Plane Curves as Pfaffians

Anita Buckley[1]

[1] Department of Mathematics
Faculty of Mathematics and Physics
University of Ljubljana
Slovenia

Workshop GeoLMI, Toulouse, France
November 19-20, 2009

Partly joint work with Tomaž Košir[1]

Outline

Notation

- k algebraically closed field
- $F(x_0, x_1, x_2)$ homogeneous polynomial of degree d
- C a smooth curve in $\mathbb{P}^2$ defined by F

Find a $2d \times 2d$ skew-symmetric matrix

$$A = \begin{bmatrix} 0 & L_{1\,2} & L_{1\,3} & \cdots & L_{1\,2d} \\ -L_{1\,2} & 0 & L_{2\,3} & \cdots & L_{2\,2d} \\ -L_{1\,3} & -L_{2\,3} & 0 & & \\ \vdots & \vdots & & \ddots & \vdots \\ -L_{1\,2d} & -L_{2\,2d} & & \cdots & 0 \end{bmatrix}$$

with linear forms $L_{ij} = a_{ij}^0 x_0 + a_{ij}^1 x_1 + a_{ij}^2 x_2$ such that

$$\operatorname{Pf} A(x_0, x_1, x_2) = c\, F(x_0, x_1, x_2) \;\text{ for some }\; c \in k, c \neq 0.$$

Definition: Pfaffian representation

Matrix A is called linear pfaffian representation of C.
Two pfaffian representations A and A' are equivalent if there exists $X \in \mathrm{GL}_{2d}(k)$ such that

$$A' = XAX^t.$$

- Its cokernel is a rank 2 vector bundle on C. Throughout the paper we identify vector bundles with locally free sheaves.

A locally free sheaf $\mathcal{E}$ of rank 2 is stable if for every invertible sheaf $\mathcal{E} \to \mathcal{F} \to 0$ holds

$$\deg \mathcal{F} > \frac{1}{2} \deg \mathcal{E}.$$

Replacing $>$ by $\geq$ defines semistable.

Definition: Pfaffian representation

Matrix A is called linear pfaffian representation of C.
Two pfaffian representations A and A' are equivalent if there exists $X \in \mathrm{GL}_{2d}(k)$ such that

$$A' = XAX^t.$$

- Its cokernel is a rank 2 vector bundle on C. Throughout the paper we identify vector bundles with locally free sheaves.

A locally free sheaf $\mathcal{E}$ of rank 2 is stable if for every invertible sheaf $\mathcal{E} \to \mathcal{F} \to 0$ holds

$$\deg \mathcal{F} > \frac{1}{2} \deg \mathcal{E}.$$

Replacing $>$ by $\geq$ defines semistable.

Definition: Pfaffian representation

Matrix A is called linear pfaffian representation of C.
Two pfaffian representations A and A' are equivalent if there exists $X \in \mathrm{GL}_{2d}(k)$ such that

$$A' = XAX^t.$$

- Its cokernel is a rank 2 vector bundle on C. Throughout the paper we identify vector bundles with locally free sheaves.

A locally free sheaf $\mathcal{E}$ of rank 2 is stable if for every invertible sheaf $\mathcal{E} \to \mathcal{F} \to 0$ holds

$$\deg \mathcal{F} > \frac{1}{2} \deg \mathcal{E}.$$

Replacing $>$ by $\geq$ defines semistable.

Definition: Determinantal representation

Study of pfaffian representations is strongly **related** to and **motivated** by determinantal representations. A linear determinantal representation of C is a $d \times d$ matrix of linear forms

$$M = x_0 M_0 + x_1 M_1 + x_2 M_2, \quad M_0, M_1, M_2 \in \mathrm{M}_d(k)$$

satisfying

$$\det M = c\,F, \ \ c \in k, \ c \neq 0.$$

Two determinantal representations M and M' are equivalent if there exist $X, Y \in \mathrm{GL}_d(k)$ such that

$$M' = XMY.$$

Determinantal representation $\leftrightarrow$ Cokernel line bundle

Theorem (Beauville, 2000)

Let C be a plane curve defined by a polynomial F of degree d and let L be a line bundle of degree $\frac{1}{2}d(d-1)$ on C with $H^0(C, L(-1)) = 0$. Then there exists a $d \times d$ linear matrix M with $\det M = F$ and an exact sequence

$$0 \to \bigoplus_{i=1}^{d} \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{M} \bigoplus_{i=1}^{d} \mathcal{O}_{\mathbb{P}^2} \to L \to 0. \quad (1)$$

Conversely, let M be a linear $d \times d$ matrix with $\det M = F$. Then its cokernel is a line bundle of degree $\frac{1}{2}d(d-1)$ and $H^0(C, \operatorname{Coker} M(-1)) = 0$.

Jacobian Variety

Corollary (Vinnikov, 1989)

All linear determinantal representations of F (up to equivalence) can be parametrised by the nonexceptional points on the Jacobian variety of C.

Analogy: parametrise all linear pfaffian representations by points in an open subset of the moduli space $M_C(2, K_C)$.

Jacobian Variety

Corollary (Vinnikov, 1989)

All linear determinantal representations of F (up to equivalence) can be parametrised by the nonexceptional points on the Jacobian variety of C.

Analogy: parametrise all linear pfaffian representations by points in an open subset of the moduli space $M_C(2, K_C)$.

Definition: Moduli Space

Definition

The moduli space $M_C(2, K_C)$ consists of semistable rank 2 vector bundles on C with canonical determinant.

- It is an irreducible, normal projective variety and for C of genus $g \geq 2$ it has dimension $3(g-1)$.

Definition: Moduli Space

Definition

The moduli space $M_C(2, K_C)$ consists of semistable rank 2 vector bundles on C with canonical determinant.

- It is an irreducible, normal projective variety and for C of genus $g \geq 2$ it has dimension $3(g-1)$.

Pfaffian representation $\leftrightarrow$ Cokernel rank 2 bundle

Theorem (Beauville, 2000)

Let C be a smooth plane curve defined by a polynomial F of degree d and let $\mathcal{E}$ be a rank 2 bundle on C with determinant $\mathcal{O}_C(d-1)$ and $H^0(C, \mathcal{E}(-1)) = 0$. Then there exists a $2d \times 2d$ skew-symmetric linear matrix A with $\operatorname{Pf} A = F$ and an exact sequence

$$0 \to \bigoplus_{i=1}^{2d} \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{A} \bigoplus_{i=1}^{2d} \mathcal{O}_{\mathbb{P}^2} \to \mathcal{E} \to 0. \tag{2}$$

Conversely, let A be a linear skew-symmetric $2d \times 2d$ matrix with $\operatorname{Pf} A = F$. Then its cokernel is a a rank 2 bundle with $\det \mathcal{E} \cong \mathcal{O}_C(d-1)$ and $H^0(C, \mathcal{E}(-1)) = 0$.

