GRADUATE COURSE ON
POLYNOMIAL METHODS FOR
ROBUST CONTROL
PART IV.1

ROBUST ANALYSIS WITH LINEAR MATRIX INEQUALITIES AND POLYNOMIAL MATRICES

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Linear matrix inequalities

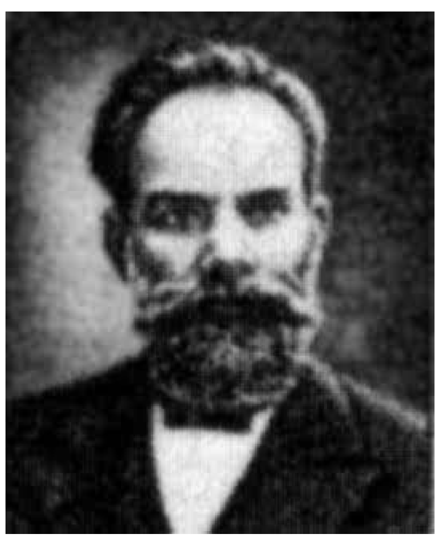
Historically, the first LMIs appeared around 1890 when Lyapunov showed that the differential equation

$$\frac{d}{dt}x(t) = Ax(t)$$

is stable (all trajectories converge to zero) iff there exists a solution to the matrix inequalities

$$A^T P + PA < 0 \quad P = P^T > 0$$

which are linear in unknown matrix P



Aleksandr Mikhailovich Lyapunov
(1857 Yaroslavl - 1918 Odessa)

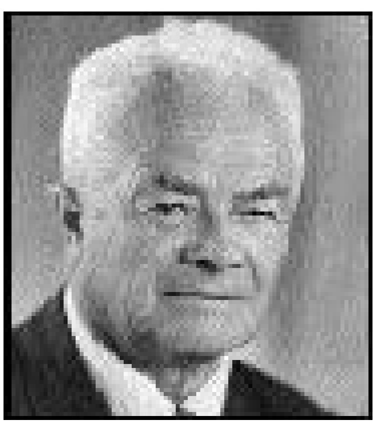
Some history

In the 1940s Lu're, Postnikov and others in Soviet Union applied Lyapunov's approach to some specific control problems with non-linearity in the actuator, they obtained stability criteria in the form of LMIs (although they did not explicitly form matrix inequalities)

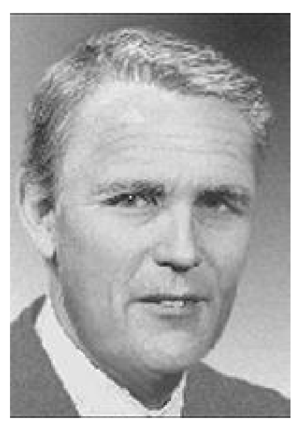
These inequalities were polynomial (frequency dependent) inequalities

In the early 1960s Yakubovich, Popov, Kalman, Anderson and others obtained the positive real lemma, which reduces the solution of the LMIs to a simple graphical criterion (Popov, circle and Tsypkin criteria)

Researchers like Willems in the 1970s then focused on solving algebraic equations such as Lyapunov's or Riccati's equations, rather than LMIs



Vladimir Andreevich Yakubovich
(1926 Novosibirsk)



Rudolf Emil Kalman
(1930 Budapest)

Convex optimization

Development of powerful and efficient polynomial-time interior-point algorithms for linear programming by Karmarkar in 1984

In 1988 Nesterov and Nemirovskii developed interior-point methods that apply directly to linear matrix inequalities (and even more)

It was then recognized than LMIs can be solved with convex optimization on a computer

In 1993 Gahinet and Nemirovskii wrote a commercial Matlab package called the LMI Toolbox for Matlab

Several freeware solvers are now available, see the next slide

Some pointers

Semidefinite programming

Christoph Helmberg's page
www.zib.de/helmberg/semidef.html

Stephen Boyd's page
www.stanford.edu/∼boyd

Laurent El Ghaoui's page
robotics.eecs.berkeley.edu/∼elghaoui

Interior-point methods online
www-unix.mcs.anl.gov/otc/InteriorPoint

Control

Optimization and Control at arXiv
arXiv.org/list/math.OC/recent

Control Theory and Engineering
www.theorem.net/control.html

Systems and Control Archive
scad.utdallas.edu

Control Engineering Virtual Library
www-control.eng.cam.ac.uk

LMI solvers

we recommend SeDuMi by Jos Sturm
fewcal.kub.nl/sturm

LMI interface by Dimitri Peaucelle
www.laas.fr/∼peaucell/SeDuMiInt.html

How does an LMI look like ?

