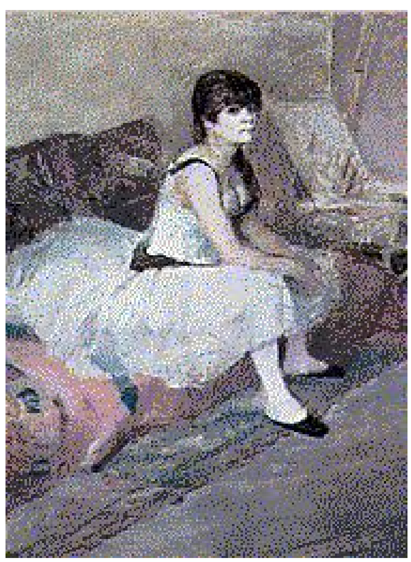
GRADUATE COURSE ON
POLYNOMIAL METHODS FOR
ROBUST CONTROL
PART III.1

ROBUST DESIGN:
POLE PLACEMENT

Didier HENRION
www.laas.fr/~henrion
henrion@laas.fr



Dancer seated on a pink divan
Henri de Toulouse-Lautrec (1864-1901)

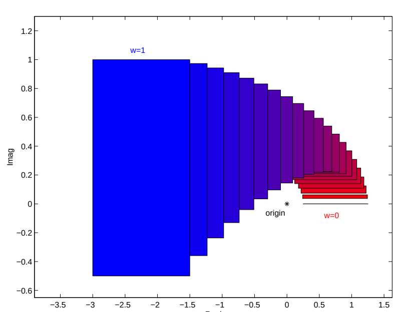
October-November 2001

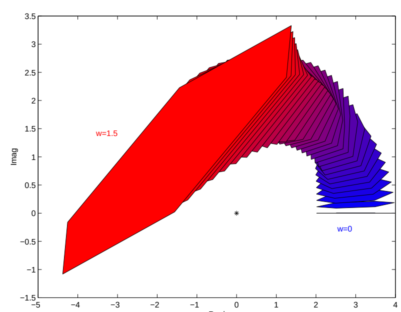
Summary

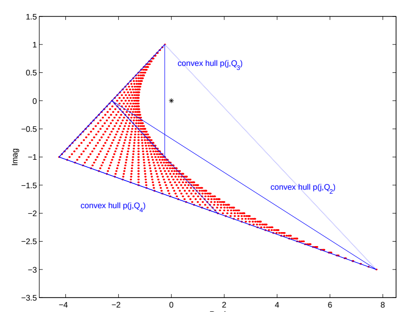
In the first part of the course we studied robust stability analysis

We saw that checking robust stability can be easy (polynomial-time algorithms) or more difficult (NP-hard problem), depending namely on the uncertainty description

We focused on polytopic uncertainty:
• Interval scalar polynomials
Kharitonov's theorem (ct only)
• Polytope of scalar polynomials
(affine polynomial families)
Edge theorem
• Interval matrix polynomials
(multiaffine polynomial families)
Mapping theorem
• Polytopes of matrix polynomials
(polynomic polynomial families)





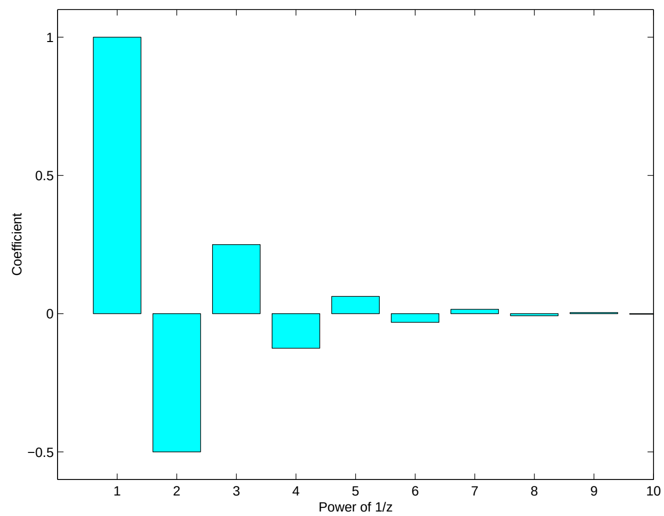


Robust design

Only in very special cases robust analysis techniques can be extended to design
(for example the 32 plant theorem)

In this third part of the course we will describe robust design algorithms based either on
• polynomial pole placement equations, or
• rational Youla-Kučera parametrization of stabilizing controllers

