GRADUATE COURSE ON
POLYNOMIAL METHODS FOR
ROBUST CONTROL
PART III.2

ROBUST DESIGN: PARAMETRIZATION OF ALL CONTROLLERS

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Disks of Newton
František Kupka (1871-1957)

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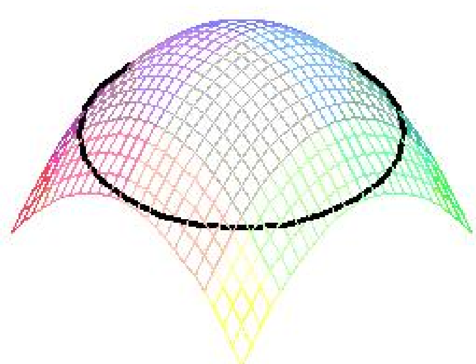
Summary

We have seen that robust design can be performed by pole placement, provided a good approximation of the non-convex stability region of a fixed degree polynomial

Alternatively, non-convexity of the stability region can also be overcome by a rational parametrization of all stabilizing controllers

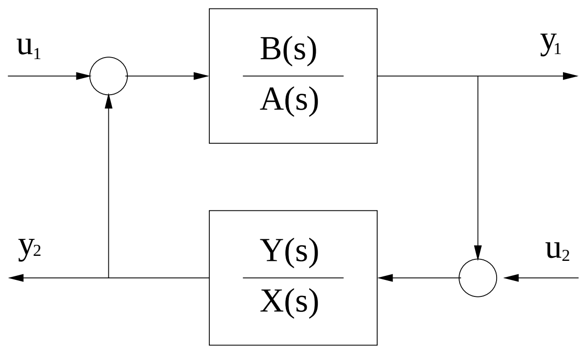
The parametrization is infinite-dimensional, but it turns out to be convex which is crucial for design



In the sequel we show that it allows to parametrize in a convex way the whole set of robustly stabilizing controllers for a specific class of uncertainty

Rational coprime factorization

We consider the controller-plant setup



where plant and controller transfer function

$$\frac{b(s)}{a(s)}, \quad \frac{y(s)}{x(s)}$$

are now ratios of coprime proper stable rational functions (and not polynomials anymore)

Example
Plant

$$\frac{b(s)}{a(s)} = \frac{s-1}{s+2}$$

can be written as

$$\frac{b(s)}{a(s)} = \frac{\left(\frac{s-1}{s+1}\right)}{\left(\frac{s+2}{s+1}\right)}$$

where both $a(s)$ and $b(s)$ are proper stable rational

Normalized coprime factorization

A suitable choice of a coprime factorization of a rational function

$$\frac{b(s)}{a(s)} = \frac{\left(\frac{b(s)}{c(s)}\right)}{\left(\frac{a(s)}{c(s)}\right)}$$

is the so-called normalized factorization, where polynomial $c(s)$ solves the quadratic spectral factorization equation

$$a^\star(s)a(s) + b^\star(s)b(s) = c^\star(s)c(s)$$

In ct $c^\star(s) = c(-s)$, in dt $c^\star(z) = c(z^{-1})$

Example

Plant

$$\frac{b(s)}{a(s)} = \frac{s-1}{s+2}$$

Denominator polynomial $c(s)$ satisfies

$$\begin{aligned}&(s-1)^\star(s-1) + (s+2)^\star(s+2)\\ &\quad = (-s-1)(s-1) + (-s+2)(s+2)\\ &\quad = 5 - s^2 = c^\star(s)c(s)\end{aligned}$$

Solving the quadratic equation, we obtain the stable polynomial

$$c(s) = \sqrt{5} + \sqrt{2}s$$

Internal stability

It turns out that the whole set of proper stable rational functions (further denoted by R) is a ring, just as the set of polynomials

We will encounter the Diophantine equation or Bézout equation (from Etienne Bézout [1730 Nemours - 1783 Fontainebleau], no photo sorry)

$$ax + by = 1$$

now defined over R

A controller y/x internally stabilizes the plant b/a iff all the 4 transfer functions

$$\begin{array}{ll} z_1 = \dfrac{bx}{ax+by}u_1 & z_1 = \dfrac{-by}{ax+by}u_2 \\ z_2 = \dfrac{-by}{ax+by}u_1 & z_2 = \dfrac{-ay}{ax+by}u_2 \end{array}$$

are stable

We see that denominator $ax + by$ must be a unit in R (i.e. reciprocal also belongs to R)

Parametrization of all stabilizing controllers

Frequently called Youla or Youla-Kučera or also Q-parametrization, published around 1975