Moduli Space

Corollary

***All** linear pfaffian representations of C (up to equivalence) can be parametrised by the open set*
$M_C(2, K_C) - \{\mathcal{K} : h^0(C, \mathcal{K}) > 0\}$.

- An **explicit** construction of all representations (from the global sections of rank 2 vector bundles with certain properties) yields an **explicit** description of the moduli space.

Moduli Space

Corollary

All *linear pfaffian representations of C (up to equivalence) can be parametrised by the open set* $M_C(2, K_C) - \{\mathcal{K} : h^0(C, \mathcal{K}) > 0\}$.

- An **explicit** construction of all representations (from the global sections of rank 2 vector bundles with certain properties) yields an **explicit** description of the moduli space.

Determinantal representation $\leftrightarrow$ decomposable Pfaffian

- There are many **more** pfaffian than determinantal representations: every determinantal representation M induces a decomposable pfaffian representation $\begin{bmatrix} 0 & M \\ -M^t & 0 \end{bmatrix}$.

Note that the equivalence relation is well defined since

$$\begin{bmatrix} 0 & XMY \\ -(XMY)^t & 0 \end{bmatrix} = \begin{bmatrix} X & 0 \\ 0 & Y^t \end{bmatrix} \begin{bmatrix} 0 & M \\ -M^t & 0 \end{bmatrix} \begin{bmatrix} X^t & 0 \\ 0 & Y \end{bmatrix}.$$

Determinantal representation $\leftrightarrow$ decomposable Pfaffian

- There are many **more** pfaffian than determinantal representations: every determinantal representation M induces a decomposable pfaffian representation $\begin{bmatrix} 0 & M \\ -M^t & 0 \end{bmatrix}$.

Note that the equivalence relation is well defined since

$$\begin{bmatrix} 0 & XMY \\ -(XMY)^t & 0 \end{bmatrix} = \begin{bmatrix} X & 0 \\ 0 & Y^t \end{bmatrix} \begin{bmatrix} 0 & M \\ -M^t & 0 \end{bmatrix} \begin{bmatrix} X^t & 0 \\ 0 & Y \end{bmatrix}.$$

Show Picture

Rank of Pfaffian Representation

Lemma

For any $x \in C$ the corank of $A(x)$ equals 2.

Denote by $\mathrm{Pf}^{ij} A$ the pfaffian of the $(2d-2) \times (2d-2)$ skew-symmetric matrix obtained by removing the ith and jth rows and columns from A. Then

$$\frac{\partial F}{\partial x_k}(x) = \frac{1}{c} \sum_{i,j} a_{ij}^k \, \mathrm{Pf}^{ij} A(x).$$

If for some $x \in C$ all $2d-2$ pfaffian minors vanish, then x must be a singular point of F. Our F is smooth, thus $\operatorname{rank} A(x) \geq 2d-2$ for all $x \in C$. The rank of skew-symmetric matrices is even and $\det A = F^2 = 0$.

Rank of Pfaffian Representation

Lemma

For any $x \in C$ the corank of $A(x)$ equals 2.

Denote by $\mathrm{Pf}^{ij} A$ the pfaffian of the $(2d-2) \times (2d-2)$ skew-symmetric matrix obtained by removing the ith and jth rows and columns from A. Then

$$\frac{\partial F}{\partial x_k}(x) = \frac{1}{c} \sum_{i,j} a_{ij}^k \, \mathrm{Pf}^{ij} A(x).$$

If for some $x \in C$ all $2d-2$ pfaffian minors vanish, then x must be a singular point of F. Our F is smooth, thus $\operatorname{rank} A(x) \geq 2d-2$ for all $x \in C$. The rank of skew-symmetric matrices is even and $\det A = F^2 = 0$.

Pfaffian Adjoint

Definition

The pfaffian adjoint of A is the skew-symmetric matrix

$$\tilde{A} = \begin{bmatrix} 0 & & & \\ & \ddots & (-1)^{i+j}\,\mathrm{Pf}^{ij} A & \\ & & \ddots & \\ & & & 0 \end{bmatrix}.$$

- By analogy with determinants the following holds

$$\tilde{A} \cdot A = \mathrm{Pf}\, A \cdot \mathrm{Id}_{2d}.$$

Pfaffian Adjoint

Definition

The pfaffian adjoint of A is the skew-symmetric matrix

$$\tilde{A} = \begin{bmatrix} 0 & & & \\ & \ddots & (-1)^{i+j}\,\mathrm{Pf}^{ij} A & \\ & & \ddots & \\ & & & 0 \end{bmatrix}.$$

- By analogy with determinants the following holds

$$\tilde{A} \cdot A = \mathrm{Pf}\, A \cdot \mathrm{Id}_{2d}.$$

Construction of the Pfaffian Representation

Theorem

Let C be a smooth plane curve of degree d. To every rank 2 vector bundle $\mathcal{E}$ on C with properties

(i) $h^0(C, \mathcal{E}) = 2d$,

(ii) $H^0(C, \mathcal{E}(-1)) = 0$,

(iii) $\det \mathcal{E} = \bigwedge^2 \mathcal{E} = \mathcal{O}_C(d-1)$

we can assign a pfaffian representation A with cokernel $\mathcal{E}$. In particular, isomorphic bundles induce equivalent representations.

Construction: Proof

Choose a basis $\{s_1, \ldots, s_{2d}\}$ for $U = H^0(C, \mathcal{E})$ and define

$$C \ni x \overset{\psi}{\mapsto} \sum_{1 \le i < j \le 2d} (s_i(x) \wedge s_j(x))(E_{ij} - E_{ji}) = \begin{bmatrix} 0 & & & \\ & \ddots & s_i(x) \wedge s_j(x) & \\ & & \ddots & \\ & & & 0 \end{bmatrix}.$$

Construction: Proof

Since $s_i \wedge s_j \in \bigwedge^2 U$, by property (iii) the map ψ extends to

$$\Psi : \mathbb{P}^2 \longrightarrow \mathbb{P}(\bigwedge^2 U)$$

given by a linear system of plane curves of degree $d-1$. In coordinates it equals to a $2d \times 2d$ skew-symmetric matrix $B(x_0, x_1, x_2)$ with entries from the space of homogeneous polynomials of degree $d-1$. This means that

$$A = \frac{1}{F^{d-2}}\,\tilde{B}.$$

Canonical Form 1

Proposition

For every pfaffian representation $A = x_0A_0 + x_2A_2 + x_2A_2$ of C there exists a basis of k^{2d} in which A has the canonical form

$$A = x_1 \begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix} - x_2 \begin{bmatrix} D_1 & 0 & \cdots & 0 \\ 0 & D_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & D_d \end{bmatrix} + x_0A_0,$$

where

$$I = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ and } D_i = \begin{bmatrix} 0 & p_i \\ -p_i & 0 \end{bmatrix}.$$

