# On the pfaffian representation of general homogeneous polynomials

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# Bibliography


E.Carlini, L.Chiantini, A.Geramita.
COMPLETE INTERSECTION POINTS ON GENERAL SURFACES IN P^3.
arXiv:0811.2233

L.Chiantini, D.Faenzi.
RANK 2 ARITHMETICALLY COHEN-MACAULAY BUNDLES ON A GENERAL QUINTIC SURFACE.
Math. Nach. to appear.

L.Chiantini, D.Faenzi.
ON GENERAL SURFACES DEFINED BY AN ALMOST LINEAR PFAFFIAN.
Geom. Ded. 142 (2009) 91-107.

E.Carlini, L.Chiantini, A.Geramita.
COMPLETE INTERSECTIONS ON GENERAL HYPERSURFACES.
Michigan Math. J. 57 (2008) 121-136.

# On the pfaffian representation of general homogeneous polynomials

**PROBLEM:** Find a representation of general forms of degree d, by means of simpler forms

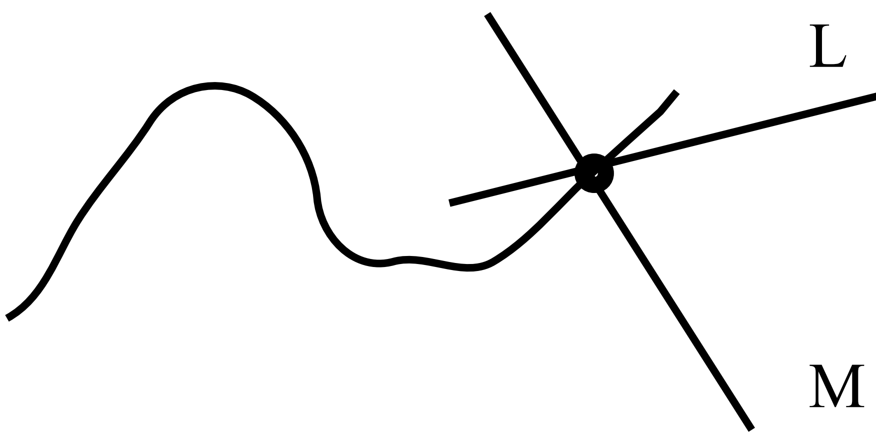
of smaller degree

The interest in the problem relies also in its geometric counterpart

**EXAMPLE:** write a general form in 3 variables, of degree d, as a determinant of a matrix of forms.

$$F_d = \det\begin{pmatrix} F' & G' \\ L & M \end{pmatrix}$$

degrees $\begin{pmatrix} d-1 & d-1 \\ 1 & 1 \end{pmatrix}$




$F_d = LF' - MG'$

L,M linear forms

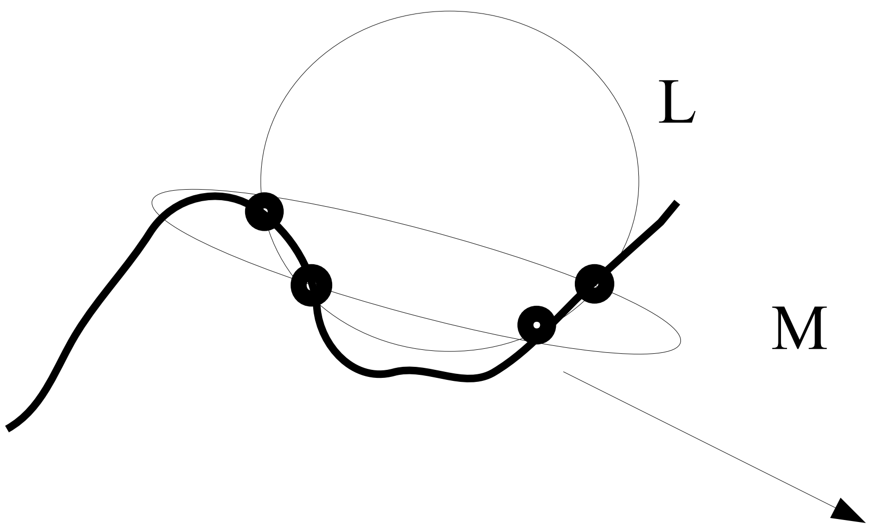
$F_d \in \langle L, M \rangle$

plane curve

**EXAMPLE:** write a general form in 3 variables, of degree d, as a determinant of a matrix of forms.

$$F_d = \det \begin{pmatrix} F' & G' \\ L & M \end{pmatrix}$$

degrees $\begin{pmatrix} d-2 & d-2 \\ 2 & 2 \end{pmatrix}$




$F_d = LF' - MG'$

L,M quadratic

$F_d \in \langle L, M \rangle$

4 general points

**EXAMPLE:** write a general form in 3 variables, of degree d, as a determinant of a matrix of forms.

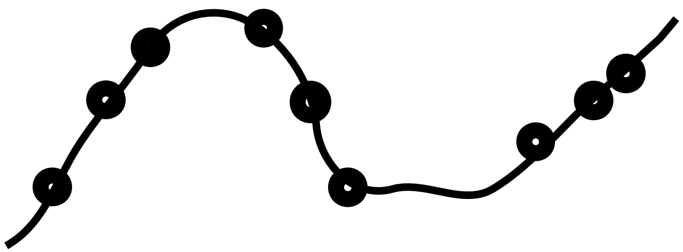
$$F_d = \det \begin{pmatrix} F' & G' \\ L & M \end{pmatrix}$$