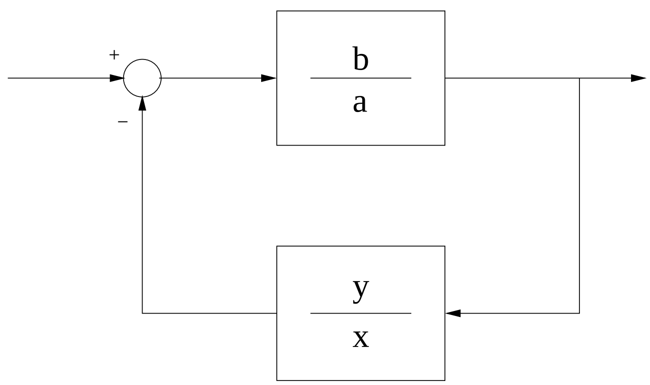
We will see that robust design is a difficult problem when looking for low-order controllers



Coefficients of $1/(1+0.5z^{-1}) = r_0 + r_1 z^{-1} + r_2 z^{-2} + \cdots$

Nominal Pole placement

We consider the SISO feedback system



Closed-loop transfer function

$$\frac{bx}{ax+by}$$

In the absence of hidden modes (a and b coprime polynomials), pole placement amounts to finding polynomials x and y solving the Diophantine equation (from Diophantus of Alexandria 200-284)

$$ax+by=c$$

where c is a given closed-loop characteristic polynomial capturing the desired system poles

Polynomial parametrization of controllers

Let x',y' be the least degree solution w.r.t y' of the Diophantine design equation

$$ax + by = c$$

All controllers achieving pole placement are given by

$$\frac{y}{x} = \frac{y' + at}{x' - bt}$$

where t is a free polynomial parameter

Example

Water tank with inflow-to-level transfer function

$$\frac{b}{a} = \frac{1}{s+1}$$

To counteract level disturbances, a PI controller is sought that places closed-loop poles at -6 and -10, i.e. we have to solve the Diophantine equation

$$(s+1)x + y = c = (s+6)(s+10)$$

Controllers

$$\frac{y}{x} = \frac{45 + (s+1)t}{s + 15 - t}$$

PI controller for $t = 15$

$$\frac{y_0}{x_0} = 15 + \frac{60}{s}$$

Pole placement

Example

Consider the rotary hydraulic test rig with identified sampled transfer function

$$\frac{b}{a} = \frac{z^{-3}(-0.0036 + 0.1718z^{-1} + 0.3029z^{-2} - 0.0438z^{-3} - 0.0775z^{-4})}{1 - 2.8805z^{-1} + 3.7827z^{-2} - 2.8269z^{-3} + 1.1785z^{-4} - 0.2116z^{-5}}$$

in the backward shift operator z^{-1}

The desired closed-loop characteristic polynomial is

$$c = (1 - 0.3z^{-1})(1 - 0.4z^{-1}) = 1 - 0.7z^{-1} + 0.12z^{-2}$$

Solving the Diophantine equation $ax + by = c$ yields the unstable controller

$$\frac{y'}{x'} = \frac{2.9023 - 6.7682z^{-1} + 7.4670z^{-2} - 4.3287z^{-3} + 1.0169z^{-4}}{1 + 2.1805z^{-1} + 2.6182z^{-2} + \cdots - 0.3725z^{-6}}$$

With the choice

$$t = 1.2222 - 0.1952z^{-1} - 0.1310z^{-2} + 0.5663z^{-3} + 0.8805z^{-4} + 0.5677z^{-5}$$

as a free polynomial parameter we obtain a stable controller of higher order

$$\frac{y''}{x''} = \frac{1.6801 - 3.0525z^{-1} + 2.4125z^{-2} + \cdots + 0.1201z^{-10}}{1 + 2.1085z^{-1} + 2.6182z^{-2} + \cdots - 0.0440z^{-12}}$$

Pole placement: numerical aspects

The polynomial Diophantine equation

$$ax + by = c$$

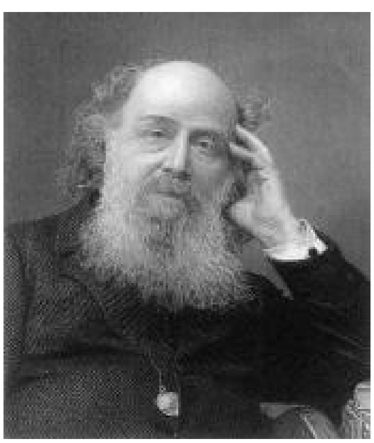
is linear in unknowns x and y, and denoting

$$a(s) = a_0 + a_1 s + \cdots + a_{d_a} s^{d_a}$$
$$x(s) = x_0 + x_1 s + \cdots + x_{d_b} x^{d_b}$$
etc..