Canonical Form 1: Proof

We can always assume that after a projective change of coordinates C intersects the line $L : x_0 = 0$ in distinct points $P_1 = (p_1, 1, 0), \dots, P_d = (p_d, 1, 0)$. By restricting to L, we obtain the pencil of skew-symmetric matrices

$$A_L = x_1 A_1 + x_2 A_2$$

with $\operatorname{Pf} A_L = F|_L = F(0, x_1, x_2) = \prod_{i=1}^{d}(x_1 - p_i x_2)$. Note that $\mathcal{E}(P_i)$ is the kernel=cokernel of $p_i A_1 + A_2$. Thus $\mathcal{E}(P_i),\ i = 1, \dots, d$ are 2-dimensional subspaces in k^{2d}. Condition $h^0(C, \mathcal{E}(-1)) = 0$ implies the following: the union of bases of the vector spaces $\mathcal{E}(P_i)$ span the whole space k^{2d}. In this basis A_L is equivalent to the canonical form above.

Canonical Form 2

Proposition

Another canonical form is

$$A = x_1 \begin{bmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{bmatrix} - x_2 \begin{bmatrix} 0 & D \\ -D & 0 \end{bmatrix} + x_0 A_0,$$

where D is the diagonal matrix $\{p_1, \ldots, p_d\}$.

- This canonical form is particularly useful since it naturally includes all the decomposable representations. The same canonical form was obtained by Lancaester and Rodman (2005), where canonical forms for matrix pairs were classified purely by the methods of linear algebra.

Canonical Form 2: Proof

The equivalence relation action $Q\,A\,Q^t$ of

$$Q = \begin{bmatrix} 1 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & -1 & & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & \cdots & & 1 & -1 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & \cdots & & 0 & 1 \end{bmatrix}$$

brings the first two matrices in the Canonical form 1 into the Canonical form 2.

Notation

The standard notation of vessels will be used (Ball and Vinnikov, 1999):

- move to affine coordinates $(x_0, x_1, x_2) \equiv (1, y_1, y_2)$,
- $\mathrm{Pf}(y_1\sigma_2 - y_2\sigma_1 + \gamma) = c\, f(y_1, y_2)$, where $\sigma_1, \sigma_2, \gamma$ are $2d \times 2d$ skew-symmetric matrices and $0 \neq c \in k$,
- $\mathcal{E}(y_1, y_2) := \mathrm{Coker}(y_1\sigma_2 - y_2\sigma_1 + \gamma) \cong \ker(y_1\sigma_2 - y_2\sigma_1 + \gamma)$.

Admissible Vectors

- $\lambda = (\lambda_1, \lambda_2)$ and $\mu = (\mu_1, \mu_2)$ distinct regular points on C.

For all $v_\lambda \in \mathcal{E}(\lambda)$, $u_\mu \in \mathcal{E}(\mu)$

$$v_\lambda^t(\lambda_1\sigma_2 - \lambda_2\sigma_1 + \gamma)u_\mu = 0 \ \text{ and } \ v_\lambda^t(\mu_1\sigma_2 - \mu_2\sigma_1 + \gamma)u_\mu = 0,$$

implies $(\lambda_1 - \mu_1)v_\lambda^t\sigma_2 u_\mu = (\lambda_2 - \mu_2)v_\lambda^t\sigma_1 u_\mu$. In other words, for any pair of complex parameters t_1, t_2,

$$K_{v_\lambda u_\mu} := \frac{1}{t_1(\lambda_1 - \mu_1) + t_2(\lambda_2 - \mu_2)} v_\lambda^t(t_1\sigma_1 + t_2\sigma_2)u_\mu.$$

is **constant** whenever the denominator is 0.

Admissible Vectors

Definition

The pair of vectors $v_\lambda \in \mathcal{E}(\lambda)$, $u_\mu \in \mathcal{E}(\mu)$ is called admissible if $K_{v_\lambda u_\mu}$ is not 0.

For an admissible pair of vectors write:

- $\tilde{\gamma} = \gamma - \frac{1}{2\,K_{v_\lambda u_\mu}}\,\sigma_1 u_\mu \wedge \sigma_2 v_\lambda + \frac{1}{2\,K_{v_\lambda u_\mu}}\,\sigma_2 u_\mu \wedge \sigma_1 v_\lambda$
- $\bar{\gamma} = \gamma + \rho\,\sigma_2 v_\lambda \wedge \sigma_1 v_\lambda$, for arbitrary constant $\rho \neq 0$

which are clearly skew–symmetric matrices,

Elementary Transformations of $y_1\sigma_2 - y_2\sigma_1 + \gamma$

Definition

The Type I elementary transformation $y_1\sigma_2 - y_2\sigma_1 + \tilde{\gamma}$ based on the admissible vectors $v_\lambda \in \mathcal{E}(\lambda)$, $u_\mu \in \mathcal{E}(\mu)$,
The Type II elementary transformation $y_1\sigma_2 - y_2\sigma_1 + \bar{\gamma}$ based on $v_\lambda \in \mathcal{E}(\lambda)$ and the constant $\rho \neq 0$.

Theorem

$y_1\sigma_2 - y_2\sigma_1 + \tilde{\gamma}$ and $y_1\sigma_2 - y_2\sigma_1 + \bar{\gamma}$ are pfaffian representations of C since
$\mathrm{Pf}(y_1\sigma_2 - y_2\sigma_1 + \tilde{\gamma}) = \mathrm{Pf}(y_1\sigma_2 - y_2\sigma_1 + \gamma) = \mathrm{Pf}(y_1\sigma_2 - y_2\sigma_1 + \bar{\gamma})$.

The Inverse of Elementary Transformation

The fact that $v_\lambda \in \tilde{\mathcal{E}}(\mu),\ u_\mu \in \tilde{\mathcal{E}}(\lambda)$ and $v_\lambda \in \bar{\mathcal{E}}(\lambda)$ implies the following

Corollary

The Type I elementary transformation of $y_1\sigma_2 - y_2\sigma_1 + \tilde{\gamma}$ based on $u_\mu \in \tilde{\mathcal{E}}(\lambda),\ v_\lambda \in \tilde{\mathcal{E}}(\mu)$ brings us back to $y_1\sigma_2 - y_2\sigma_1 + \gamma$. The same way the Type II elementary transformation of $y_1\sigma_2 - y_2\sigma_1 + \bar{\gamma}$ based on $v_\lambda \in \bar{\mathcal{E}}(\lambda)$ and $-\rho$ brings us back to $y_1\sigma_2 - y_2\sigma_1 + \gamma$.