Canonical form

$$F(x) = F_0 + \sum_{i=1}^{m} x_i F_i > 0$$

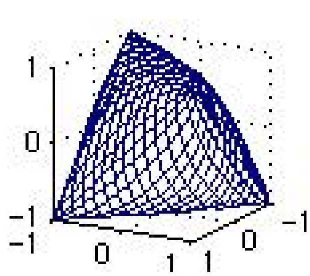
where x is a vector of m decision variables and matrices $F_i = F_i^\star$ are given

The inequality sign means positive definite, i.e. all the eigenvalues are positive

We can also encounter non-strict LMIs when some eigenvalues can be zero

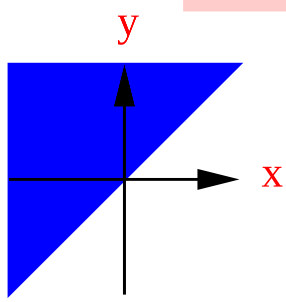
$$F(x) \geq 0$$

LMI is a convex constraint on x

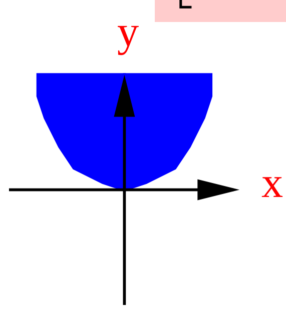


Simple LMIs in the plane

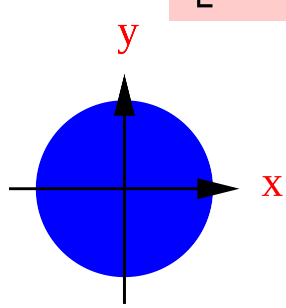
$y > x$ $\qquad$ $y - x > 0$



$y > x^2$ $\qquad$ $\begin{bmatrix} 1 & x \\ x & y \end{bmatrix} > 0$



$x^2 + y^2 < 1$ $\qquad$ $\begin{bmatrix} 1 & x & y \\ x & 1 & 0 \\ y & 0 & 1 \end{bmatrix} > 0$



Various formulations

Besides the canonical form, there exists also another formulation of an LMI, which resembles more linear programming

$$
\begin{array}{ll}
\text{min} & c^\star x \\
\text{s.t.} & Ax = b \\
 & x \in K
\end{array}
$$

K is a semidefinite cone, i.e. entries of vector x belong to semidefinite matrices, this is why LMI optimization is sometimes called semidefinite programming

In this form it is easy to build the dual LMI

$$
\begin{array}{ll}
\text{max} & b^\star y \\
\text{s.t.} & c - A^\star y \in K^\star
\end{array}
$$

where $K^\star = K$ is the self-dual semidefinite cone

One must be careful when studying duality of semidefinite programs, in general we have

$$c^\star x \geq b^\star y$$

where equality holds when the primal and the dual are strictly feasible (positive definite), this is Slater's constraint qualification

Example of duality gap

Example

Consider the primal semidefinite program

$$\begin{array}{ll} \min & x_1 \\ \text{s.t.} & \begin{bmatrix} 0 & x_1 & 0 \\ x_1 & x_2 & 0 \\ 0 & 0 & 1+x_1 \end{bmatrix} \geq 0 \end{array}$$

with dual

$$\begin{array}{ll} \max & y_1 \\ \text{s.t.} & \begin{bmatrix} -y_2 & (1+y_1)/2 & -y_3 \\ (1+y_1)/2 & 0 & -y_4 \\ -y_3 & -y_4 & -y_1 \end{bmatrix} \geq 0 \end{array}$$

In the primal necessarily $x_1 = 0$ (x_1 appears in a row with zero diagonal entry) so the primal optimum is

$$\hat{x_1} = 0$$

Similarly, in the dual necessarily $(1+y_1)/2 = 0$ so the dual optimum is

$$\hat{y_1} = -1$$

There is a non-zero duality gap here

Matrices as variables

Generally, in control problems we do not encounter the LMI in canonical or semidefinite form but rather with matrix variables

Lyapunov's inequality

$$A^\star P + PA < 0 \quad P = P^\star > 0$$

can be written in canonical form

$$F(x) = F_0 + \sum_{i=1}^{m} F_i x_i > 0$$

with the notations

$$F_0 = 0 \quad F_i = -A^\star B_i - B_i A$$

where B_i, $i = 1, \ldots, n(n+1)/2$ are matrix basis for symmetric matrices of size n

Most software packages for solving LMIs however work with the canonical or semidefinite forms, so that a (sometimes time-consuming) pre-processing step is required

Tricks with LMIs

We can use the Schur complement to convert a non-linear (convex) matrix inequality into an LMI

$$\begin{bmatrix} A(x) & B(x) \\ B^\star(x) & C(x) \end{bmatrix} > 0 \iff \begin{array}{l} C(x) > 0 \\ A(x) - B(x)C^{-1}(x)B^\star(x) > 0 \end{array}$$

Matrix norm constraint as an LMI

$$\|Z(x)\|_2 < 1 \iff \begin{bmatrix} I & Z(x) \\ Z^\star(x) & I \end{bmatrix} > 0$$