All stabilizing controllers are generated by all solution pairs to Bézout equation

$$\frac{y}{x} = \frac{y_0 - aq}{x_0 + bq}$$

where x_0, y_0 is a particular solution and q is any proper stable rational function such that $x_0 + bq \neq 0$

This fundamental result launched the entire area of research on polynomial methods

Design in three steps:

1. Express plant as ratio of elements in R
2. Find particular solution to Bézout equation
3. Use degrees of freedom in q to optimize some criterion or reach some performance

Affine parametrization

Main advantage of YK parametrization:
it is affine in the parameter q

Take our 4 stable transfer functions

$$z_1 = \frac{bx}{ax+by}u_1 \quad z_1 = \frac{-by}{ax+by}u_2$$
$$z_2 = \frac{-by}{ax+by}u_1 \quad z_2 = \frac{-ay}{ax+by}u_2$$

Recall that

$$ax + by = 1$$
$$x = x_0 + bq$$
$$y = y_0 - aq$$

so that our transfers can be written as

$$z_1 = b(x_0 + bq)u_1 \quad z_1 = -b(y_0 - aq)u_2$$
$$z_2 = -b(y_0 - aq)u_1 \quad z_2 = -a(x_0 + bq)u_2$$

All of them depend affinely in the parameter

Convex constraints on transfer functions = convex constraints on parameter q

Parametrization of all stabilizing controllers

Basic example
Integrator plant

$$\frac{b}{a} = \frac{1}{s} = \frac{\left(\frac{1}{s+1}\right)}{\left(\frac{s}{s+1}\right)}$$

One stabilizing controller

$$\frac{y_0}{x_0} = 1$$

particular solution to Bézout equation

$$\frac{s}{s+1}x + \frac{1}{s+1}y = 1$$

All stabilizing controllers

$$\frac{y}{x} = \frac{1 - \frac{s}{s+1}q}{1 + \frac{1}{s+1}q}$$

all solutions to Bézout equation

Another stabilizing controller

$$\frac{y_1}{x_1} = \frac{1 - \frac{s}{s+1}\frac{s+1}{s+2}}{1 + \frac{1}{s+1}\frac{s+1}{s+2}} = \frac{2}{s+3}$$

can be obtained with the choice

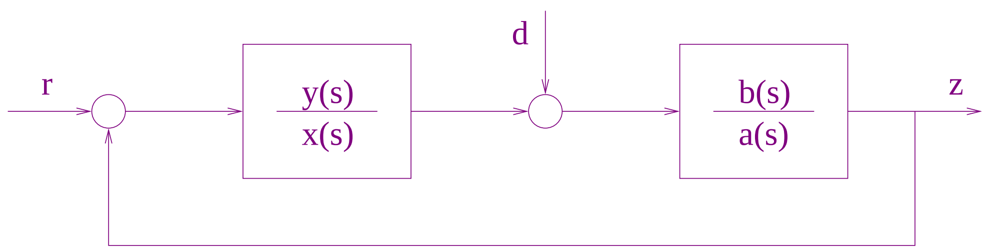
$$q = \frac{s+1}{s+2}$$

Parametrization of all stabilizing controllers

More interesting example
Consider the plant

$$\frac{b}{a} = \frac{1}{(s-1)(s-2)}$$

in the unity-feedback configuration



We seek a controller y/x ensuring
Spec 1: internal stability
Spec 2: $z \to 1$ when r unit step and $d = 0$
Spec 3: $z \to 0$ when d sinusoid 10 rad/s and $r = 0$

All controllers satisfying spec 1 are given by $y/x = (y_0 - aq)/(x_0 + bq)$ for q proper stable rational function

Transfer function from r to $z = b(y - aq)$ hence spec 2 holds iff

$$b(0)(y(0) - a(0)q(0)) = 1$$

Transfer function from d to $z = b(x + bq)$ hence spec 3 holds iff

$$b(j10)(x(j10) + b(j10)q(j10)) = 0$$

Design problem = algebraic problem of finding q such that the above equations hold

Finding the Youla-Kučera parameter

Plant factorization

$$\frac{b}{a} = \frac{\left(\frac{1}{(s+1)^2}\right)}{\left(\frac{(s-1)(s-2)}{(s+1)^2}\right)}$$

Bézout equation $ax+by=1$ yields a stabilizing controller

$$\frac{y_0}{x_0} = \frac{\left(\frac{19s-11}{s+1}\right)}{\left(\frac{s+6}{s+1}\right)}$$

Algebraic constraints on Youla-Kučera parameter

$$\begin{aligned} q(0) &= 6 \\ \operatorname{Re} q(10j) &= -94 \\ \operatorname{Im} q(10j) &= 70 \end{aligned}$$