The Type I and II elementary transformations are special rank 2 cases of "the concrete interpolation problem for meromorphic bundle maps" studied by Ball and Vinnikov, 1999.

The Inverse of Elementary Transformation

The fact that $v_\lambda \in \tilde{\mathcal{E}}(\mu),\ u_\mu \in \tilde{\mathcal{E}}(\lambda)$ and $v_\lambda \in \bar{\mathcal{E}}(\lambda)$ implies the following

Corollary

The Type I elementary transformation of $y_1\sigma_2 - y_2\sigma_1 + \tilde{\gamma}$ based on $u_\mu \in \tilde{\mathcal{E}}(\lambda),\ v_\lambda \in \tilde{\mathcal{E}}(\mu)$ brings us back to $y_1\sigma_2 - y_2\sigma_1 + \gamma$. The same way the Type II elementary transformation of $y_1\sigma_2 - y_2\sigma_1 + \bar{\gamma}$ based on $v_\lambda \in \bar{\mathcal{E}}(\lambda)$ and $-\rho$ brings us back to $y_1\sigma_2 - y_2\sigma_1 + \gamma$.

The Type I and II elementary transformations are special rank 2 cases of "the concrete interpolation problem for meromorphic bundle maps" studied by Ball and Vinnikov, 1999.

Dfinition: Elem. Transf. of Vector Bundles

Definition (Maruyama, 1973; Abe, 2007)

Let $\mathcal{E}$ be a rank 2 vector bundle over C. Take an effective reduced divisor Z on C and consider the canonical surjection

$$\mathcal{E} \to k(Z) \to 0,$$

where $k(Z)$ is a skyscraper sheaf at Z, i.e. rank 1 $\mathcal{O}_Z$–module. Its kernel is a rank 2 vector bundle on C called the elementary transformation of $\mathcal{E}$ at Z. We denote it by $\mathcal{E}' = \text{elem}_Z(\mathcal{E})$.

The Inverse of Elementary Transformation

There exists a skyscraper sheaf $k(Z)'$ that fits into the commutative diagram

$$\begin{array}{ccccc} \mathcal{E} \otimes \mathcal{O}_C(-Z) & & & & \\ \downarrow g & & & & \\ \mathcal{E}' & \xrightarrow{e} & \mathcal{E} & \rightarrow & k(Z) \; . \\ \downarrow & & & & \\ k(Z)' & & & & \end{array}$$

Up to tensoring line bundles, i.e. on the level of ruled surfaces, these two elementary transformations are inverse to each other.

Ruled Surface ≡ Rank 2 V.B. ≡ Normal Scroll

On C it is equivalent to consider:

(a) Ruled surface $\pi : S \to C$ together with a base-point-free unisecant complete linear system $|H|$;

(b) Rank 2 vector bundle $\mathcal{E}$ over C for which $S = \mathbb{P}\mathcal{E}$ and $\mathcal{E} \cong \pi_* \mathcal{O}_{\mathbb{P}\mathcal{E}}(H)$;

(c) Linearly normal scroll R obtained as the image of the birational map $\phi_H : S \to R \subset \mathbb{P}^N$ defined by $|H|$.

Ruled Surface $\equiv$ Rank 2 V.B. $\equiv$ Normal Scroll

On C it is equivalent to consider:

(a) Ruled surface $\pi : S \to C$ together with a base-point-free unisecant complete linear system $|H|$;

(b) Rank 2 vector bundle $\mathcal{E}$ over C for which $S = \mathbb{P}\mathcal{E}$ and $\mathcal{E} \cong \pi_* \mathcal{O}_{\mathbb{P}\mathcal{E}}(H)$;

(c) Linearly normal scroll R obtained as the image of the birational map $\phi_H : S \to R \subset \mathbb{P}^N$ defined by $|H|$.

Ruled Surface $\equiv$ Rank 2 V.B. $\equiv$ Normal Scroll

On C it is equivalent to consider:

(a) Ruled surface $\pi : S \to C$ together with a base-point-free unisecant complete linear system $|H|$;

(b) Rank 2 vector bundle $\mathcal{E}$ over C for which $S = \mathbb{P}\mathcal{E}$ and $\mathcal{E} \cong \pi_* \mathcal{O}_{\mathbb{P}\mathcal{E}}(H)$;

(c) Linearly normal scroll R obtained as the image of the birational map $\phi_H : S \to R \subset \mathbb{P}^N$ defined by $|H|$.

Ruled Surface $\equiv$ Rank 2 V.B. $\equiv$ Normal Scroll

Analogously we can define elementary transformation at a point $x \in C$ on each of the above:

(a) On the ruled surface S we choose a point $s \in \pi^{-1}(x)$. Denote by B the blow-up of S at s. By Castelnuovo theorem we can contract the starting fibre $\pi^{-1}(x)$ in B and obtain a new ruled surface $\pi' : S' \to C$;

(b) $\mathcal{E}' = \mathrm{elem}_{\{x\}}(\mathcal{E})$ at the divisor $\{x\}$ on C;

(c) Pick a point $r = \phi_H(s)$ on the scroll R such that $\pi(s) = x$. Projection from r yields a scroll $R' \subset \mathbb{P}^{N-1}$.

Ruled Surface ≡ Rank 2 V.B. ≡ Normal Scroll

Analogously we can define elementary transformation at a point $x \in C$ on each of the above:

(a) On the ruled surface S we choose a point $s \in \pi^{-1}(x)$. Denote by B the blow-up of S at s. By Castelnuovo theorem we can contract the starting fibre $\pi^{-1}(x)$ in B and obtain a new ruled surface $\pi' : S' \to C$;

(b) $\mathcal{E}' = \text{elem}_{\{x\}}(\mathcal{E})$ at the divisor $\{x\}$ on C;

(c) Pick a point $r = \phi_H(s)$ on the scroll R such that $\pi(s) = x$. Projection from r yields a scroll $R' \subset \mathbb{P}^{N-1}$.

Ruled Surface $\equiv$ Rank 2 V.B. $\equiv$ Normal Scroll

Analogously we can define elementary transformation at a point $x \in C$ on each of the above:

(a) On the ruled surface S we choose a point $s \in \pi^{-1}(x)$. Denote by B the blow-up of S at s. By Castelnuovo theorem we can contract the starting fibre $\pi^{-1}(x)$ in B and obtain a new ruled surface $\pi' : S' \to C$;

(b) $\mathcal{E}' = \text{elem}_{\{x\}}(\mathcal{E})$ at the divisor $\{x\}$ on C;

(c) Pick a point $r = \phi_H(s)$ on the scroll R such that $\pi(s) = x$. Projection from r yields a scroll $R' \subset \mathbb{P}^{N-1}$.