degrees $\begin{pmatrix} d-3 & d-3 \\ 3 & 3 \end{pmatrix}$

$F_d = LF' - MG'$

L,M cubics

$F_d \in \langle L, M \rangle$



The 9 points are NOT general

(9 general points lie only in 1 cubic)

QUESTION: are there 9 points, on a general curve of degree d, which are c.i. of two cubics?

Representation of forms ⟺ Existence of certain structures (subvarieties) on a general hypersurface

Geometric problem → Understand the geometry of subvarieties of a general hypersurface

→ Find which varieties are there inside a general hypersurface

e.g.
Find which sets of points one can find inside a general plane curve

⇓

**EXAMPLE:** write a general form in 3 variables, of degree d, as a determinant of a matrix of forms.

**THEOREM** For any choice of a,b <d, a general plane curve of degree d contains a set of points which is complete intersection of type a,b.

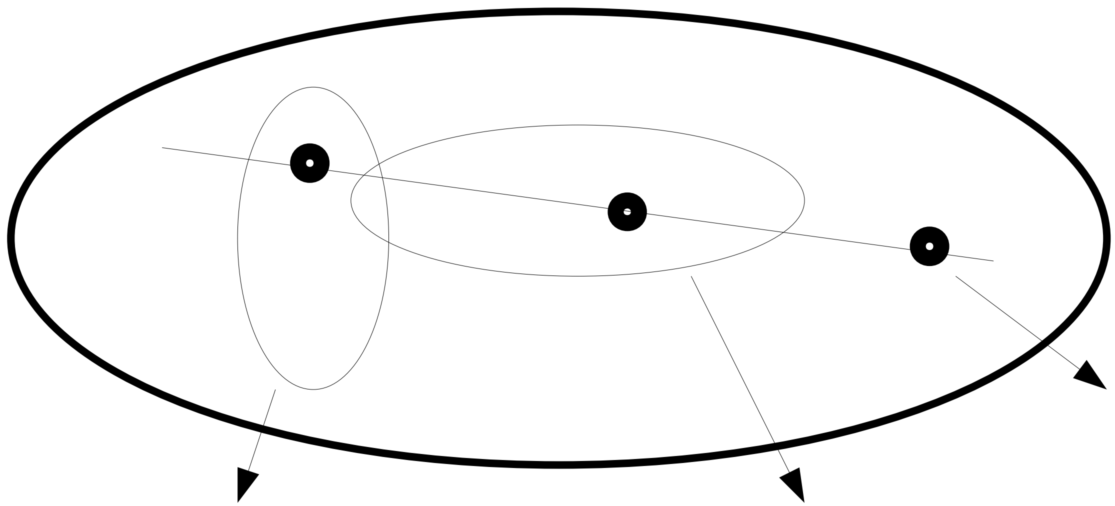
## Idea for the proof

$$F_d = LF' - MG'$$

deg(L)=a, deg(M)=b

$\mathbb{P}(d)$

Projective space parameterizing curves of degree d



F = GENERAL FORM

Forms splitting in a product LF', with deg(L) = a

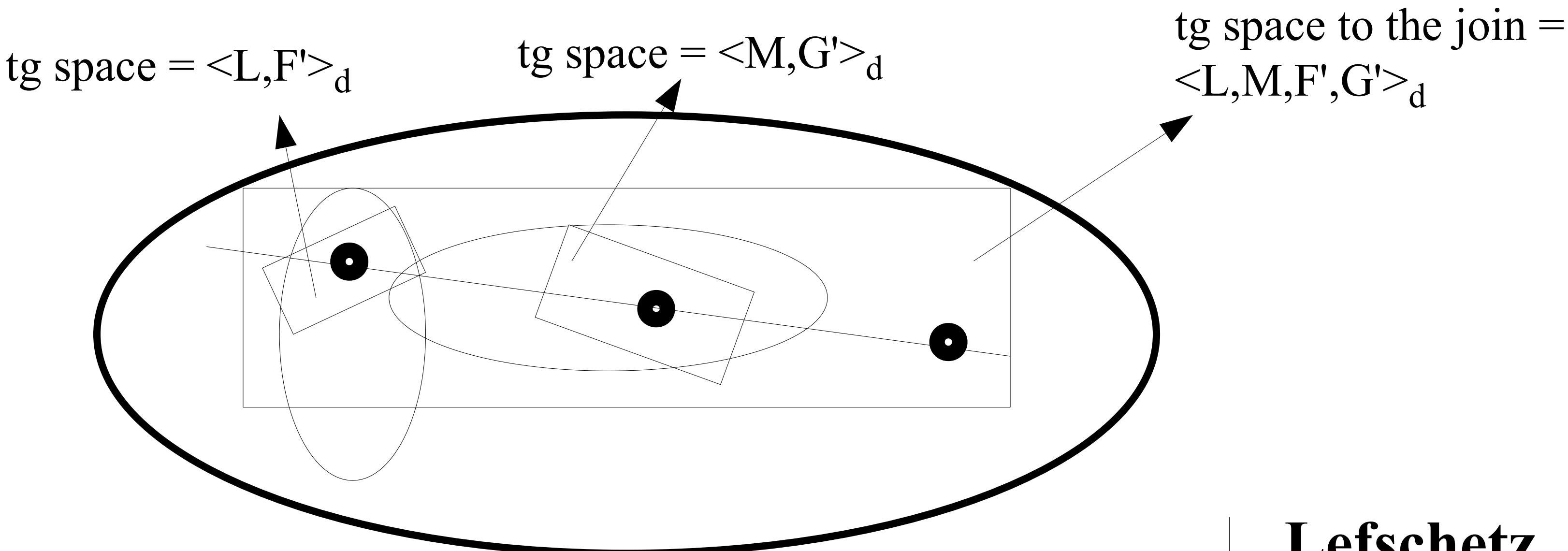
Forms splitting in a product MG', with deg(M) = b

Join Varieties

image of $\mathbb{P}(a) \times \mathbb{P}(d-a)$ under Segre's map +projection

**THEOREM** For any choice of $a,b < d$, a general plane curve of degree d contains a set of points which is complete intersection of type a,b.

**Terracini's Lemma** tg space to the join = join of tg spaces




**QUESTION:** for generic forms L,M,F',G' is $\langle L,M,F',G'\rangle_d = \text{Ring}_d$ ?

**YES!** ← **Lefschetz Hard Theorem** (Stanley)

# Generalization to $\mathbb{P}^3$

$F_d = LF' - MG'$