To remove decision variables we can use the elimination lemma

$$\begin{array}{c} A(x) + B(x)XC^\star(x) + C(x)X^\star B^\star(x) > 0 \\ \iff \\ \tilde{B}^\star(x)A(x)\tilde{B}(x) > 0 \quad \tilde{C}^\star(x)A(x)\tilde{C}(x) > 0 \end{array}$$

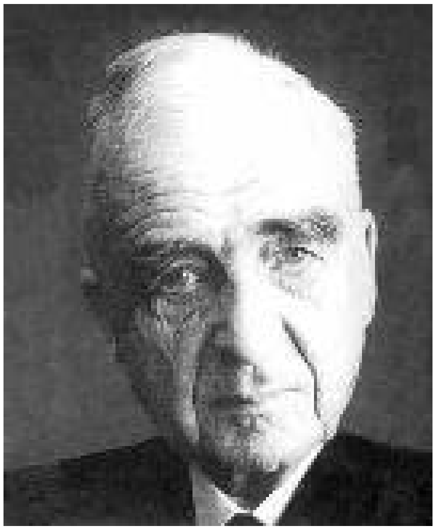
where $\tilde{B}$ and $\tilde{C}$ are orthogonal complements of B and C respectively, and x is a decision variable independent of matrix X

Finsler's lemma

A very useful trick in robustness in the so-called Finsler's lemma, a specialized version of the elimination lemma

The following statements are equivalent

$x^\star A x > 0$ for all $x \neq 0$ s.t. $Bx = 0$
$\tilde{B}^\star A \tilde{B} > 0$ where $B\tilde{B} = 0$
$A + \rho B^\star B > 0$ for some scalar ρ
$A + XB + B^\star X^\star > 0$ for some matrix X



Paul Finsler
(1894 Heilbronn - 1970 Zurich)

The S-procedure

Another useful trick in robustness is Yakubovich's S-procedure which relates constraints on quadratic forms with an LMI

Consider the quadratic forms

$$q_i(x) = \begin{bmatrix} x \\ 1 \end{bmatrix}^\star \begin{bmatrix} A_i & b_i^\star \\ b_i & c_i \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = x^\star A_i x + 2 b_i^\star x + c_i$$

for $i = 0, 1, \ldots, p$, then

$$q_0(x) \geq 0$$

for all vectors x such that

$$q_i(x) \geq 0 \quad i = 1, \ldots, p$$

if there exist scalars $\rho_i \geq 0$ satisfying the LMI

$$\begin{bmatrix} A_0 & b_0^\star \\ b_0 & c_0 \end{bmatrix} - \sum_{i=1}^{p} \rho_i \begin{bmatrix} A_i & b_i^\star \\ b_i & c_i \end{bmatrix} \geq 0$$

Most importantly, when $p = 1$ (and $p \leq 2$ for complex quadratic forms) then the converse also holds

Positive real lemma

Besides Lyapunov's inequality, we encounter classical LMIs in control theory, such as the positive real lemma

The linear system

$$\begin{array}{rcl} \dot{x} & = & Ax + Bu \\ y & = & Cx + Du \end{array}$$

is passive, i.e.

$$\int_0^T u^\star(t)y(t)dt \geq 0$$

iff its transfer matrix

$$G(s) = C(sI - A)^{-1}B + D$$

is positive real, i.e.

$$G(s) + G^\star(s) \geq 0 \quad \text{for all} \quad \text{Re}\, s > 0$$

iff there is a solution to the LMI

$$\left[\begin{array}{cc} A^\star P + PA & PB - C^\star \\ B^\star P - C & -D^\star - D \end{array}\right] \leq 0 \quad P = P^\star > 0$$

iff there is a positive definite symmetric solution to the Riccati inequality

$$A^\star P + PA + (PB - C^\star)(D + D^\star)^{-1}(PB - C^\star)^\star \leq 0$$

iff (under some additional assumptions) there is a symmetric matrix satisfying the algebraic Riccati equation

$$A^\star P + PA + (PB - C^\star)(D + D^\star)^{-1}(PB - C^\star)^\star = 0$$

Bounded real lemma

Very similarly, we can also encounter in some control problems (H_∞ optimization) the bounded real lemma

The linear system

$$\begin{array}{rcl} \dot{x} & = & Ax + Bu \\ y & = & Cx + Du \end{array}$$

is non-expansive, i.e.

$$\int_0^T y^\star(t)y(t)dt \leq \int_0^T u^\star(t)u(t)dt$$

iff its transfer matrix

$$G(s) = C(sI - A)^{-1}B + D$$

is bounded real, i.e.