Pfaffian Representation $\leftrightarrow$ Cokernel Bundle

Theorem

Let C be defined by $F = \mathrm{Pf}(x_1\sigma_2 - x_2\sigma_1 + x_0\gamma)$ and let $x_1\sigma_2 - x_2\sigma_1 + x_0\tilde{\gamma},\ x_1\sigma_2 - x_2\sigma_1 + x_0\bar{\gamma}$ be elementary transformations of Type I and II respectively. Denote by $\mathcal{E}(x), \tilde{\mathcal{E}}(x), \bar{\mathcal{E}}(x)$ the corresponding cokernels. The relating morphisms P, S, T, R, Q can be expressed by elementary transformations of vector bundles,

$$
\begin{array}{c}
\tilde{\mathcal{E}} \\
S\uparrow \ \ R^t\downarrow \\
\mathcal{E}' \\
P\uparrow \ \ T^t\downarrow \\
\mathcal{E}
\end{array}
\quad \text{and} \quad
Q\left\uparrow
\begin{array}{c}
\bar{\mathcal{E}} \\
\downarrow \\
\mathcal{E}'' \\
\uparrow \\
\mathcal{E}
\end{array}\right. .
$$

Matrices with rational elements:

$$T(x) = \mathrm{Id} + \frac{x_0}{K_{v_\lambda u_\mu}\left(t_1(x_1 - \lambda_1 x_0) + t_2(x_2 - \lambda_2 x_0)\right)}(t_1\sigma_1 + t_2\sigma_2)u_\mu v_\lambda^t,$$

$$S(x) = \mathrm{Id} + \frac{x_0}{K_{v_\lambda u_\mu}\left(t_1(x_1 - \lambda_1 x_0) + t_2(x_2 - \lambda_2 x_0)\right)}u_\mu v_\lambda^t(t_1\sigma_1 + t_2\sigma_2),$$

$$P(x) = \mathrm{Id} + \frac{x_0}{K_{v_\lambda u_\mu}\left(t_1(x_1 - \mu_1 x_0) + t_2(x_2 - \mu_2 x_0)\right)}v_\lambda u_\mu^t(t_1\sigma_1 + t_2\sigma_2),$$

$$R(x) = \mathrm{Id} + \frac{x_0}{K_{v_\lambda u_\mu}\left(t_1(x_1 - \mu_1 x_0) + t_2(x_2 - \mu_2 x_0)\right)}(t_1\sigma_1 + t_2\sigma_2)v_\lambda u_\mu^t$$

$$\text{and } Q(x) = \mathrm{Id} + \frac{2\rho x_0}{t_1(x_1 - \lambda_1 x_0) + t_2(x_2 - \lambda_2 x_0)}v_\lambda v_\lambda^t(t_1\sigma_1 + t_2\sigma_2).$$

- $S(x)P(x)\,\mathcal{E}(x) = \tilde{\mathcal{E}}(x)$ and $Q(x)\,\mathcal{E}(x) = \bar{\mathcal{E}}(x)$.

Show Picture

Bridgeing Pfaffian Representations

Theorem

From any given pfaffian representation of C we can build all the nonequivalent pfaffian representations of C by finite sequences of Type I and Type II elementary transformations.

The idea of the proof 1

Step 1: bridge the cokernel with a **decomposable** vector bundle by applying a finite number of Type II elementary transformations.

A finite sequence of m Type II elementary transformations by recursion yields a new representation $x_1\sigma_2 - x_2\sigma_1 + x_0\gamma_m$, where $\gamma_m = \gamma + \sum_{j=1}^{m} \rho_j \, \sigma_2 v_{\lambda^j} \wedge \sigma_1 v_{\lambda^j}$. The above constants $\rho_j \in k$ and points $\lambda^j \in C$ are arbitrary and $v_{\lambda^j} \in \mathcal{E}_{j-1}(\lambda^j) := \mathrm{Coker}(\lambda^j_1\sigma_2 - \lambda^j_2\sigma_1 + \lambda^j_0\gamma_{j-1})$ with $\gamma_0 = \gamma$.

Since every union $\{\mathcal{E}_j(x)\}_{x\in C}$ **spans the whole** k^{2d}, we can (by suitable choices of v_{λ^j}) generate enough independent rank 2 matrices $\sigma_2 v_{\lambda^j} \wedge \sigma_1 v_{\lambda^j}$ whose linear combination will yield a decomposable matrix γ_m.

The idea of the proof 2

Step 2: bridge any two decomposable cokernels by applying a finite number of Type I elementary transformations.

- decomposable cokernel bundles $\equiv$
- decomposable pfaffian representations $\equiv$
- determinantal representations

Vinnikov, 1990: any two determinantal representations can be bridged by a a finite sequence of elementary transformations.
Induction: Type I elementary transformation based on

$$\begin{bmatrix} v_n \\ 0 \end{bmatrix} \in {}_d\mathcal{E}_{n-1}(\lambda^n), \quad \begin{bmatrix} 0 \\ u_n \end{bmatrix} \in {}_d\mathcal{E}_{n-1}(\mu^n).$$

The idea of the proof 2

Step 2: bridge any two decomposable cokernels by applying a finite number of Type I elementary transformations.

- decomposable cokernel bundles $\equiv$
- decomposable pfaffian representations $\equiv$
- determinantal representations

Vinnikov, 1990: any two determinantal representations can be bridged by a a finite sequence of elementary transformations.
Induction: Type I elementary transformation based on

$$\begin{bmatrix} v_n \\ 0 \end{bmatrix} \in {}_d\mathcal{E}_{n-1}(\lambda^n), \begin{bmatrix} 0 \\ u_n \end{bmatrix} \in {}_d\mathcal{E}_{n-1}(\mu^n).$$

The idea of the proof 2

Step 2: bridge any two decomposable cokernels by applying a finite number of Type I elementary transformations.

- decomposable cokernel bundles $\equiv$
- decomposable pfaffian representations $\equiv$
- determinantal representations

Vinnikov, 1990: any two determinantal representations can be bridged by a a finite sequence of elementary transformations. Induction: Type I elementary transformation based on

$$\begin{bmatrix} v_n \\ 0 \end{bmatrix} \in {}_d\mathcal{E}_{n-1}(\lambda^n), \begin{bmatrix} 0 \\ u_n \end{bmatrix} \in {}_d\mathcal{E}_{n-1}(\mu^n).$$

Introduction

Definition

A nonsingular plane quartic C is a non-hyperelliptic genus 3 curve embedded by its canonical linear system $|K_C|$.