Three constraints, so three coefficients to be found

$$q(s) = q_0 + q_1 \frac{1}{s+1} + q_2 \frac{1}{(s+1)^2}$$

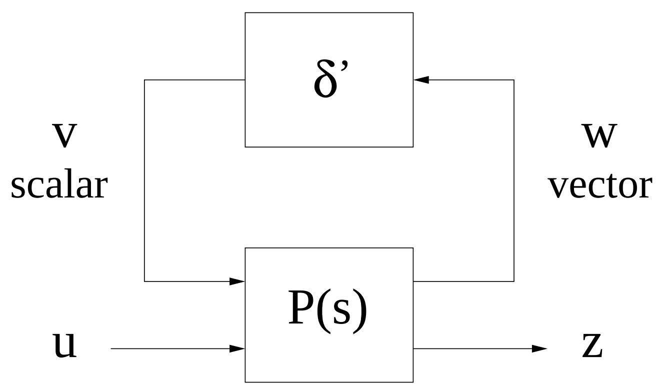
Report the solution $q_0 = -79$, $q_1 = -723$ and $q_2 = 808$ into the controller parametrization

$$\frac{y}{x} = \frac{y_0 - aq}{x_0 + bq} = \frac{-60s^4 - 598s^3 + 2515s^2 - 1794s + 1}{s(s^2+100)(s+9)}$$

Rank-one uncertainty

Using the Youla-Kučera parametrization of all stabilizing controllers, we will now describe a convex parametrization of all robust controllers obtained by Rantzer and Megretski in 1994

It is valid for a specific (yet useful) class of systems with rank-one LFT uncertainty



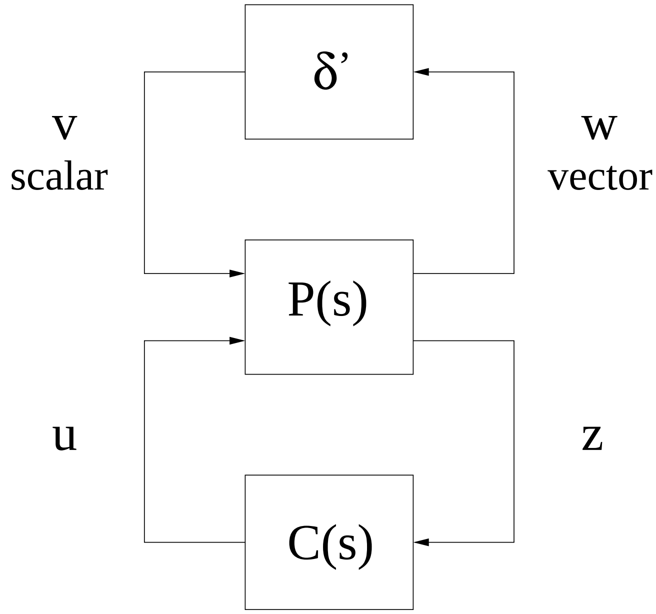
where uncertainty vector δ satisfies

$$\|\delta\| \leq 1$$

LFT = linear fractional transformation
rank-one = norm-bounded or unstructured
$\|.\| = l_1$, l_2 (Euclidean) or l_∞ norm

Controller parametrization

Now consider the closed-loop system



From the Youla-Kučera parametrization, the transfer function from v to w is given by