we can identify powers of the indeterminate s to build a linear system of equations

$$\left[\begin{array}{ccc|ccc} a_0 & & & b_0 & & \\ a_1 & \ddots & & b_1 & \ddots & \\ \vdots & & a_0 & \vdots & & b_0 \\ a_{d_a} & & a_1 & b_{d_b} & & b_1 \\ & \ddots & \vdots & & \ddots & \vdots \\ & & a_{d_a} & & & b_{d_b} \end{array}\right] \left[\begin{array}{c} x_0 \\ x_1 \\ \vdots \\ x_{d_x} \\ \hline y_0 \\ y_1 \\ \vdots \\ y_{d_y} \end{array}\right] = \left[\begin{array}{c} c_0 \\ c_1 \\ \vdots \\ c_{d_c} \end{array}\right]$$

The above matrix is called Sylvester matrix, it has a special Toeplitz banded structure that can be exploited when solving the equation



James J Sylvester
(1814 London - 1897 London)

Otto Toeplitz
(1881 Breslau - 1940 Jerusalem)

Pole placement for MIMO systems

Pole placement can be performed similarly for a plant left MFD

$$A^{-1}(s)B(s)$$

with a controller right MFD

$$Y(s)X^{-1}(s)$$

The Diophantine equation to be solved is now over polynomial matrices

$$A(s)X(s) + B(s)Y(s) = C(s)$$

and right hand-side matrix $C(s)$ captures information on invariant polynomials and eigenstructure

For example $C(s)$ may contain H_2 or H_∞ optimal dynamics (obtained with spectral factorization)

Robust pole placement

Now assume that the plant transfer function

$$\frac{b(q)}{a(q)}$$

contains some uncertain parameter q

The problem of robust pole placement will then consists in finding a controller

$$\frac{y}{x}$$

such that the uncertain closed-loop charact. polynomial

$$a(q)x + b(q)y = c(q)$$

is robustly stable

How can we find x, y to ensure robust stability of $c(q)$ for all admissible uncertainty q ?

Coefficients of c are linear in x and y, but are stability conditions also linear in c ?

Stability criteria

Given a continuous-time polynomial

$$p(s) = p0 + p1s + \cdots + p_{n-1}s^{n-1} + p_n s^n$$

with $p_n > 0$ we define its $n \times n$ Hurwitz matrix

$$H(p) = \begin{bmatrix} p_{n-1} & p_{n-3} & & 0 & 0 \\ p_n & p_{n-2} & & \vdots & \vdots \\ 0 & p_{n-1} & \ddots & 0 & 0 \\ 0 & p_n & & p_0 & 0 \\ \vdots & \vdots & & p_1 & 0 \\ 0 & 0 & & p_2 & p_0 \end{bmatrix}$$

Hurwitz stability criterion: Polynomial $p(s)$ is stable iff all principal minors of $H(p)$ are > 0

Example

For $n = 4$ the stability conditions are

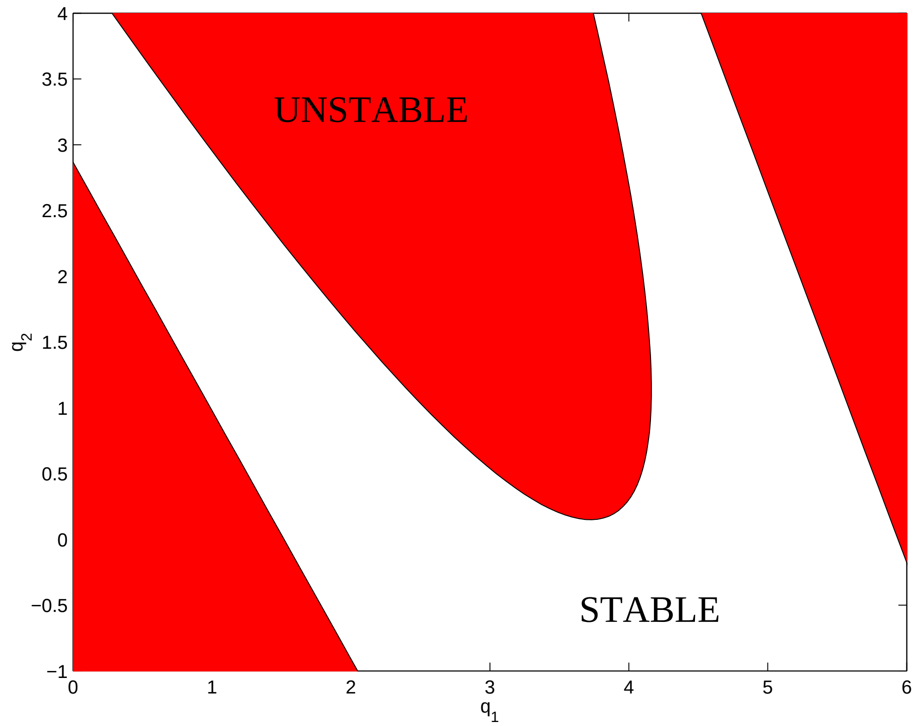
$$\begin{array}{l} p_3 > 0 \\ p_2p_3 - p_1p_4 > 0 \\ p_1p_2p_3 - p_0p_3^2 - p_1^2p_4 > 0 \\ p_0p_1p_2p_3 - p_0^2p_3^2 - p_0p_1^2p_4 > 0 \end{array}$$

Highly non-linear inequalities

Similar criteria for discrete-time polynomials (Jury matrix) or any other stability region (sector, parabola etc..)