The moduli space $M_C(2, \mathcal{O}_C(1)) \cong M_C(2, \mathcal{O}_C)$ can be embeded as a **Coble quartic hypersurface** in $\mathbb{P}^7$ with singularities along the 3–dimensional **Kummer variety** $\mathcal{K}_C$.

- Using the canonical pfaffian representations of C, we can explicitly parametrise $M_C(2, \mathcal{O}_C(1)) \setminus \Theta_{2,\mathcal{O}_C(1)}$.

Introduction

Definition

A nonsingular plane quartic C is a non-hyperelliptic genus 3 curve embedded by its canonical linear system $|K_C|$.

The moduli space $M_C(2, \mathcal{O}_C(1)) \cong M_C(2, \mathcal{O}_C)$ can be embeded as a **Coble quartic hypersurface** in $\mathbb{P}^7$ with singularities along the 3–dimensional **Kummer variety** $\mathcal{K}_C$.

- Using the canonical pfaffian representations of C, we can explicitly parametrise $M_C(2, \mathcal{O}_C(1)) \setminus \Theta_{2,\mathcal{O}_C(1)}$.

Introduction

Definition

A nonsingular plane quartic C is a non-hyperelliptic genus 3 curve embedded by its canonical linear system $|K_C|$.

The moduli space $M_C(2, \mathcal{O}_C(1)) \cong M_C(2, \mathcal{O}_C)$ can be embeded as a **Coble quartic hypersurface** in $\mathbb{P}^7$ with singularities along the 3–dimensional **Kummer variety** $\mathcal{K}_C$.

- Using the canonical pfaffian representations of C, we can explicitly parametrise $M_C(2, \mathcal{O}_C(1)) \setminus \Theta_{2,\mathcal{O}_C(1)}$.

Even Theta Characteristic

Definition

An even theta characteristic of C is a line bundle $\mathcal{L}_\vartheta$ with the property

$$\mathcal{L}_\vartheta^{\otimes 2} \cong \omega_C \cong \mathcal{O}_C(1) \text{ and } \dim H^0(C, \mathcal{L}_\vartheta) \text{is even.}$$

- Dolgachev: There are exactly 36 even theta characteristics on a smooth plane quartic, all with $\dim H^0(C, \mathcal{L}_\vartheta) = 0$.

Even Theta Characteristic

Definition

An even theta characteristic of C is a line bundle $\mathcal{L}_\vartheta$ with the property

$$\mathcal{L}_\vartheta^{\otimes 2} \cong \omega_C \cong \mathcal{O}_C(1) \text{ and } \dim H^0(C, \mathcal{L}_\vartheta) \text{is even.}$$

- Dolgachev: There are exactly 36 even theta characteristics on a smooth plane quartic, all with $\dim H^0(C, \mathcal{L}_\vartheta) = 0$.

Even Theta Characteristic $\equiv$ Symmetric Determinantal Representation

For a line bundle $\mathcal{L}$ on a nonsingular plane quartic C with $H^0(C, \mathcal{L}) = 0$ the following are equivalent:

- $\mathcal{L}$ is an even theta characteristics on C,
- $\mathcal{L} \cong \mathcal{L}^{-1} \otimes \mathcal{O}_C(1)$,
- $\mathcal{L} = \text{Coker}\, M \otimes \mathcal{O}_C(-1)$ where M is a **symmetric** determinantal representation of C with the property $\text{Coker}\, M \cong \text{Coker}\, M^t$.

Example

$$27x_0^3x_1 - 432x_0x_1^3 - x_1^4 - 72\,107^{1/3}x_0^2x_1x_2 - 9\,107^{1/3}x_0x_1^2x_2 + 81\,107^{-1/3}x_0^2x_2^2 - 108x_0x_2^3 - 27x_1x_2^3$$

$$M_\vartheta = x_1\,\mathrm{Id}_4 - x_2\,\mathrm{Diag}\left[0, -3, 3(-1)^{1/3}, -3(-1)^{2/3}\right] + x_0\begin{bmatrix} 4 & -24.296` & 23.685`+0.336`i & -23.685`+0.336`i \\ & \frac{428}{3}-107^{1/3} & -141.449`+2.004`i & 141.449`+2.004`i \\ & & \frac{428}{3}-107^{1/3}(-1)^{2/3} & -145.099` \\ & & & \frac{428}{3}+107^{1/3}(-1)^{1/3} \end{bmatrix}$$

Definition: the Scorza Map

Definition

The Scorza map between plane quartics

$$F \mapsto \text{the Clebsch covariant quartic } S(F)$$

is

$$F \mapsto \text{polar cubic } P_x(F) \text{ at } x \in \mathbb{P}^2 \mapsto \text{Aronhold invariant}(P_x(F)) .$$

Note that in this notation the coefficients w_{ijk} of the cubic $P_{(x_0,x_1,x_2)}(F)$ are linear in x_0, x_1, x_2.

Definition: the Scorza Map

Definition

The Scorza map between plane quartics

$$F \mapsto \text{the Clebsch covariant quartic } S(F)$$

is

$$F \mapsto \text{polar cubic } P_x(F) \text{ at } x \in \mathbb{P}^2 \mapsto \text{Aronhold invariant}(P_x(F)) \, .$$

Note that in this notation the coefficients w_{ijk} of the cubic $P_{(x_0,x_1,x_2)}(F)$ are linear in x_0, x_1, x_2.

The Scorza Map

Theorem (Dolgachev and Kanev, 1993)

The curve $S(F)$ carries a unique even theta characteristic ϑ, more precisely, the Scorza map

$$F \mapsto (S(F), \vartheta)$$

is an injective birational isomorphism and the natural projection to the first component is an unramified covering of degree 36.

Aronhold Invariant

Ottaviani, 2009: The Aronhild invariant evaluated in

$$w_{000}x^3 + w_{111}y^3 + w_{222}z^3 + 6w_{012}xyz + 3w_{001}x^2y + 3w_{002}x^2z + 3w_{011}xy^2 + 3w_{022}xz^2 + 3w_{112}y^2z + 3w_{122}yz^2$$

equals Pf Ar of the Aronhold pfaffian representation

$$Ar = \begin{bmatrix} 0 & w_{222} & -w_{122} & 0 & w_{112} & 0 & w_{022} & -w_{012} \\ & 0 & w_{022} & w_{122} & -w_{012} & -w_{022} & 0 & w_{002} \\ & & 0 & -w_{112} & 0 & w_{012} & -w_{002} & 0 \\ & & & 0 & -w_{111} & 0 & -w_{012} & w_{011} \\ & & & & 0 & -w_{011} & w_{001} & 0 \\ & & & & & 0 & w_{002} & -w_{001} \\ & & & & & & 0 & w_{000} \\ & & & & & & & 0 \end{bmatrix}.$$

Aronhold Bundles

Pauly, 2002:

- The Aronhold bundles (cokernels of Aronhold representations) are in 1-to-1 correspondence with the 288 unordered Aronhold sets of bitangents on C.