$$G^\star(s)G(s) \leq I \quad \text{for all} \quad \text{Re}\, s > 0$$

or equivalently satisfying the H_∞ norm constraint

$$\|G(s)\|_\infty \leq 1$$

iff there is a solution to the LMI

$$\left[\begin{array}{cc} A^\star P + PA + C^\star C & PB + C^\star D \\ B^\star P + D^\star C & D^\star D - I \end{array}\right] \leq 0 \quad P = P^\star > 0$$

iff (under some addition assumptions) there is a symmetric matrix satisfying the algebraic Riccati equation

$$A^\star P + PA + C^\star C + (PB + C^\star D)(I - D^\star D)^{-1}(PB + C^\star D)^\star = 0$$

Positive polynomials

Recently, Nesterov from Université Catholique de Louvain showed that we can express positivity of a polynomial matrix as an LMI

`www.auto.ucl.ac.be/~vdooren`

This gives a new, unified insight into several problems frequently encountered in control:

- spectral factorization of polynomials
  (H_2 and H_∞ optimal control)
- global optimization over polynomials
  (robust stability analysis)
- positive real and bounded real lemma
  (non-linear systems, H_∞ control)
- sufficient stability conditions for polynomials
  (robust analysis and design)

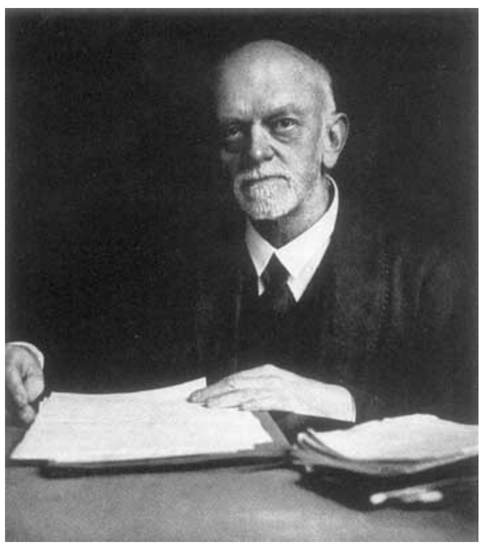
We will use Nesterov's result to derive several new results in robustness analysis and design, in both state-space and polynomial frameworks

Positive polynomials

The set of polynomials that are positive on the real axis is a convex set that can be described by an LMI

Idea originating from Shor (1987), related with Hilbert's 17th pb about algebraic sum-of-squares decompositions, see Parrilo's PhD thesis (2000)

`www.cds.caltech.edu/~pablo`



David Hilbert
(1862 Königsberg - 1943 Göttingen)

Can be proved with cone duality (Nesterov) or with theory of moments (Lasserre)

LMI formulation of positivity

The even polynomial

$$r(s) = r_0 + r_1 s + \cdots + r_{2n} s^{2n}$$

satisfies $r(s) \geq 0$ for all real s if and only if

$$r_k = \sum_{i+j=k} P_{ij}, \quad k = 0, 1, \ldots, 2n$$

for some matrix $P = P^\star \geq 0$

Proof

The expression of r_k comes from

$$r(s) = [1 \quad s \quad \cdots \quad s^n] P [1 \quad s \quad \cdots \quad s^n]^\star$$

hence $P \geq 0$ naturally implies $r(s) \geq 0$

Conversely, the existence of P for any polynomial $r(s) \geq 0$ follows from the existence of a decomposition as a sum-of-squares of

$$r(s) = \sum_k p_k^2(s) \geq 0$$

matrix P having entries $P_{ij} = \sum_k p_{k_i} p_{k_j}$

Positive polynomials and LMIs

Example

Global minimization of the polynomial

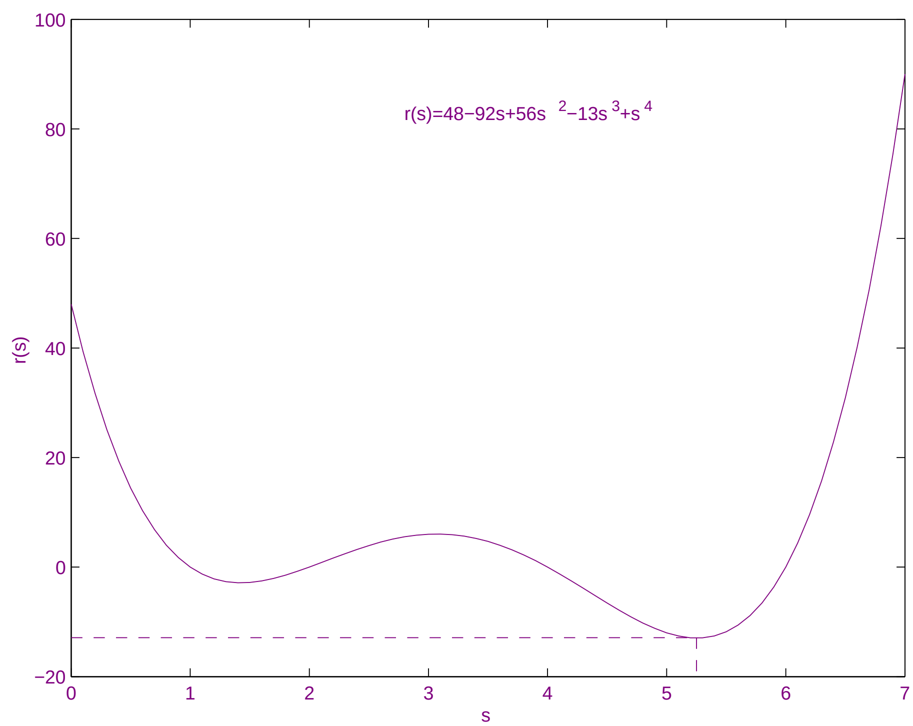
$$r(s) = 48 - 92s + 56s^2 - 13s^3 + s^4$$