Non-convexity of stability domain

Main problem: the stability domain in the space of polynomial coefficients p_i is non-convex in general



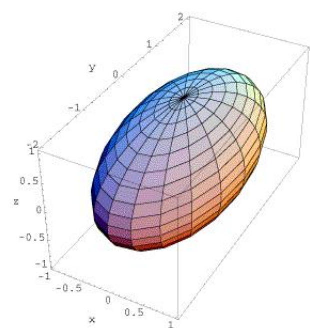
Discrete-time stability domain in (q_1, q_2) plane for polynomial $p(z,q) = (-0.825 + 0.225q_1 + 0.1q_2) + (0.895 + 0.025q_1 + 0.09q_2)z + (-2.475 + 0.675q_1 + 0.3q_2)z^2 + z^3$

How can we overcome the non-convexity of the stability conditions in the coefficient space ?

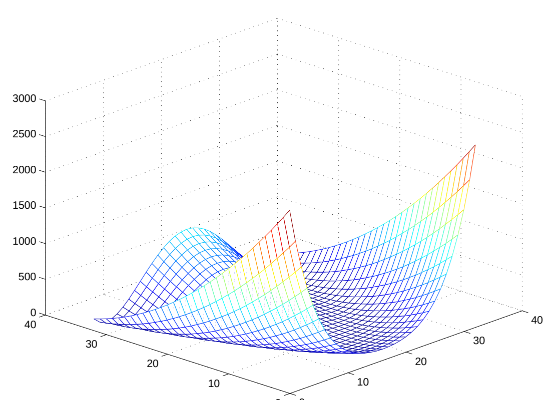
Handling non-convexity

Basically, we can pursue two approaches:

• we can approximate the non-convex stability domain with a convex domain (segment, polytope, sphere, ellipsoid, LMI)



• we can address the non-convexity with the help of non-convex optimization (global or local optimization)



Approximation of the stability domain

From the tools of robust stability analysis, we can build around a nominally stable polynomial

- stability segments (eigenvalue criterion)
- stability boxes (Kharitonov's theorem)
- stability polytopes (Edge theorem)

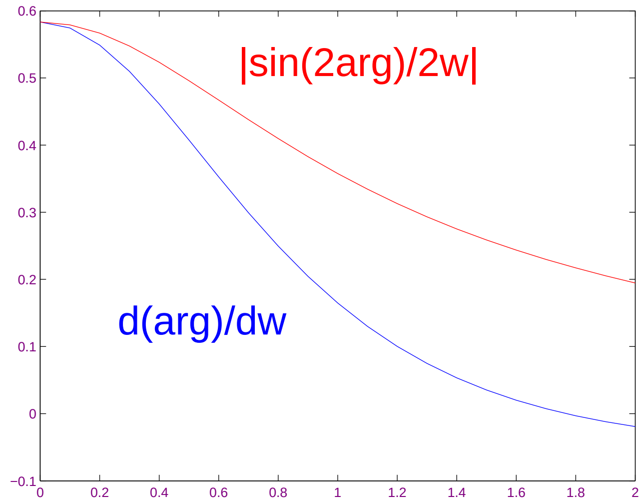
There exists other results, such as Rantzer's growth condition: a polynomial $g(s)$ is a convex direction iff

$$\frac{d}{d\omega}\arg g(j\omega) \leq \left|\frac{\sin 2\arg g(j\omega)}{2\omega}\right|, \quad \omega > 0$$

It means that given any stable $f(s)$ such that $f(s)+g(s)$ is stable then the whole segment $[f(s),\, g(s)]$ is stable

Example

$g(s) = 24 + 14s - 13s^2 - 2s^3 + s^4$ is a growth direction



Stability polytopes

Largest hyper-rectangle around a nominally stable polynomial

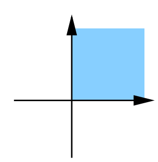
$$p(s) + r \sum_{i=0}^{n} [-\varepsilon_i, \varepsilon_i] s^i$$

obtained with the eigenvalue criterion applied on the 4 Kharitonov polynomials

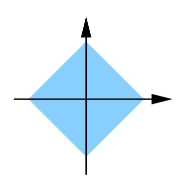
In general, there is no systematic way to obtain more general stability polytopes, namely because of computational complexity
(no analytic formula for the volume of a polytope)