They are stable (thus indecomposable) rank 2 bundle with canonical determinant $\mathcal{O}_C(1)$

Aronhold Bundles

Pauly, 2002:

- The Aronhold bundles (cokernels of Aronhold representations) are in 1-to-1 correspondence with the 288 unordered Aronhold sets of bitangents on C.

They are stable (thus indecomposable) rank 2 bundle with canonical determinant $\mathcal{O}_C(1)$

Aronhold Bundle $\leftrightarrow$ Theta Characteristic

Proposition

From the Aronhold pfaffian representation of $S(F)$ it is possible to ***explicitly*** *recover the unique theta characteristic on $S(F)$.*

Proof

The Scorza correspondence is

$$R_\vartheta := \{(\lambda, \mu) \in S(F) \times S(F) \ : \ h^0(\vartheta + \lambda - \mu) > 0\}.$$

- $\lambda \, R_\vartheta \, \mu$ iff $v^t \, M_\vartheta(x) \, u \equiv 0$ for all $v \in \operatorname{Coker} M_\vartheta(\lambda), \ u \in \operatorname{Coker} M_\vartheta(\mu), \ x \in \mathbb{P}^2$.
- $\lambda, \mu \in S(F)$ are related if the second polar $P_{\lambda,\mu}(F) = g_i^2$ for some $i = 1, 2, 3$ such that $P_\lambda(F) = g_1^3 + g_2^3 + g_3^3$.

Example

The Scorza map sends $F = x^4 + x^3y - y^4 - yz^3 + 107^{1/3}xy^2z$ to $S(F) = \mathrm{Pf}[Ar]$ given in our first Example. We will compute the unique theta characteristic on $S(F)$. We calculate in `Wolfram Mathematica` to precision 10^{-10}.

Example

For $\lambda = (1, 0, \frac{3}{4}107^{-1/3}) \in S(F)$ we get $P_\lambda(F) = g_1^3 + g_2^3 + g_3^3$ for

$$\begin{aligned} g_1 &= (4x + y)(-0.198\text{`} - 0.344\text{`}i), \\ g_2 &= (0.002\text{`} - 2.089\text{`}i)y + (-0.181\text{`} + 0.104\text{`}i)z, \\ g_3 &= (0.002\text{`} + 2.089\text{`}i)y + (-0.181\text{`} - 0.104\text{`}i)z, \end{aligned}$$

which is explicitly obtained from the equality $\det \operatorname{Hess}(P_\lambda(F)) = g_1 g_2 g_3$.

Example

The intersections

$$\mu^1 = g_2 \cap g_3 = (1, 0, 0),$$
$$\mu^2 = g_1 \cap g_3 = (1, -4, -20.034` + 34.609`i),$$
$$\mu^3 = g_1 \cap g_2 = (1, -4, -20.034` - 34.609`i)$$

determine the polar triangle $R(\lambda)$ of λ. This proves that λ is in relation with μ^1, μ^2 and μ^3 on $S(F)$.

Example

On the other hand it is easy to compute all the 36 symmetric determinantal representations of $S(F)$. For M_ϑ in Example 1 we have

$$v_\lambda = [-0.006-0.009i, -0.335-0.482i, -0.571+0.04i, -0.236+0.521i]^t \in \mathrm{Coker}\, M_\vartheta(\lambda$$
$$v_{\mu^1} = [0, -0.543-0.164i, -0.419+0.404i, 0.124+0.569i]^t \in \mathrm{Coker}\, M_\vartheta(\mu^1),$$
$$v_{\mu^2} = [0.602-0.73i, -0.186-0.025i, -0.124+0.147i, 0.062+0.172i]^t \in \mathrm{Coker}\, M_\vartheta(\mu^2)$$
$$v_{\mu^3} = [0.613+0.72i, -0.185+0.028i, -0.059+0.173i, 0.127+0.145i]^t \in \mathrm{Coker}\, M_\vartheta(\mu^3)$$

We check that

$$v_\lambda^t\, M_\vartheta(x_0, x_1, x_2)\, v_{\mu^i} = 0 \ \text{ for any } \ (x_0, x_1, x_2) \in \mathbb{P}^2, \ i = 1, 2, 3.$$

This proves that the corresponding $\mathcal{L}_\vartheta$ is the unique theta characteristic on $S(F)$ that we were looking for.

Definition: Pfaffian

Let $A = [a_{ij}]$ be a $2n \times 2n$ skew–symmetric matrix

$$\begin{matrix} a_{1\,2} & a_{1\,3} & \cdots & a_{1\,2n} \\ & a_{2\,3} & & a_{2\,2n} \\ & & \ddots & \vdots \\ & & & a_{2n-1\,2n} \end{matrix} \quad .$$

Definition: Pfaffian

Consider permutations

$$P_{2n} := \{\sigma \in S_{2n} \, : \, \sigma(2i-1) < \sigma(2i) \text{ and } \sigma(2i-1) < \sigma(2i+1)\} .$$

Definition

The Pfaffian of A is

$$\begin{aligned} \mathrm{Pf}(A) &= \sum_{\sigma \in P_{2n}} \mathrm{sgn}(\sigma)\, a_{\sigma(1)\sigma(2)} \cdot \ldots \cdot a_{\sigma(2n-1)\sigma(2n)} \\ &= \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \mathrm{sgn}(\sigma)\, a_{\sigma(1)\sigma(2)} \cdot \ldots \cdot a_{\sigma(2n-1)\sigma(2n)} . \end{aligned}$$

Alternative Definition: Pfaffian

To A one can associate a bivector

$$\omega = \sum_{i<j} a_{ij}\, e_i \wedge e_j,$$

where $e_1, e_2, \ldots, e_{2n}$ is the standard basis of k^{2n}.