Global optimum r^*: maximum value of r_{low} such that $r(s) - r_{\text{low}}$ is a positive polynomial

We just have to solve the LMI

$$\begin{array}{ll} \min & 48 - 92s_1 + 56s_2 - 13s_3 + s_4 \\ \text{s.t.} & \begin{bmatrix} 1 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{bmatrix} \geq 0 \end{array}$$

to obtain $r^* = r(5.25) = -12.89$



Strict positive realness

Let

$$\mathcal{S} = \{s \; : \; \begin{bmatrix} 1 \\ s \end{bmatrix}^\star \underbrace{\begin{bmatrix} a & b \\ b^\star & c \end{bmatrix}}_{S} \begin{bmatrix} 1 \\ s \end{bmatrix} < 0\}$$

be a stability region in the complex plane where Hermitian matrix S has inertia $(1, 0, 1)$

Let $\partial\mathcal{S}$ denote the 1-D boundary of $\mathcal{S}$

Standard choices are

$$S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad S = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

for the left half-plane and the unit disk resp.

We say that a rational matrix $G(s)$ is strictly positive real (SPR for short) when

$$G(s) > 0 \quad \text{for all } s \in \partial\mathcal{D}$$

Stability and strict positive realness

Consider two square polynomial matrices of size n and degree d

$$\begin{array}{rcl} N(s) & = & N_0 + N_1 s + \cdots + N_d s^d \\ D(s) & = & D_0 + D_1 s + \cdots + D_d s^d \end{array}$$

Polynomial matrix $N(s)$ is stable iff there exists a stable polynomial $D(s)$ such that rational matrix $N(s)D^{-1}(s)$ is strictly positive real

Proof

From the definition of SPRness, $N(s)D^{-1}(s)$ SPR with $D(s)$ stable implies $N(s)$ stable

Conversely, if $N(s)$ is stable then the choice $D(s) = N(s)$ makes rational matrix $N(s)D^{-1}(s) = I$ obviously SPR

Thus we obtain a sufficient stability condition of a polynomial matrix

It turns out that this condition can be characterized by an LMI, as shown in the next slide

SPRness as an LMI

Let $N = [N_0 \quad N_1 \cdots N_d]$, $D = [D_0 \quad D_1 \cdots D_d]$ and

$$\Pi = \begin{bmatrix} I & & & 0 \\ & \ddots & & \vdots \\ & & I & 0 \\ 0 & I & & \\ \vdots & & \ddots & \\ 0 & & & I \end{bmatrix}$$

Given a stable $D(s)$, $N(s)$ ensures SPRness of $N(s)D^{-1}(s)$ iff there exists a matrix $P = P^\star$ of size dn such that

$$D^\star N + N^\star D - S(P) > 0$$

where

$$S(P) = \Pi^\star (S \otimes P)\Pi = \Pi^\star \begin{bmatrix} aP & bP \\ b^\star P & cP \end{bmatrix} \Pi$$

Proof

Similar to the proof on positivity of a polynomial, based on the decomposition as a sum-of-squares

Stability as an LMI

From the previous results, we obtain a sufficient LMI condition for stability of a polynomial matrix

Polynomial matrix $N(s)$ is stable iff there exists a polynomial matrix $D(s)$ and a matrix $P = P^\star > 0$ satisfying the LMI

$$D^\star N + N^\star D - S(P) > 0$$

Example
Consider the second-degree discrete-time polynomial

$$n(z) = n_0 + n_1 z + z^2$$

We will study the shape of the LMI stability region for the following stable polynomial

$$d(z) = z^2$$

We can show that feasibility of the above LMI is equivalent to existence of a matrix $P = P^\star \geq 0$ satisfying

$$\begin{array}{rcl} p_{00} + p_{11} + p_{22} & = & 1 \\ p_{10} + p_{01} + p_{21} + p_{22} & = & n_1 \\ p_{20} + p_{02} & = & n_0 \end{array}$$

which is an LMI in the primal semidefinite programming form

Stability as an LMI

We use a result on semidefinite programming duality to show that infeasibility of the primal LMI is equivalent to the existence of a Farkas vector satisfying the dual LMI

$$y_0 + n_1 y_1 + n_0 y_2 < 0 \quad Y = \begin{bmatrix} y_0 & y_1 & y_2 \\ y_1 & y_0 & y_1 \\ y_2 & y_1 & y_0 \end{bmatrix} \geq 0$$