large degree

# FORGET IT!

$F_d$ cannot belong to $= \langle L,F' \rangle_d$,
deg(L),deg(F') < d,
because all of its curves are complete intersection of $F_d$

**NOETHER – LEFSCHETZ principle**

**not a 2x2 determinant**

Larger determinants?

Same problem

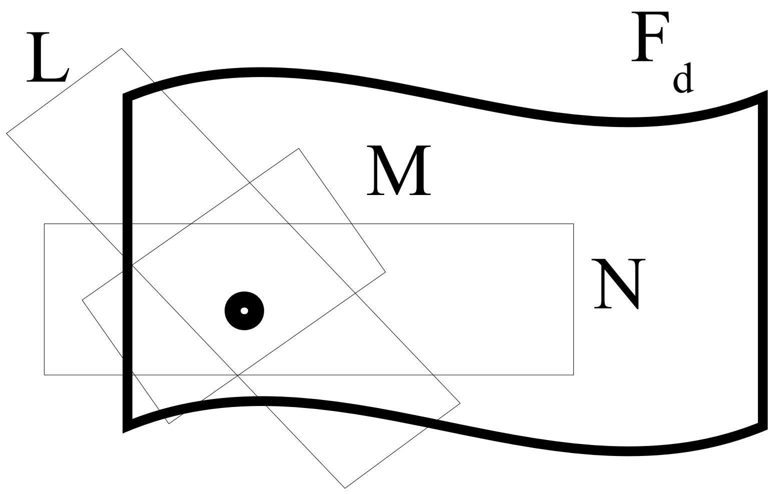
# Generalization to $\mathbb{P}^3$

**Problem**: find a decomposition of type

$F_d = AL + BM + CN$ with deg(L), deg(M), deg(N) < d

for a general form of degree d in 4 variables.

$F_d \in \langle L, M, N \rangle$




**e.g.**

deg(L)=deg(M)=deg(N)=1

done

On a general surface of degree >3, there are “few curves”,
but many sets of points!

**Problem**: find a decomposition of type

$F_d = AL + BM + CN$ with $\deg(L), \deg(M), \deg(N) < d$

for a general form of degree d in 4 variables.

⇧

Which complete intersection sets of points are there on a general surface of degree $d \gg 0$ in $\mathbb{P}^3$?

**assume** $\deg(L) \leq \deg(M) \leq \deg(N)$

**DEFINITION** We say that a triple (deg(L), deg(M), deg(N)) is **asymptotic** if for all $d > d_0$ a general surface of degree d contains a complete intersection set of points of type (deg(L), deg(M), deg(N))

**Example** (1,1,1) is asymptotic

**Problem**: find a decomposition of type
$F_d = AL + BM + CN$ with deg(L), deg(M), deg(N) < d
for a general form of degree d in 4 variables.

