quasi-homogeneous pfaffians
  (n, M homogeneous of different degrees)
  - M matrix of linear forms
    (---, Faenzi)

$d = \alpha + 1$

d = $\alpha$ + 1

d = 16

$\alpha = 10$

$\alpha = 15$

Linear matrix of size $(2\alpha-3) \times (2\alpha-3)$

$\begin{pmatrix} 0 & n \\ n^T & M \end{pmatrix}$

$t = 2\alpha - 2$

Degree matrix of M

$\begin{pmatrix} d-\alpha+2 & \ldots & d-\alpha+2 \\ d-\alpha+2 & \ldots & d-\alpha+2 \\ \vdots & \ddots & \vdots \\ d-\alpha+2 & \ldots & d-\alpha+2 \end{pmatrix}$
HINT of proofs for non-existence

\[ I = \{(F,Z): \ Z \subseteq F\} \]

but BEWARE: for \( d = 16 \) \( \dim(I) > \dim(\mathbb{P}(d)) \)

nevertheless \( p \) is not dominant

REASON: \( Z \) determines on \( F \) a rank 2 bundle \( E \) with \( \dim(H^0(E)) \) large, so \( Z \) moves in a high dimensional family on \( F \) and \( p \) has general fibers of large dimension
HINT of proofs for existence

\[ \dim(I) \geq \dim(\mathbb{P}(d)) \]

for sparse results, compute the dimension of the tangent space to \( p(I) \), which is generated by submaximal pfaffians

For asymptotic results

Assume \( H_Z(d) = \deg(Z) \)

\[ \dim(I) \geq \dim(\mathbb{P}(d)) \]

plane

\[ \text{then use induction} \]

For sparse results, compute the dimension of the tangent space to \( p(I) \), which is generated by submaximal pfaffians
If the general surface is NOT pfaffian

Pfaffian rank

$P(d)$

Pfaf = \{pfaffians\}

in general $H = F_1 + \ldots + F_k$

minimum $k$?

e.g. $H = F + G \implies H \in \text{Secant variety}$

Sec(Pfaf) expected dimensions?

TERRACINI'S LEMMA

the tg space of the secant variety at $H$

is the span of the tg spaces to Pfaf at $F,G$
If the general surface is NOT pfaffian

Pfaffian rank

Secant varieties of Pfaff

do they have the expected dimensions?

TERRACINI'S LEMMA
the tg space of the secant variety at H
is the span of the tg spaces to Pfaf at F,G

study the Hilbert function of a union
of two arithmetically Gorenstein sets

work in progress

thank you for your attention