The eigenvalues of Toeplitz matrix Y are

$$y_0 - y_2 \quad \text{and} \quad (2y_0 + y_2 \pm \sqrt{y_2^2 + 8y_1^2})/2$$

so it is positive definite iff y_1 and y_2 belong to the interior of a bounded parabola scaled by y_0

The corresponding values of n_0 and n_1 belong to the interior of the envelope generated by the curve

$$(2\lambda_2 - 1)n_0 + (2\lambda_1 - 1)\sqrt{\lambda_2} n_1 + 1 > 0 \quad 0 \leq \lambda_i \leq 1$$

The implicit equation of the envelope is given by

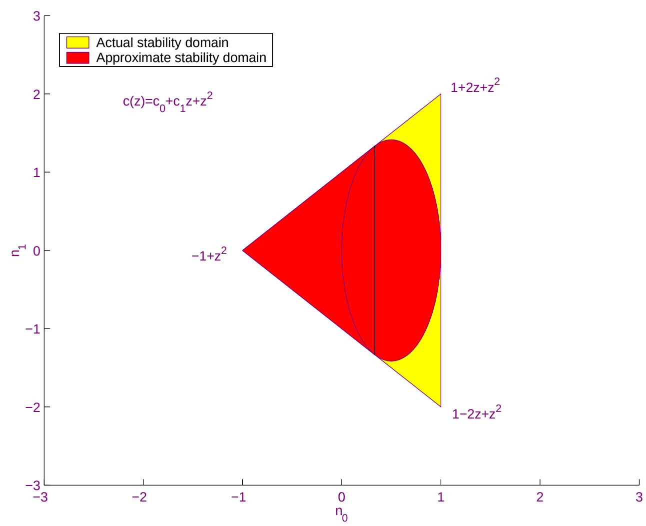
$$8n_0^2 - 8n_0 + n_1 = 0 \quad \text{or} \quad (2n_0 - 1)^2 + (\frac{\sqrt{2}}{2} n_1)^2 = 1$$

which is a scaled circle

The LMI stability region is then the union of the interior of the circle with the interior of the triangle delimited by the two lines

$$n_0 \pm n_1 + 1 = 0$$

tangent to the circle, with vertices $[-1, 0]$, $[1/3, 4/3]$ and $[1/3, -4/3]$



Application to robust stability analysis

Assume that $N(s, \lambda)$ is a polynomial matrix with multi-linear dependence in a parameter vector λ belonging to a polytope Λ

Denote by $N_i(s)$ the vertices obtained by enumerating each vertex in Λ for $i = 1, \ldots, m$

Polytopic polynomial matrix $N(s, \lambda)$ is robustly stable if there exists a matrix D and matrices $P_i = P_i^\star > 0$ satisfying the LMI

$$D^\star N_i + N_i^\star D - S(P_i) > 0 \quad i = 1, \ldots, m$$

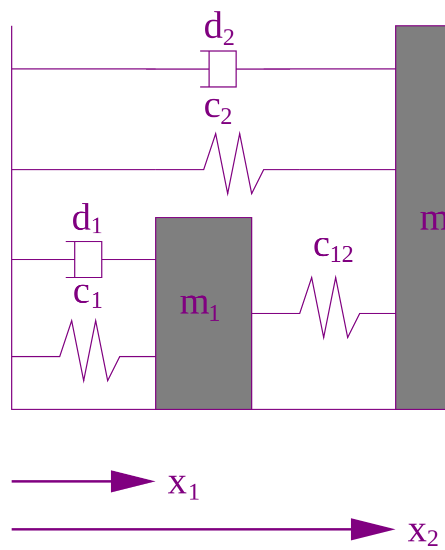
Proof

Since the LMI is linear in D - matrix of coefficients of polynomial matrix $D(s)$ - it is enough to check the vertices to prove stability in the whole polytope

Robust stability of polynomial matrices

Example

Consider the following mechanical system



It is described by the polynomial MFD

$$\begin{bmatrix} m_1 s^2 + d_1 s + c_1 + c_{12} & -c_{12} \\ -c_{12} & m_2 s^2 + d_2 s + c_2 + c_{12} \end{bmatrix} \begin{bmatrix} x_1(s) \\ x_2(s) \end{bmatrix} = \begin{bmatrix} 0 \\ u(s) \end{bmatrix}$$

System parameters $\lambda = [\, m_1 \quad d_1 \quad c_1 \quad m_2 \quad d_2 \quad c_2 \,]$
belong to the uncertainty hyper-rectangle
$\Lambda = [1,\, 3] \times [0.5,\, 2] \times [1,\, 2] \times [2,\, 5] \times [0.5,\, 2] \times [2,\, 4]$
and we set $c_{12} = 1$