⇧

Which complete intersection sets of points are there on a general surface of degree $d \gg 0$ in $\mathbb{P}^3$?

**assume** $\deg(L) \le \deg(M) \le \deg(N)$

**Theorem** ( Carlini, --- , Geramita)

If $\deg(L) \le 4$, the triple (deg(L), deg(M), deg(N)) is asymptotic

If $\deg(L) > 6$, the triple (deg(L), deg(M), deg(N)) is not asymptotic

If $5 \le \deg(L) \le 6$, the asymptotic triples are:

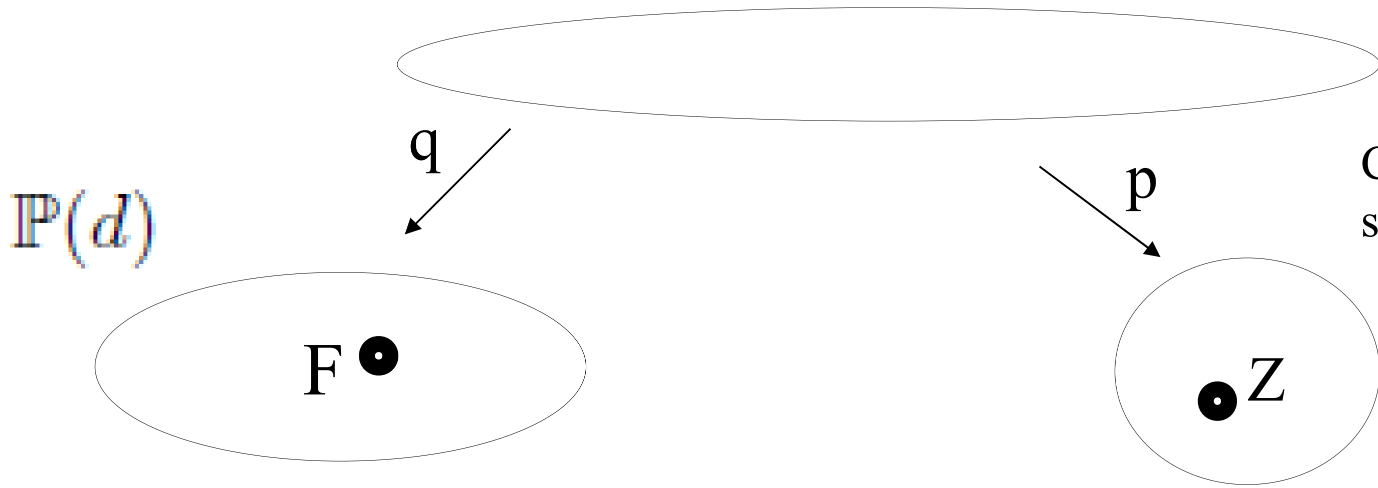
$\deg(L) = 5 \quad \deg(M) < 12$
$\deg(L) = 6 \quad \deg(M) < 8$

$\deg(L) = 5 \quad \deg(M) = 12 \quad \deg(N) = 12$
$\deg(L) = 6 \quad \deg(M) = 8 \quad \deg(N) = 8,9$

# BASIC CONSTRUCTION

for non-existence

$$I = \{(F,Z):\ Z \subseteq F\}$$




Complete intersections sets of points

general fiber of $p = \mathbb{P}(H^0\mathcal{I}_Z(d))$

known from the Hilbert function of Z

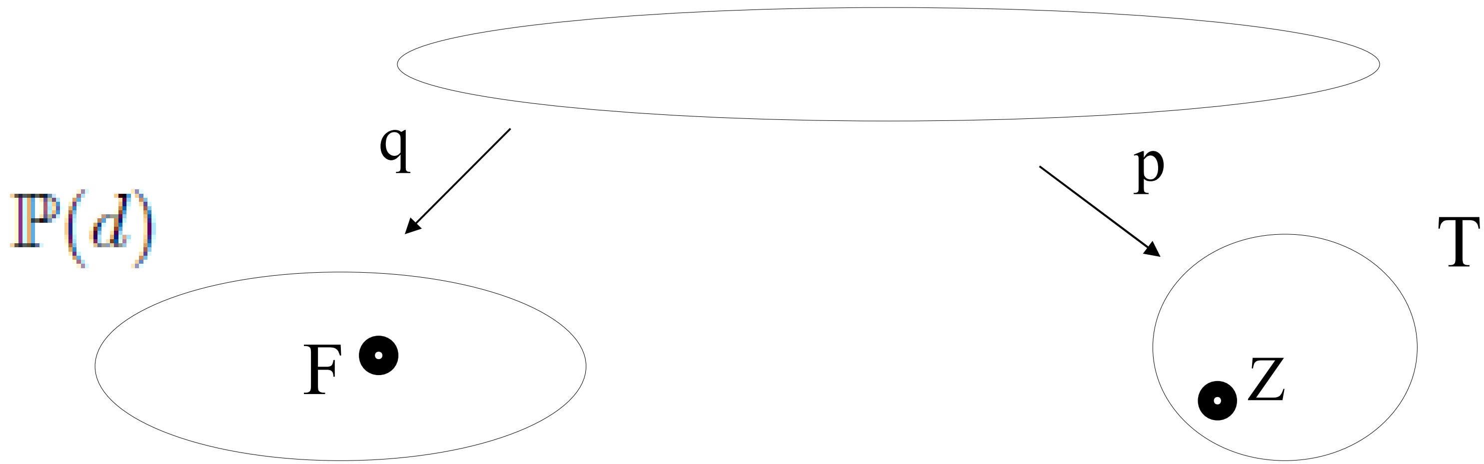
***Problem:*** is q DOMINANT?