$$t_1 + t_2 q$$

where t_1 and t_2 are given elements in R and q is a free parameter in R

Closed-loop system is robustly stable iff

$$\frac{1}{1 + \delta'(t_1 + t_2 q)}$$

belongs to R for all admissible uncertainty $\|\delta\| \leq 1$

In this form, non-convex condition in q

Convex parametrization

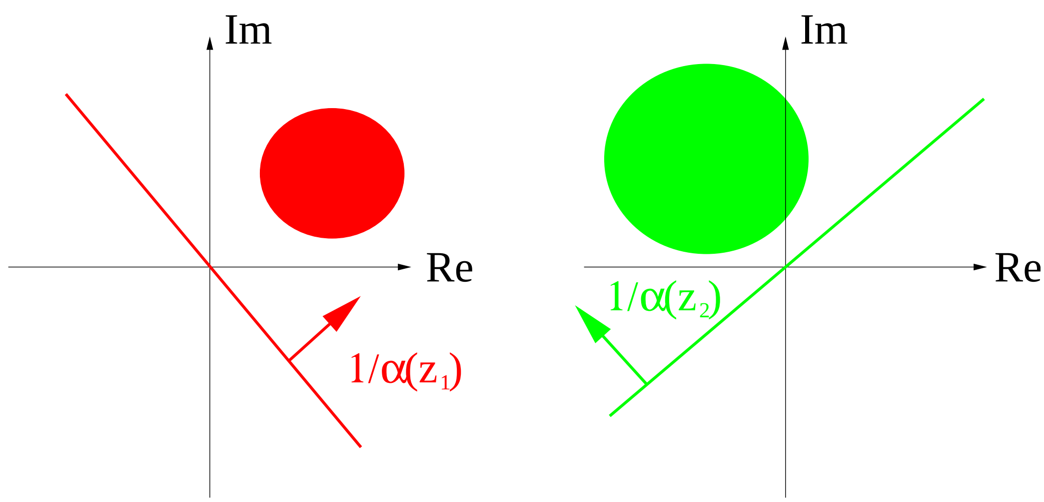
Stability is lost along stability boundary D (imaginary axis or unit circle) so the condition

$$\frac{1}{1+\delta'(t_1+t_2q)} \in R \text{ for all } \|\delta\| \leq 1$$

is equivalent to the zero exclusion condition on the value set of the rational denominator

$$1+\delta'(t_1+t_2q)(z) \neq 0 \text{ for all } z \in D \text{ for all } \|\delta\| \leq 1$$

For a fixed frequency z the above value set V is convex hence there exists a hyperplane separating V from the origin



Equivalently, for all fixed z in D there exists a complex number $\alpha(z)$ in R such that

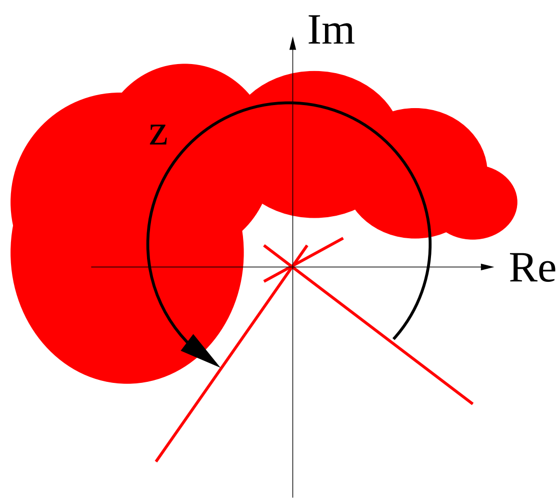
$$\text{Re}\,(\alpha(z)[1+\delta'(t_1+t_2q)(z)]) > 0 \text{ for all } \|\delta\| \leq 1$$

$\alpha(z)$ is orthogonal to the separating hyperplane

Convex parametrization

Only the argument of $\alpha(z)$ matters, so by proper choice of $\|\alpha(z)\|$ (technical part of the proof) it can be shown that this is equivalent to the existence of a $\alpha(z)$ in R valid over the whole frequency range

$$\text{Re}\,(\alpha(z)[1+\delta'(t_1+t_2 q)(z)]) > 0 \text{ for all } z \in D \text{ and } \|\delta\| \leq 1$$



Coprime factorization over R of parameter

$$q(z) = \frac{\beta(z)}{\alpha(z)}$$

yields the equivalent condition

$$\text{Re}\,\alpha(z) + \delta' \text{Re}\,(t_1(z)\alpha(z) + t_2(z)\beta(z)) > 0$$

valid for all $z \in D$ and $\|\delta\| \leq 1$

Convex parametrization

The last step is in using the definition of the dual norm

$$\|x\|_{\text{dual}} = \max\{x'y \ : \ \|y\| \leq 1\}$$

so that the condition

$$\text{Re}\,\alpha(z) + \delta'\text{Re}\,(t_1(z)\alpha(z) + t_2(z)\beta(z)) > 0$$

for all $z \in D$ and $\|\delta\| \leq 1$ is equivalent to

$$\|\text{Re}\,(t_1(z)\alpha(z) + t_2(z)\beta(z))\|_{\text{dual}} < \text{Re}\,\alpha(z)$$

for all $z \in D$ (dependence on δ is removed)

The above condition is convex in design parameters $\alpha(z)$ and $\beta(z)$, so we have proved the main result

> The whole set of robustly stabilizing controllers for rank-one LFT uncertainties is described by a convex frequency-dependent inequality

We can solve the frequency dependent condition with the help of optimization over linear matrix inequalities (LMIs), see the last part of the course