This mechanical system is passive so it must be open-loop stable (when $u(s) = 0$) independently of the values of the masses, springs, and dampers

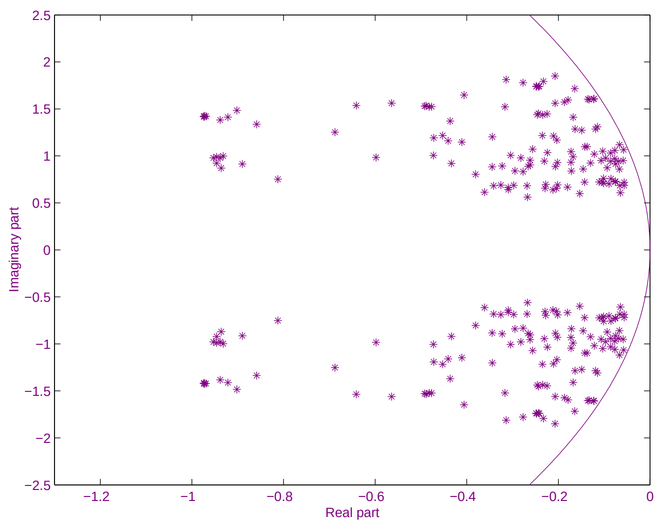
Robust stability of polynomial matrices

However, it is a non-trivial task to know whether the open-loop system is robustly stable in some stability region $\mathcal{S}$ ensuring a certain damping. Here we choose the disk of radius 12 centered at -12

$$\mathcal{S} = \{s : (s+12)^2 < 12^2\} = \{s : \begin{bmatrix} 1 \\ s \end{bmatrix}^\star \begin{bmatrix} 0 & 12 \\ 12 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ s \end{bmatrix} < 0\}$$

The robust stability analysis problem amounts to assessing whether the second degree polynomial matrix in the MFD has its zeros in $\mathcal{S}$ for all admissible uncertainty in a polytope with $m = 2^6 = 64$ vertices

With the SeDuMi solver we find the LMI problem feasible after about 3.5 seconds of CPU time



Only zeros of the 64 polynomial matrix vertices are represented because an accurate 6-dimensional brute force gridding of the root-locus would be necessary to check robust stability graphically, since there is no vertex or edge result available for interval polynomial matrices

Polytope of polynomials

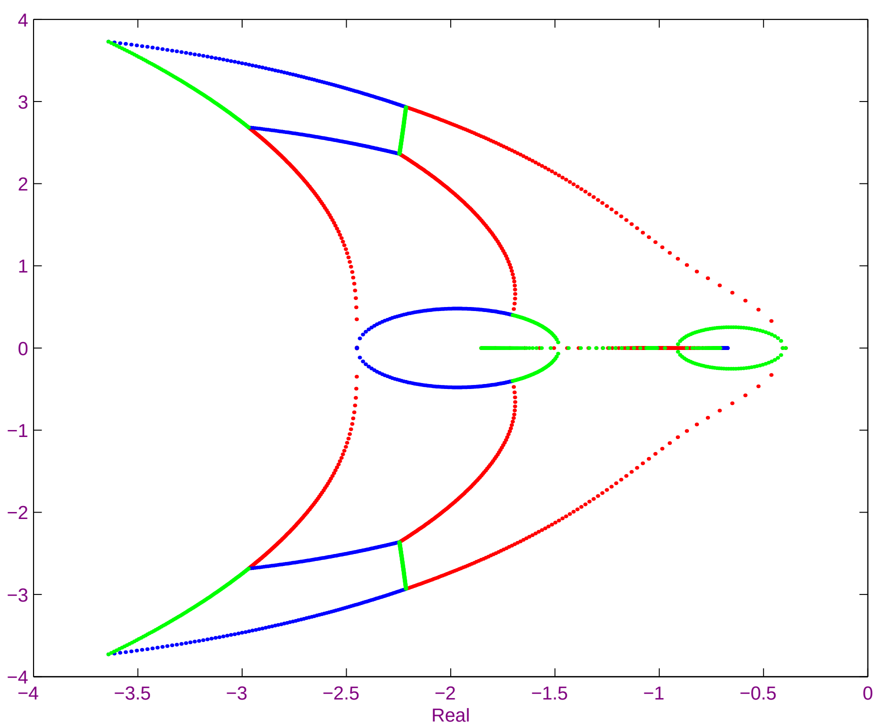
We can also check robust stability of polytopes of polynomials without using the edge theorem or the graphical value set

Example

Continuous-time polytope of degree 3 with 3 vertices

$$\begin{aligned} n_1(s) &= 28.3820 + 34.7667s + 8.3273s^2 + s^3 \\ n_2(s) &= 0.2985 + 1.6491s + 2.6567s^2 + s^3 \\ n_3(s) &= 4.0421 + 9.3039s + 5.5741s^2 + s^3 \end{aligned}$$