if $\dim(I) < \dim(\mathbb{P}(d))$, return: NO

if $\dim(I) \geq \dim(\mathbb{P}(d))$, compute tg spaces

# GENERALIZING the BASIC CONSTRUCTION

$$I = \{(F,Z):\ Z \subseteq F\}$$




**T** = type of variety = irreducible component of some Hilbert scheme

$$\dim(I) = \dim(T) + (\dim(\mathbb{P}(d)) - H_Z(d))$$

***NB:*** if T parameterizes objects of dim > 0, no chance of ASYMPTOTIC positive answer

***Problem:*** is q DOMINANT? DOMINANT $\rightarrow$ $\dim(T) \geq H_Z(d)$

Back to the representation of forms F having a c.i. set of points Z of type a,b,c

$$0 \to \mathcal{O}(-a-b-c) \to \begin{matrix}\mathcal{O}(-a-b)\\ \oplus \\ \mathcal{O}(-a-c)\\ \oplus \\ \mathcal{O}(-b-c)\end{matrix} \xrightarrow{M} \begin{matrix}\mathcal{O}(-a)\\ \oplus \\ \mathcal{O}(-b)\\ \oplus \\ \mathcal{O}(-c)\end{matrix} \to \mathcal{I}_Z \to 0$$

F

$$\mathcal{O}(-d) \quad = \quad \mathcal{O}(-d)$$

rank 2 bundle on F

$M^{\vee}$ $$0 \to \mathcal{O}(-a-b-c) \to \mathcal{O}(-a) \oplus \mathcal{O}(-b) \oplus \mathcal{O}(-c) \oplus \mathcal{O}(-d) \oplus \to \mathcal{E} \to 0$$

$$0 \to \mathcal{O}_F \to \mathcal{E} \to \mathcal{I}_{Z,F}(a+b+c+d-4) \to 0$$

indeed $(LA+MB+NC)^2$ is the determinant of a skew-symmetric matrix

F = (LA + MB + NC) is a 4x4 PFAFFIAN

# Back to the representation of forms

$$
\begin{array}{ccccccccc}
 & & & \mathcal{O}(-a-b) & & \mathcal{O}(-a) & & & \\
 & & & \oplus & & \oplus & & & \\
0 \to & \mathcal{O}(-a-b-c) & \to & \mathcal{O}(-a-c) & \xrightarrow{M} & \mathcal{O}(-b) & \to & \mathcal{I}_Z & \to 0 \\
 & & & \oplus & & \oplus & & \uparrow & \\
 & & & \mathcal{O}(-b-c) & & \mathcal{O}(-c) & & & \\
 & & & & & \uparrow & & & \\
 & & & & & \mathcal{O}(-d) & = & \mathcal{O}(-d) &
\end{array}
$$

**N.B.** In $\mathbb{P}^3$ a representation is always possible, for $a = b = c = 1$: every surface contains one point!

$$
\deg(M) = \begin{pmatrix} 0 & d-1 & d-1 & d-1 \\ d-1 & 1 & 1 & 1 \\ d-1 & 1 & 1 & 1 \\ d-1 & 1 & 1 & 1 \end{pmatrix}
$$

**N.B.** In $\mathbb{P}^n$, $n \geq 4$, false in high degree.

# GENERAL PFAFFIANS

Z = arithmatically Gorenstein 0-dimensional set $\subseteq$ F

$$\begin{array}{ccccccccccc} & & & & & & \mathcal{O}(-d) & = & \mathcal{O}(-d) & & \\ & & & & & & \downarrow & & \downarrow \ \text{F} & & \\ 0 & \rightarrow & \mathcal{P}_0 & \rightarrow & \mathcal{P}_1 & \xrightarrow{M} & \mathcal{P}_2 & \rightarrow & \mathcal{I}_Z & \rightarrow 0 \end{array}$$