Well-known candidates:

- ct LHP: outer approximation (necessary stab cond)
  positive cone $p_i > 0$



- dt unit disk: inner approximation (sufficient stab cond)
  diamond $|p_0| + |p_1| + \cdots + |p_{n-1}| < 1$



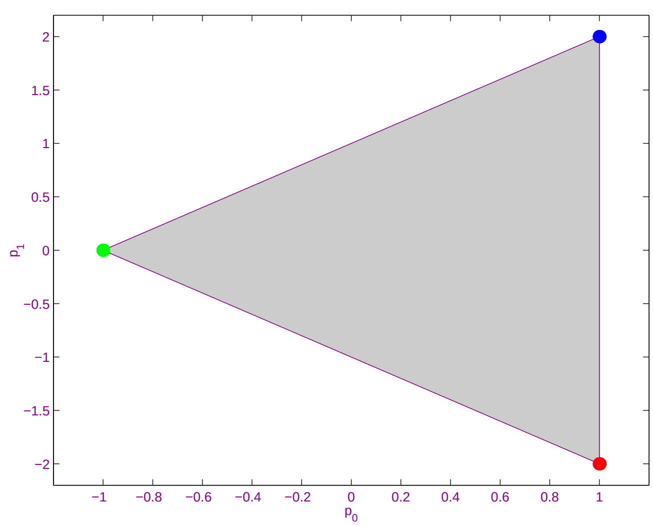
Stability polytopes

Necessary stab cond in dt: convex hull of stability domain is a polytope whose $n+1$ vertices are polynomials with roots $+1$ or -1

Example

When $n = 2$: triangle with vertices

$$
\begin{aligned}
(z+1)(z+1) &= 1+2z+z^2 \\
(z+1)(z-1) &= -1+z^2 \\
(z-1)(z-1) &= 1-2z+z^2
\end{aligned}
$$



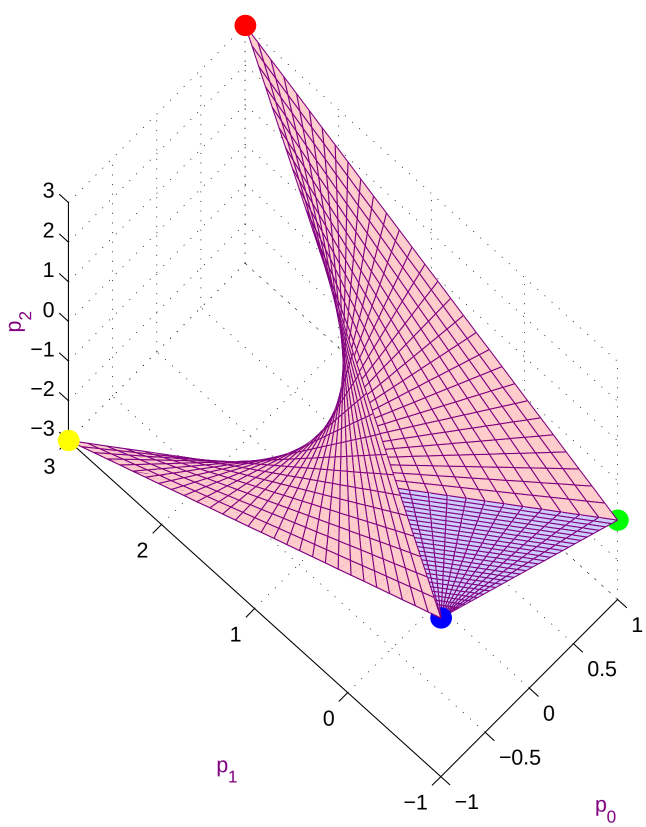
In this simple case it is the exact stability domain

Stability polytopes

Example

Third degree dt polynomial: two hyperplanes and a non-convex hyperbolic paraboloid with a saddle point at $p(z) = p_0 + p_1 z + p_2 z^2 + z^3 = z(1+z^2)$

$$\begin{aligned} (z+1)(z+1)(z+1) &= 1 + 3z + 3z^2 + z^3 \\ (z+1)(z+1)(z-1) &= -1 - z + z^2 + z^3 \\ (z+1)(z-1)(z-1) &= 1 - z - z^2 + z^3 \\ (z-1)(z-1)(z-1) &= -1 + 3z - 3z^2 + z^3 \end{aligned}$$