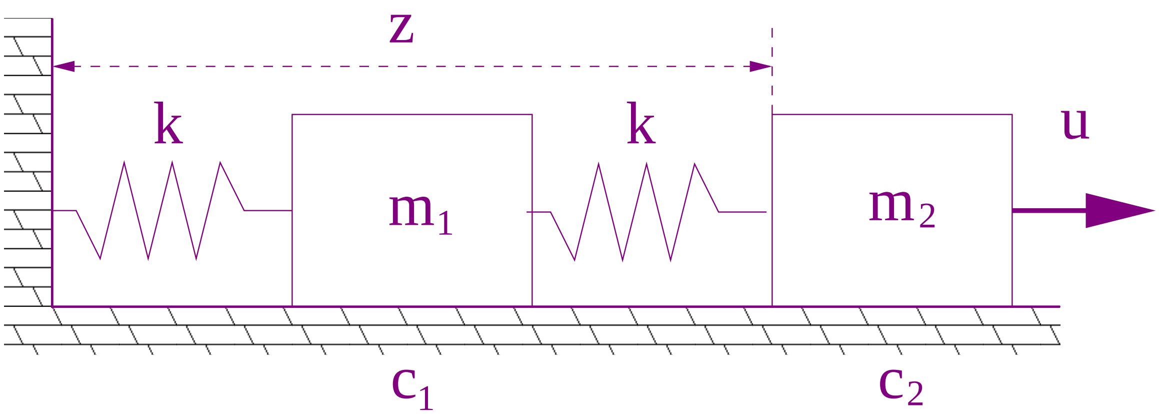
Main strength: the condition is convex

Main drawback: the condition is infinite-dimensional since the unknowns live in R, and design may result in a controller of arbitrarily high degree

Mass-spring system

Example

Consider the mass-spring system



where m_i is the mass, c_i the damping coefficient, k the spring constant, output z the position of the second mass and input u the control force

$$m_1 = 2.25\text{kg},\ c_1 = 3.25\text{Ns/m},\ k = 423\text{N/m}$$
$$m_2 = 2.07\text{kg},\ c_2 = 8.18\text{Ns/m}$$

l_1 norm-bounded uncertainty

$$m_2 = \hat{m}_2 + \delta_1,\ c_2 = \hat{c}_2 + \delta_2,\ \|\delta\|_1 \leq 2$$

Open loop transfer function

$$\frac{b(s)}{a(s)} = \frac{k}{(m_1 s^2 + c_1 s + k)(m_2 s^2 + c_2 s + k) - k^2} = \frac{k}{g_1(s) g_2(s) - k^2}$$

Mass-spring system

YK controller parametrization

$$\frac{y(s)}{x(s)} = \frac{-k - \frac{g_1 g_2 - k^2}{g_1 g_2} q}{1 - \frac{k}{g_1 g_2} q}$$

Affine closed-loop parametrization

$$t_1(s) + t_2(s) q(s) = \begin{bmatrix} \frac{s^2}{g_2(s)} \\ \frac{s}{g_2(s)} \end{bmatrix} + \begin{bmatrix} -\frac{ks^2}{g_1(s) g_2^2(s)} \\ -\frac{ks}{g_1(s) g_2^2(s)} \end{bmatrix} q(s)$$

Convex parametrization of robustly stabilizing controllers in real ω

$$\|\mathrm{Re} \begin{bmatrix} \frac{-\omega^2}{g_2(j\omega)} \\ \frac{j\omega}{g_2(j\omega)} \end{bmatrix} \left(\alpha(j\omega) - \frac{k}{g_1(j\omega) g_2(j\omega)} \beta(j\omega) \right) \|_\infty < \mathrm{Re}\,\alpha(j\omega)$$

Applying a primal-dual LP algorithm, Ghulchak finds the YK parameter

$$q = \frac{-1500s^5 - 2290s^4 + 2599s^3 + 8838s^2 + 8379s + 2933}{s^5 + 2.025s^4 + 1.294s^3 - 1.211s^2 - 2.051s - 1.06}$$

see www.control.lth.se/~ghulchak

Other design methods

Besides Rantzer and Megretski's approach there are several other robust design methods based on polynomials or rational fractions and the YK parameter

- mixed sensitivity problem [Kwakernaak]
  H_∞ optimization, spectral factorization
  www.math.utwente.nl/∼twhuib
- loop shaping [Doyle, Francis et al]
  model matching, Nevanlinna-Pick interpolation
  www.control.utoronto.ca/people/profs/francis/dft.html
- convex optimization [Boyd]
  linear and quadratic programming, LMIs
  www.stanford.edu/∼boyd

A finite-dimensional approximation of the infinite-dimensional but convex set R of proper stable rational functions must be used when implementing algorithms (convex programming, duality, lower and upper bounds, truncation)