$\mathcal{P}_0$ = line bundle $\qquad$ $\mathcal{P}_1 \quad \mathcal{P}_2$ = free

F = pfaffian of a matrix N = $\underbrace{\begin{pmatrix} 0 & n \\ n^T & M \end{pmatrix}}_{t} \Big| \; t$

n = vector of forms of degree < d

Pfaffian representations of F $\Longleftrightarrow$ Find arithmatically Gorenstein 0-dimensional sets $\subseteq$ F

# SITUATION FOR PFAFFIAN SURFACES IN $\mathbb{P}^3$

A general surface F is the pfaffian of a skew-symmetric matrix when:

- Homogeneous pfaffians (n, M homogeneous of the same degree)
  - degree 1 (linear pfaffians)

    $\deg(F) \leq 15$ (Adler – Beauville - Schreyer)
  - degree 2 (quadratic pfaffians)

    $\deg(F) \leq 15$ (Faenzi)
  - higher degree

    $\deg(F) \leq 8$ ( --- , Faenzi)

Never asymptotic

F = pfaffian of a matrix $N = \begin{pmatrix} 0 & n \\ n^T & M \end{pmatrix}$ t

t

# SITUATION FOR PFAFFIAN SURFACES IN $\mathbb{P}^3$

A general surface F is the pfaffian of a skew-symmetric matrix when:

- Quasi-homogeneous pfaffians
  (n, M homogeneous of different degrees) ( --- , Faenzi)

  - M matrix of forms of degrees b > 1
    deg(F) ≤ 8

    Non asymptotic

  - M matrix of linear forms

    Some is asymptotic (depends on n)

    F = pfaffian of a matrix $N = \underbrace{\begin{pmatrix} 0 & n \\ n^T & M \end{pmatrix}}_{t}$ t

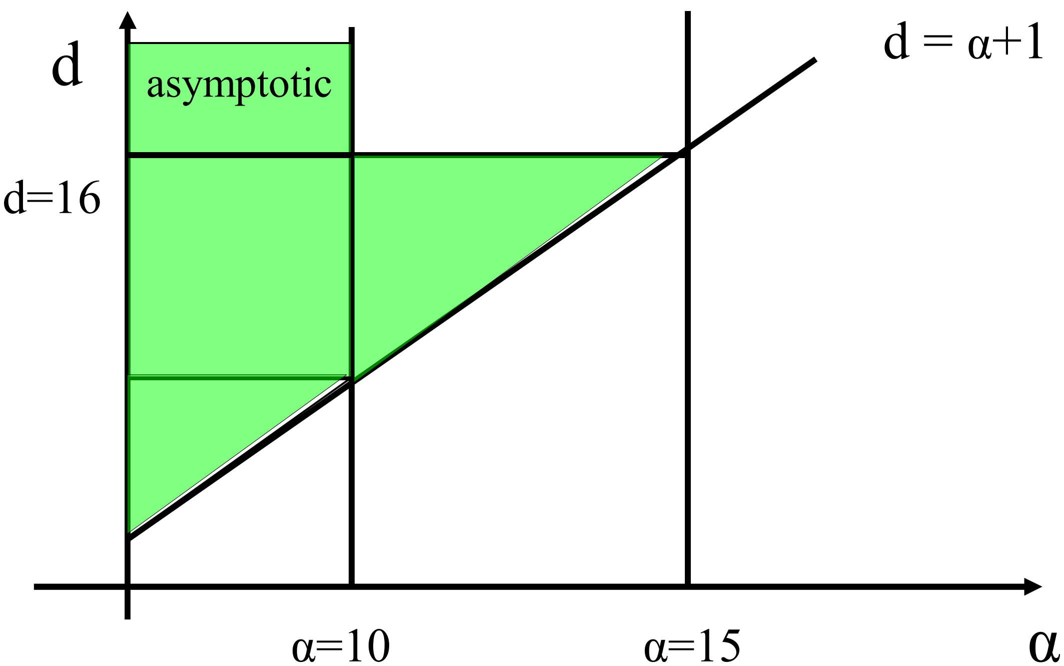
# SITUATION FOR PFAFFIAN SURFACES IN $\mathbb{P}^3$