The convex hull is made of four hyperplanes

Stability hyper-spheres

Largest hyper-sphere around a nominally stable polynomial

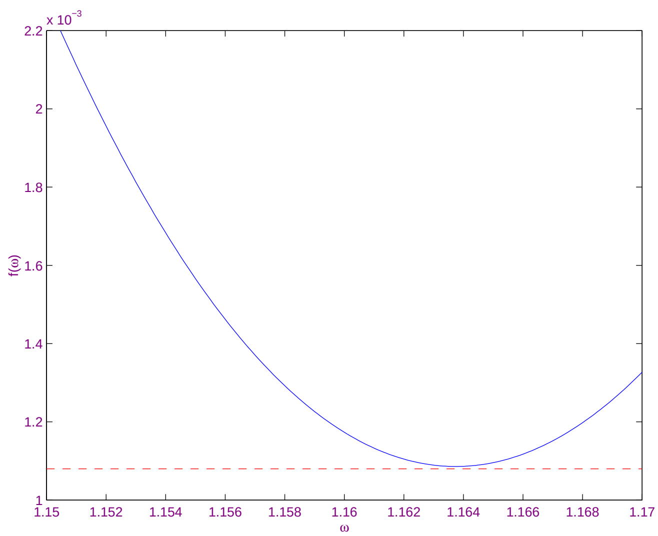
$$p(s) + \sum_{i=0}^{n} q_i s^i, \ \|q\| \leq r$$

has radius

$$r_{\text{max}} = \min\{|p_0|, |p_n|, \inf_{\omega>0} \sqrt{\frac{(\text{Re}\, p(j\omega))^2}{1 + w^4 + w^8 ..} + \frac{(\text{Im}\, p(j\omega))^2}{w^2 + w^6 + ..}}\}$$

Example

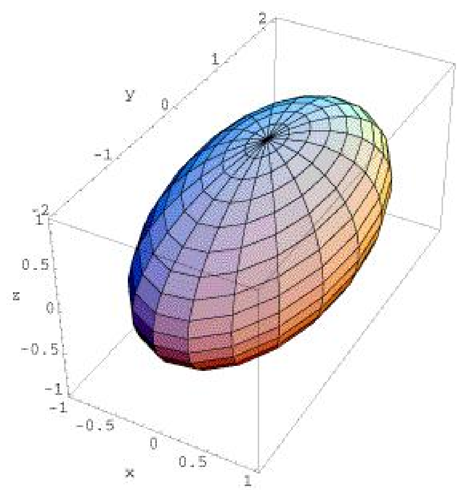
$(2 + q_0) + (1.4 + q_1)s + (1.5 + q_2)s^2 + (1 + q_3)s^3, \ \|q\| \leq r$



$r_{\text{max}} = \min\{2, \ 1, \ \inf_{\omega>0} f(w)\} = 1.08 \cdot 10^{-3}$

Stability ellipsoids

A weighted and rotated hyper-sphere is an ellipsoid



Using recently developed algorithms based on LMI optimization one can obtain ellipsoidal inner approximation of stability domains

$$E = \{p \,:\, (p-\bar{p})^T P (p-\bar{p}) \leq 1\}$$

where

p	coef vector of polynomial $p(s)$
$\bar{p}$	center of ellipsoid
P	positive definite matrix

Example

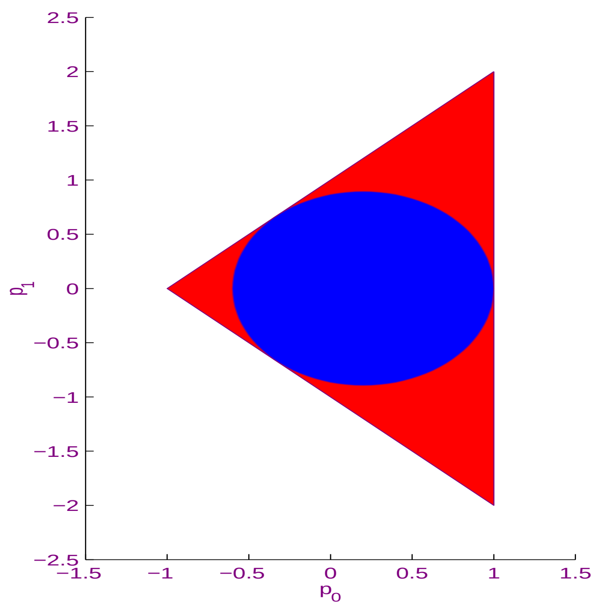
Discrete-time second degree polynomial

$$p(z) = p_0 + p_1 z + z^2$$

We solve the LMI problem and we obtain

$$P = \begin{bmatrix} 1.5625 & 0 \\ 0 & 1.2501 \end{bmatrix} \quad \bar{p} = \begin{bmatrix} 0.2000 \\ 0 \end{bmatrix}$$

which describes an ellipse E inscribed in the exact triangular stability domain S