Definition

The Pfaffian of A is given by

$$\frac{1}{n!}\underbrace{\omega \wedge \cdots \wedge \omega}_{n} = \mathrm{Pf}(A)\, e_1 \wedge e_2 \wedge \cdots \wedge e_{2n}.$$

Hyper Matrix

An r-dimensional $rn \times \cdots \times rn$ matrix $A = [a_{i_1 \dots i_r}]_{1 \leq i_j \leq rn}$ is skew–symmetric, if for any permutation σ_r of r elements $i_1, \dots, i_r$ holds

$$a_{i_1 \dots i_r} = \operatorname{sgn}(\sigma_r)\, a_{\sigma_r(i_1) \dots \sigma_r(i_r)}.$$

- A can be presented by the upper r-tetraheder, obtained as the intersection of $r - 1$ diagonal hyperplanes.

Hyper Matrix

An r-dimensional $rn \times \cdots \times rn$ matrix $A = [a_{i_1 \ldots i_r}]_{1 \leq i_j \leq rn}$ is skew–symmetric, if for any permutation σ_r of r elements $i_1, \ldots, i_r$ holds

$$a_{i_1 \ldots i_r} = \operatorname{sgn}(\sigma_r)\, a_{\sigma_r(i_1) \ldots \sigma_r(i_r)}.$$

- A can be presented by the upper r-tetraheder, obtained as the intersection of $r-1$ diagonal hyperplanes.

Definition: HyperPfaffian

Consider the permutations

$$P_{rn} := \left\{ \sigma \in S_{rn} \ : \ \begin{array}{l} \sigma(ri - r + 1) < \cdots < \sigma(ri) \text{ for } 1 \le i \le n, \quad \text{and} \\ \sigma(ri - r + 1) < \sigma(ri + 1) \text{ for } 1 \le i \le n - 1 \end{array} \right\}.$$

Definition

For a skew–symmetric A, define the HyperPfaffian by

$$\begin{aligned} \operatorname{HyPf} A &= \frac{1}{(r!)^n \, n!} \sum_{\sigma \in S_{rn}} \operatorname{sgn}(\sigma)\, a_{\sigma(1)\ldots\sigma(r)} \cdot \ldots \cdot a_{\sigma(rn-r+1)\ldots\sigma(rn)} \\ &= \begin{cases} \sum_{\sigma \in P_{rn}} \operatorname{sgn}(\sigma)\, a_{\sigma(1)\ldots\sigma(r)} \cdot \ldots \cdot a_{\sigma(rn-r+1)\ldots\sigma(rn)}, & \text{if } r \text{ eve} \\ 0, & \text{if } r \text{ odd} \end{cases} \end{aligned}$$

Alternative Definition: HyperPfaffian

The wedge product in $\bigwedge V$ is anticommutative:
$v_1 \wedge v_2 = (-1)^{l_1 l_2}\, v_2 \wedge v_1, \;$ for $v_1 \in \bigwedge^{l_1} V, \; v_2 \in \bigwedge^{l_2} V.$

This gives another description of A and HyPf A in the standard basis $e_1, e_2, \ldots, e_{rn}$ of k^{rn}:

- A corresponds to an r–vector $\omega = \sum_{i_1 < \cdots < i_r} a_{i_1 \ldots i_r}\, e_{i_1} \wedge \cdots \wedge e_{i_r} \;\in \bigwedge^r k^{rn}$
- the HyperPfaffian is given by the equation

$$\frac{1}{n!}\underbrace{\omega \wedge \cdots \wedge \omega}_{n} = \mathrm{HyPf}(A)\, e_1 \wedge e_2 \wedge \cdots \wedge e_{rn}.$$

Examples

- A $3n \times 3n \times 3n$ matrix $A = [a_{ijk}]_{1 \leq i,j,k \leq 3n}$ with $a_{ijk} = a_{jki} = a_{kij} = -a_{jik} = -a_{kji} = -a_{ikj}$ can be presented by the upper tetraheder, obtained as intersection of two diagonal planes.
- In a $9 \times 9 \times 9$ matrix $A_{(3 \cdot 3)^3}$ the HyperPfaffian contains 280 cubic monomials that sum into 0:

$$\text{HyPf}\, A_{(3 \cdot 3)^3} = a_{123}a_{456}a_{789} - a_{123}a_{457}a_{689} + \cdots + a_{148}a_{236}a_{579} \pm \cdots$$

Examples

- In a $8 \times 8 \times 8 \times 8$ matrix HyPf $A_{(4\cdot 2)^4} =$

$$a_{1234}a_{5678} - a_{1235}a_{4678} + a_{1236}a_{4578} - a_{1237}a_{4568} + a_{1238}a_{4567} -$$
$$a_{1245}a_{3678} + a_{1246}a_{3578} - a_{1247}a_{3568} + a_{1248}a_{3567} -$$
$$a_{1256}a_{3478} + a_{1257}a_{3468} - a_{1258}a_{3467} + a_{1267}a_{3458} - a_{1268}a_{3457} + a_{127}$$
$$a_{1345}a_{2678} + a_{1346}a_{2578} - a_{1347}a_{2568} + a_{1348}a_{2567} -$$
$$a_{1356}a_{2478} + a_{1357}a_{2468} - a_{1358}a_{2467} + a_{1367}a_{2458} - a_{1368}a_{2457} + a_{137}$$
$$a_{1456}a_{2378} + a_{1457}a_{2368} - a_{1458}a_{2367} + a_{1467}a_{2358} - a_{1468}a_{2357} + a_{147}$$
$$a_{1567}a_{2348} + a_{1568}a_{2347} - a_{1578}a_{2346} + a_{1678}a_{2345}.$$

Sub–HyperPfaffian $\mathrm{HyPf}^{i_1 \ldots i_r}$

Definition

Given an r–dimensional skew–symmetric matrix A of size $rn \times \cdots \times rn$, let $A^{i_1 \ldots i_r}$ denote the r–dimensional skew–symmetric matrix of size $r(n-1) \times \cdots \times r(n-1)$ obtained from A by removing all $a_{j_1 \ldots j_r}$ for which at least one of $j_l \in \{i_1, \ldots, i_r\}$. Denote the HyperPfaffian of $A^{i_1 \ldots i_r}$ by $\mathrm{HyPf}^{i_1 \ldots i_r}$.

Interesting Questions

- For a fixed integer j, $1 \leq j \leq rn$, prove that

$$\mathrm{HyPf}\, A = \sum_{j \in \{i_1 < \cdots < i_r\}} a_{i_1 \ldots i_r}\, \mathrm{HyPf}^{i_1 \ldots i_r}.$$

- Define the adjoint of A via $\mathrm{HyPf}^{i_1 \ldots i_r}$.
- Show that $A * \mathrm{adj}\, A = \mathrm{HyPf}\, A \cdot \mathrm{Id}_{\underbrace{rn \times \cdots \times rn}_{2(r-1)}}$.
- What does it mean for adj A to have rank r?
- Use other immanants instead of $\mathrm{sgn}(\sigma)$.