- Quasi-homogeneous pfaffians
  (n, M homogeneous of different degrees)
  - M matrix of linear forms
    ( --- , Faenzi)

$$\begin{pmatrix} 0 & n \\ n^T & M \end{pmatrix}$$

t | t

t = 2□ - 2




degree matrix of M

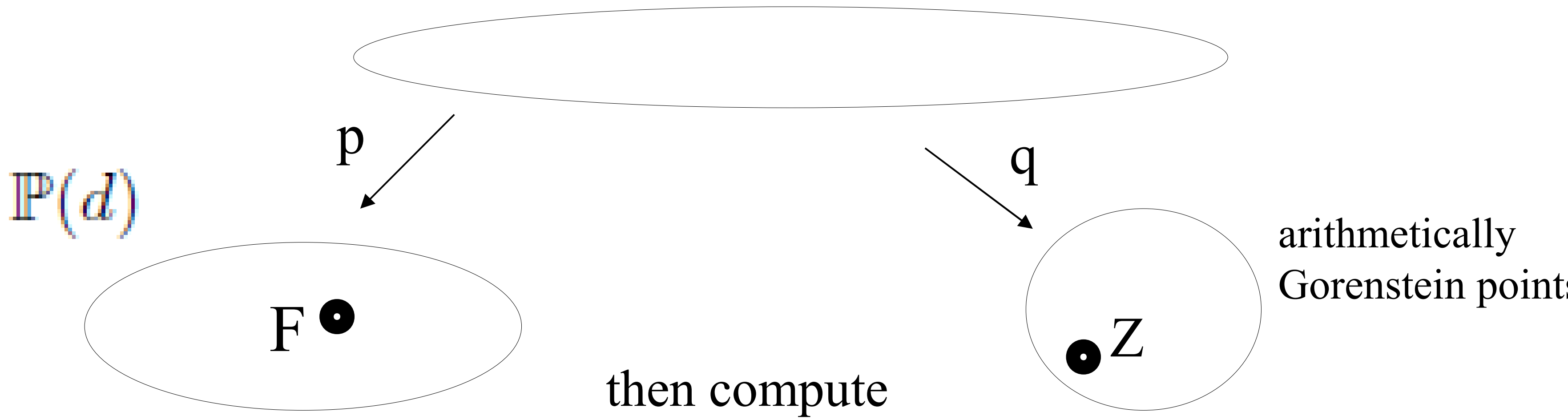
$$\begin{pmatrix} 0 & d-\alpha+2 & \dots & d-\alpha+2 \\ d-\alpha+2 & & & \\ \dots & & & \\ d-\alpha+2 & & & \end{pmatrix}$$

↓

linear matrix of size
(2α-3) x (2α-3)

HINT of proofs for non-existence

$I = \{(F,Z):\ Z \subseteq F\}$

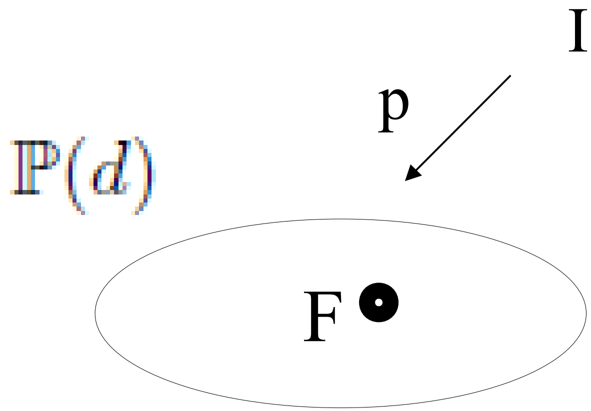



but BEWARE: for d = 16 dim(I) > dim( $\mathbb{P}(d)$ )
nevertheless p is not dominant

p has not max rank

REASON: Z determines on F a rank 2 bundle E with $\dim(H^0(E))$ large,
so Z moves in a high dimensional family on F
and p has general fibers of large dimension