The LMI problem is feasible, so the polytope is robustly stable – see robust root locus below



Interval polynomial matrices

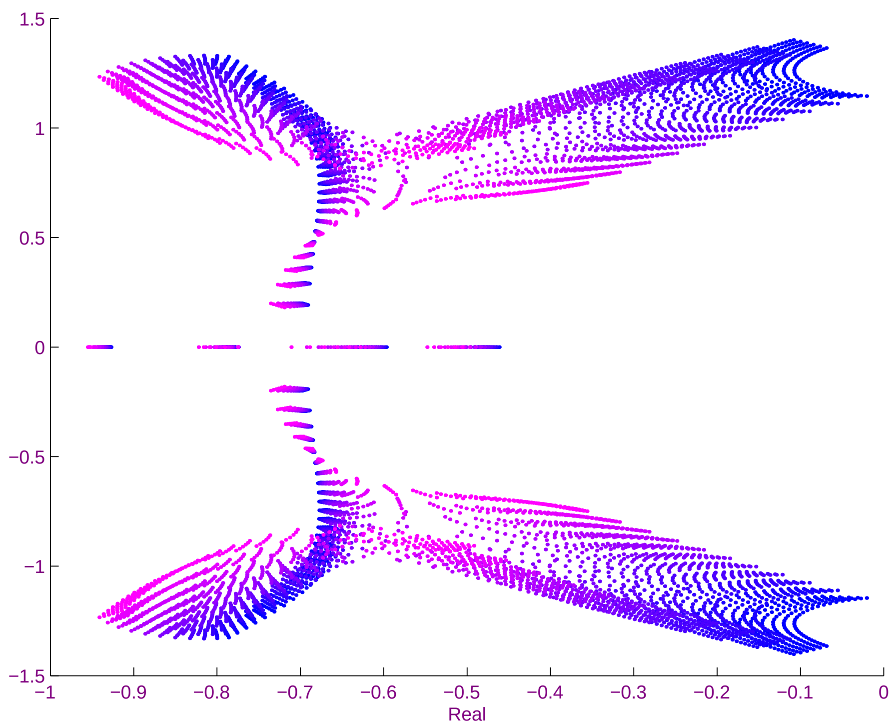
Similarly, we can assess robust stability of interval polynomial matrices, an NP-hard problem in general

Example

Continuous-time interval polynomial matrix of degree 2 with $2^3 = 8$ vertices

$$\begin{bmatrix} [7.7 - 2.3s + 4.3s^2, \; 3.7 + 2.7s + 4.3s^2] & [-3.1 - 6s - 2.2s^2, \; -4.1 - 7s - 2.2s^2] \\ 3.6 + 6.4s + 4.3s^2 & [3.2 + 11s + 8.2s^2, \; 16 + 12s + 8.2s^2] \end{bmatrix}$$

After less than 1 second of CPU time, SeDuMi finds a feasible solution to the LMI, so the matrix is robustly stable – see robust root locus below



State-space systems

One advantage of our approach is that state-space results can be obtained as simple by-products, since stability of a constant matrix A is equivalent to stability of the pencil matrix

$$N(s) = sI - A$$

Matrix A is stable iff there exists a matrix F and a matrix $P = P^\star > 0$ solving the LMI

$$\begin{bmatrix} F^\star A + A^\star F - aP & -A^\star - F^\star - bP \\ -A - F - b^\star P & 2I - cP \end{bmatrix} > 0$$

Proof

Just take $D(s) = sI - F$ and notice that the LMI can be also written more explicitly as

$$\begin{bmatrix} -F^\star \\ I \end{bmatrix} \begin{bmatrix} -A & I \end{bmatrix} + \begin{bmatrix} -A^\star \\ I \end{bmatrix} \begin{bmatrix} -F & I \end{bmatrix} - \begin{bmatrix} aP & bP \\ b^\star P & cP \end{bmatrix} > 0$$

Robust stability of state-space systems

We recover the new LMI stability conditions obtained by Geromel and de Oliveira in 1999

Nice decoupling between Lyapunov matrix P and additional variable F allows for construction of parameter-dependent Lyapunov matrix

Assume that uncertain matrix $A(\lambda)$ has multilinear dependence on polytopic uncertain parameter λ and denote by A_i the corresponding vertices

Matrix $A(\lambda)$ is robustly stable if there exists a matrix F and matrices $P_i = P_i^\star > 0$ solving the LMI

$$\left[\begin{array}{cc} F^\star A_i + A_i^\star F - aP_i & -A_i^\star - F^\star - bP_i \\ -A_i - F - b^\star P_i & 2I - cP_i \end{array}\right] > 0$$

Proof

Consider the parameter-dependent Lyapunov matrix $P(\lambda)$ built from vertices P_i