Stability ellipsoids

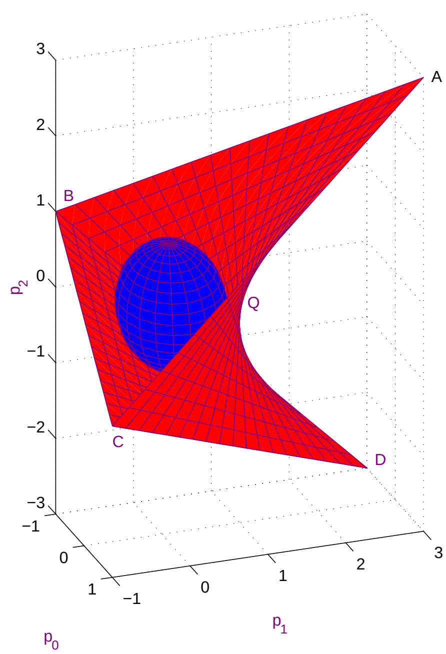
Example
Discrete-time third degree polynomial

$$p(z) = p_0 + p_1 z + p_2 z^2 + z^3$$

We solve the LMI problem and we obtain

$$P = \begin{bmatrix} 2.3378 & 0 & 0.5397 \\ 0 & 2.1368 & 0 \\ 0.5397 & 0 & 1.7552 \end{bmatrix} \quad \bar{x} = \begin{bmatrix} 0 \\ 0.1235 \\ 0 \end{bmatrix}$$

which describes a convex ellipse E inscribed in the exact stability domain S delimited by the non-convex hyperbolic paraboloid



Very simple scalar sufficient stability condition

$$2.4166 p_0^2 + 2.2088 p_1^2 + 1.8143 p_2^2 - 0.5458 p_1 + 1.1158 p_0 p_2 \leq 1$$

Volume of stability ellipsoid

In the discrete-time case, the well-known sufficient stability condition defines a diamond
$R = \{p : |p_0| + |p_1| + \cdots + |p_{n-1}| < 1\}$

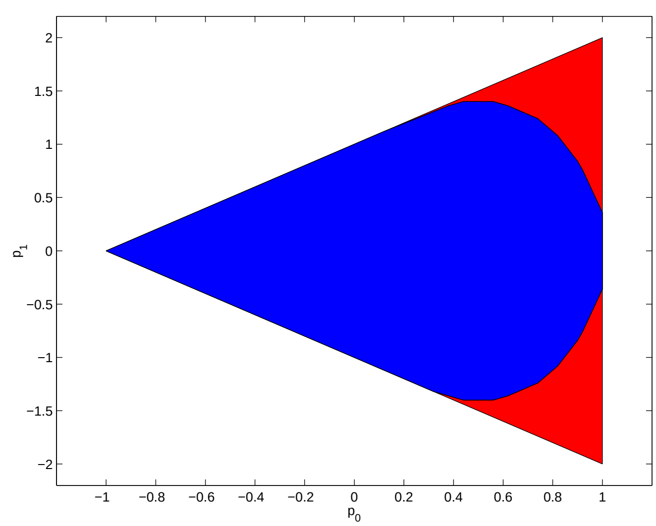
For different values of degree n, we compared volumes of exact stability domain S, ellipsoid E and diamond D

	$n = 2$	$n = 3$	$n = 4$	$n = 5$
Stability domain S	4.0000	5.3333	7.1111	7.5852
Ellipsoid E	2.2479	1.4677	0.7770	0.3171
Diamond D	2.0000	1.3333	0.6667	0.2667

E is "larger" than D, yet very small wrt S

LMI stability regions

In the fourth and last part of this course, we will propose better LMI inner approximations of the stability domain



Robust pole placement

Once we have a convex approximation of the stability region, we can perform design with

- linear programming (polytopes)
- quadratic programming (spheres, ellipsoids)
- semidefinite programming (LMIs)

Complexity of design algorithm increases
Conservatism of control law decreases



Application in robust design

First example

MIMO plant with right MFD

$$B(s)A^{-1}(s) = \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s+1 & 0 \\ 0 & s+1 \end{bmatrix}^{-1}$$

with uncertainty in parameter

$$b \in [0.5,\ 1.5]$$

We seek a proper first order controller

$$X^{-1}(s)Y(s) = \begin{bmatrix} s+x_1 & x_2 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} y_1 s + y_2 & y_3 s + y_4 \\ 0 & y_5 \end{bmatrix}$$