# HINT of proofs



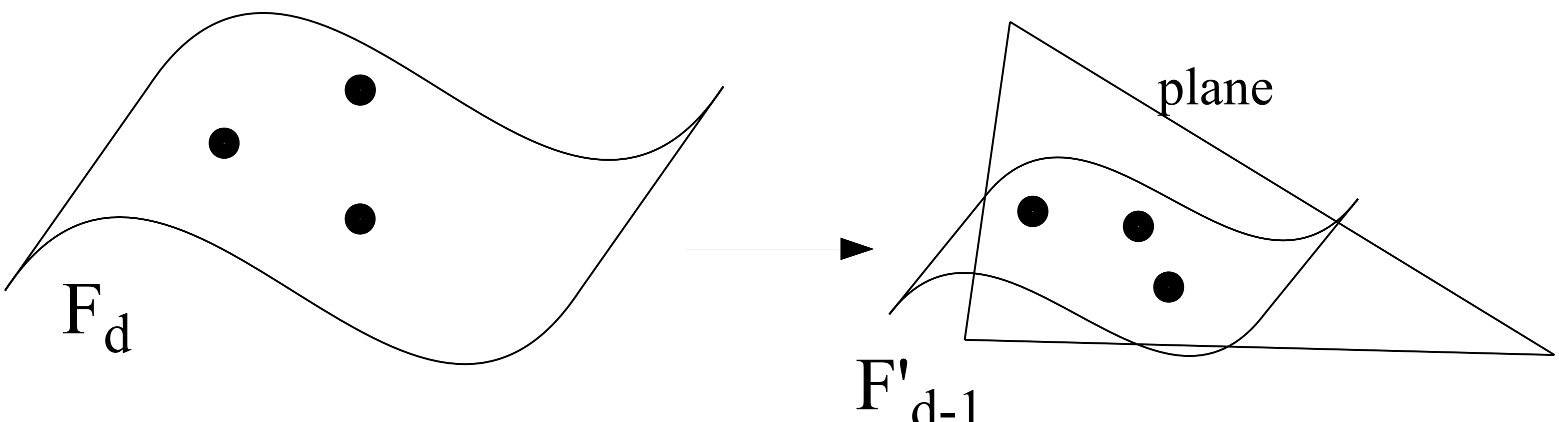

for existence

$$\dim(I) \geq \dim(\ \mathbb{P}(d)\ )$$

for sparse results, compute the dimension
of the tg space to p(I),
which is generated by submaximal pfaffians

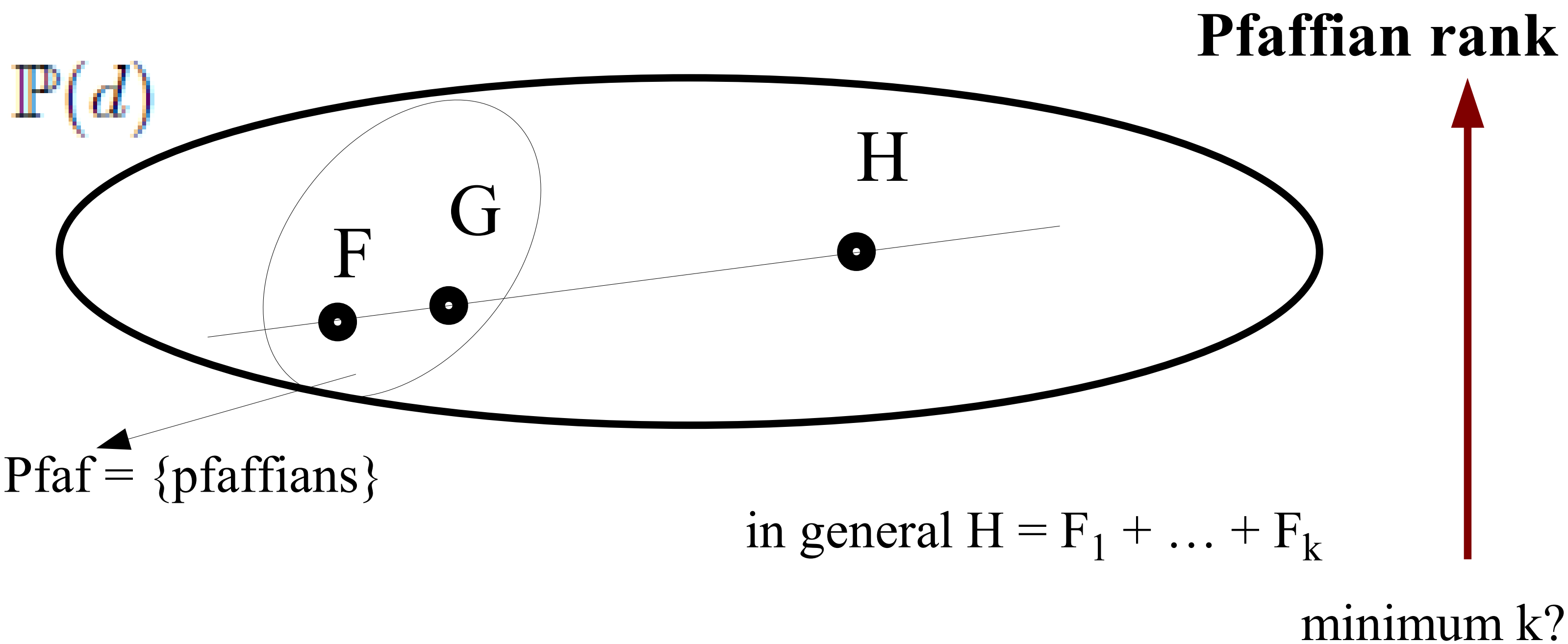
## For asymptotic results

Assume $H_Z(d) = \deg(Z)$




then use induction

## If the general surface is NOT pfaffian




e.g. H = F + G ⟹ **H ∈ Secant variety**

Sec(Pfaf) expected dimensions?

**TERRACINI'S LEMMA**

the tg space of the secant variety at H
is the span of the tg spaces to Pfaf at F,G

If the general surface is NOT pfaffian

**Pfaffian rank**

**Secant varieties of Pfaff**

do they have the expected dimensions?

**TERRACINI'S LEMMA**
the tg space of the secant variety at H
is the span of the tg spaces to Pfaf at F,G

study the Hilbert function of a union
of two arithmetically Gorenstein sets

**work in progress**

# thank you for your attention