GRADUATE COURSE ON
POLYNOMIAL METHODS FOR
ROBUST CONTROL
PART III.3

ROBUST DESIGN: SIMULTANEOUS STABILIZATION

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Summary

So far we have investigated two approaches for robust design

- pole placement, where non-convexity of the stability region can be overcome by a convex approximation, which generally results in low-order controllers

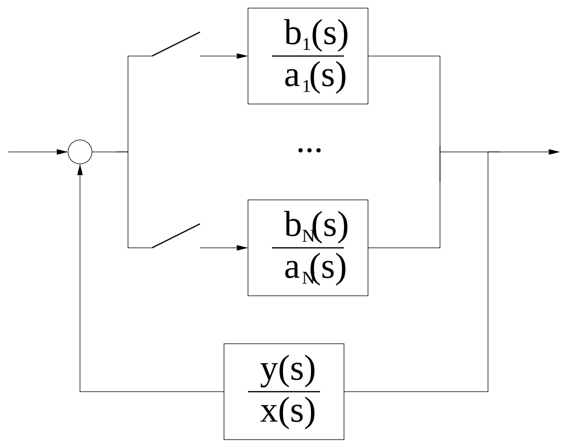
- Youla-Kučera parametrization, which is convex but where infinite-dimensionality must be overcome by a finite-dimensional approximation, which generally results in high-order controllers

In the last part of the course we will mix both techniques with the help of optimization over linear matrix inequalities

But before, we will survey briefly another basic but still open robust control problem, namely simultaneous stabilization

Simultaneous stabilization

Consider the system



where the same controller $y(s)/x(s)$ must stabilize simultaneously N plants $b_i(s)/a_i(s)$

Useful configuration if a sensor or actuator fails, or is turned off during start-up or shutdown

Despite the simplicity of the problem, it turns out that

> There is no algorithmic solution to
> the simultaneous stabilization problem

We know that the problem is undecidable, i.e. there is no algorithm with rational operations that can detect whether the system is stabilizable or not

The problem can be solved in special cases

- two plants
- static output feedback

but remains open in its full generality, and only heuristic algorithms have been designed

Strong stabilization

Consider two plants $P_1(s)$, $P_2(s)$ with no common unstable real poles and define the difference plant

$$P(s) = P_1(s) - P_2(s)$$

> P_1 and P_2 are simultaneously stabilizable iff P is stabilizable with a stable controller, or strongly stabilizable

More generally, the problem of knowing whether a given plant is strongly stabilizable is solved as follows

> P is strongly stabilizable iff
> it has an even number of real poles
> between every pair of real zeros
> in the closed right half-plane

There exists a more elaborate version for three plants:

Three plants P_1, P_2, P_3 are simultaneously stabilizable iff some plant P built from the P_is is stabilizable by a stable controller whose inverse is also stable

Strong stabilization

First example

$$P(s) = \frac{s-1}{s(s-2)}$$

Positive real zeros at $s = 1$ and $s = \infty$, with one pole at $s = 2$ in between, so the plant is not strongly stabilizable

Second example

$$P(s) = \frac{(s-1)^2(s^2-s+1)}{(s-2)^2(s+1)^3}$$

Positive real zeros at $s = 1$ (two) and $s = \infty$, with two poles at $s = 2$ in between, so the plant is strongly stabilizable

Third example

$$P_1(s) = \frac{1}{s+1} \quad P_2(s) = \frac{1}{(s+1)(s-1)}$$

Two plants simultaneously stabilizable iff

$$P(s) = P_2(s) - P_1(s) = -\frac{s-2}{(s+1)(s-1)}$$

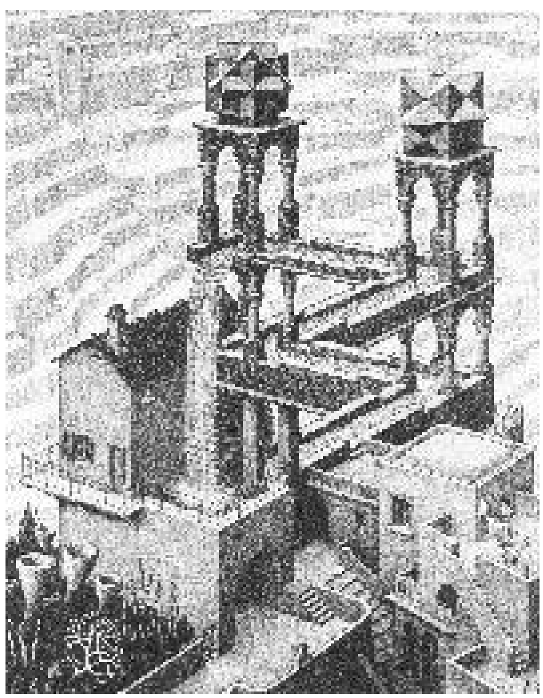
strongly stabilizable, which is the case since it has two zeros at $s = 2$ and $s = \infty$ and one unstable pole at $s = 1$

Strong stabilization

Whether a plant is strongly stabilizable or not can be checked very easily: either we compute the poles and zeros, or we use a modified version of the Routh table

Finding a stable controller is more difficult, available algorithms are based on interpolation of units in R and generally result in high-order controllers

It is has been shown that there is actually no upper bound on the degree of the strongly stabilizing controller



Infinite stream
Maurits Cornelis Escher
(Leeuwarden 1898 - Baarn 1972)

Simultaneous stabilization with static output feedback

In the special case of static output feedback the problem can be solved with the help of Hermite's stability criterion

Charles Hermite
(Dieuze 1822 - Paris 1901)

The polynomial $c(s) = c_0 + c_1 s + \cdots + c_n s^n$ is stable iff the symmetric matrix

$$H(c) = \sum_{i=0}^{n} \sum_{j=0}^{n} c_i c_j H_{ij}$$

is positive definite, where matrices H_{ij} are given and depend on the root clustering region only

Example
discrete-time stability and $n = 3$

$$H(c) = \begin{bmatrix} c_3^2 - c_0^2 & c_2 c_3 - c_0 c_1 & c_1 c_3 - c_0 c_2 \\ c_2 c_3 - c_0 c_1 & c_2^2 + c_3^2 - c_0^2 - c_1^2 & c_2 c_3 - c_0 c_1 \\ c_1 c_3 - c_0 c_2 & c_2 c_3 - c_0 c_1 & c_3^2 - c_0^2 \end{bmatrix}$$

Simultaneous stabilization with static output feedback

We build the Hermite matrices corresponding to each closed-loop characteristic polynomials

$$c_i(s) = a_i(s) + k b_i(s) \quad i = 1, \ldots, N$$

They are quadratic polynomial matrices in the static output feedback gain

$$H_i(k) = H_{i0} + H_{i1}k + H_{i2}k^2$$

and all of them must be positive definite, i.e.

$$H(k) = \text{block diag}_{i=1,\ldots,N}\, H_i(k) > 0$$

So if we arrange all distinct real zeros of matrix $H(k)$ in increasing order

$$k_0 = -\infty \leq k_1 \leq \cdots \leq k_m \leq k_{m+1} = +\infty$$

where $m \leq 2nN$ and we denote by π_i the number of positive eigenvalues of H in the interval $I_i =]k_i, k_{i+1}[$ we obtain the following result

> The static output feedback simultaneous stabilization problem is solved iff
> $k \in I_i$ when $\pi_i = nN$

Computing the eigenvalues of a quadratic polynomial matrix can be done efficiently

Simultaneous stabilization with static output feedback

Example

We want to simultaneously stabilize $N = 4$ operating points of the longitudinal short period mode of the F4E fighter aircraft

$$\begin{aligned} b_1(s)/a_1(s) &= (-351.1 - 367.6s)/(-113.0 + 51.46s + 31.84s^2 + s^3) \\ b_2(s)/a_2(s) &= (-676.5 - 346.6s)/(-31.50 + 38.53s + 31.32s^2 + s^3) \\ b_3(s)/a_3(s) &= (-455.4 - 978.4s)/(-262.5 + 84.85s + 33.12s^2 + s^3) \\ b_4(s)/a_4(s) &= (-538.7 - 790.3s)/(576.7 + 71.46s + 31.74s^2 + s^3) \end{aligned}$$

We obtain the following eigenvalue and inertia pattern

i	0	1	2	3	4
k_i	$-\infty$	-0.5764	-0.3219	-0.0466	0.0689
π_i	12	11	10	9	7

i	5	6	7	8	9
k_i	0.0962	0.1216	0.1543	1.0705	$+\infty$
π_i	7	7	7	8	

In the interval I_0 there are $nN = 12$ positive eigenvalues so the 4 plants are simultaneously stabilizable by a static output feedback

$$k < -0.5764$$

F-4E fighter

Simultaneous stabilization

In general, when the compensator is not static and when there are three plants or more to be stabilized, the problem is still open

Applying Hermite's stability criterion on each closed-loop characteristic polynomial

$$c_i(s) = a_i(s)x(s) + b_i(s)y(s) \quad i = 1, \ldots, N$$

and gathering controller coeffs x and y into a vector z, we obtain the algebraic condition

$$H(c_i) = \sum_j \sum_k z_j z_k H_{ijk} > 0 \quad i = 1, \ldots, N$$

which is a bilinear matrix inequality (BMI) problem non-convex in unknown vector

BMI problems arise frequently in robustness, this is the most difficult problem we can face

LMI relaxations or heuristics can be devised

Open problems related to
simultaneous stabilization

Vincent Blondel from Université Catholique de Louvain in Belgium

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is offering a bottle of good French champagne to the first author who answers the problem

For what values of δ is the system

$$\frac{s^2 - 1}{s^2 - 2\delta s + 1}$$

stabilizable by a stable controller whose inverse is also stable ?

Open problems related to simultaneous stabilization

One kilogram of famous Belgian chocolates are offered for the solution of the previous problem when $\delta = 0.9$

In the same vein, it is known that the 3 dt plants

$$P_1(z) = \frac{1}{z} \quad P_2(z) = \frac{1}{z} + \beta \quad P_3(z) = \frac{1}{z} - \beta$$

are simultaneously stabilizable iff $|\beta| < \frac{\Gamma^4(1/4)}{4\pi^2} = 4.377$

Two bottles of whisky are offered by Blondel for general stabilizability conditions for three arbitrary first order systems