assigning robustly the closed-loop polynomial matrix

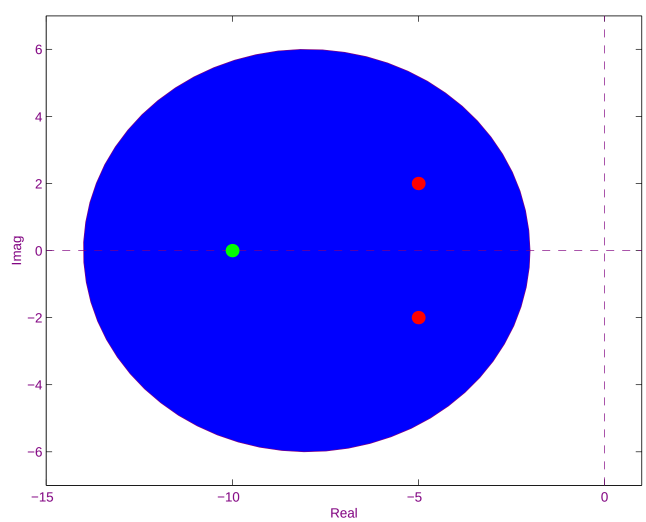
$$C(s) = \begin{bmatrix} s^2 + \alpha s + \beta & \delta(s) \\ 0 & s + \gamma \end{bmatrix}$$

whose coefficients live in the polytope

$$\begin{bmatrix} -14 & 1 & 0 \\ 16 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} > \begin{bmatrix} -196 \\ 56 \\ -4 \\ 2 \\ -14 \end{bmatrix}$$

These specifications amounts to assigning the poles within the disk

$$|s+8| < 6$$



Application in robust design

First example (end)

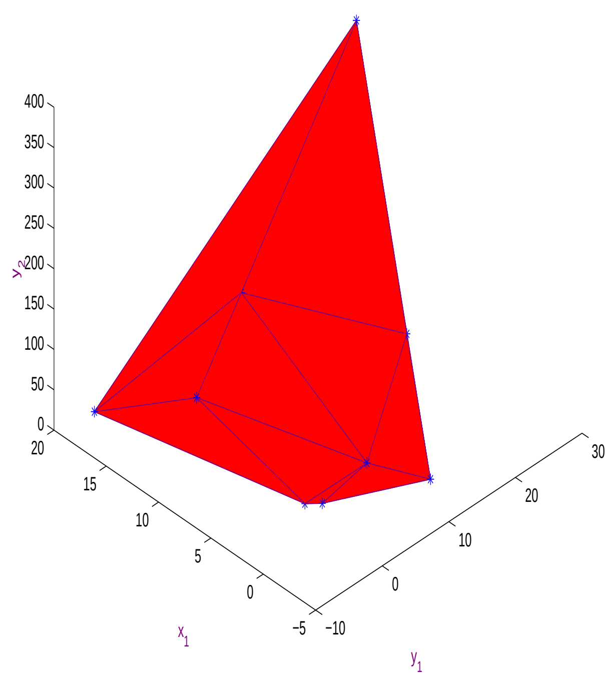
Equating powers of indeterminate s in the polynomial matrix Diophantine equation

$$X(s)A(s) + Y(s)B(s) = C(s)$$

we obtain the design inequalities

$$\begin{bmatrix} -13 & -7 & 0.5 \\ 14 & 8 & -1 \\ -1 & -1 & 0.5 \\ -13 & -21 & 1.5 \\ 14 & 24 & -3 \\ -1 & -3 & 1.5 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ y_2 \end{bmatrix} > \begin{bmatrix} -182 \\ 40 \\ -2 \\ -182 \\ 40 \\ -2 \end{bmatrix}$$

characterizing all parameters x_1, y_1 and y_2 of admissible robust controllers

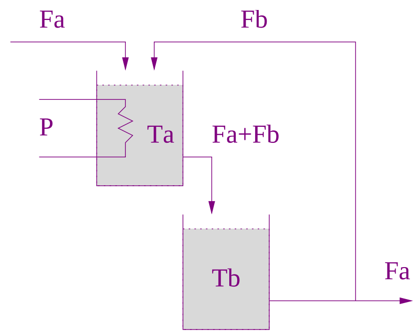


Corresponding polytope with 9 vertices

Application in robust design

Second example

We consider two mixing tanks arranged in cascade with recycle stream



The controller must be designed to maintain temp T_b of the second tank at desired set point by manipulating power P delivered by heater located in first tank - only available measurement is temp T_b

Nominal plant identification is carried out with least-squares method, resulting in 2nd order dt model

$$\frac{b(z)}{a(z)} = \frac{a_1 + a_2 z}{a_3 + a_4 z + z^2}$$

with nominal plant vector

$$\bar{a} = \left[\begin{array}{cccc} 0.0038 & 0.0028 & 0.2087 & -1.1871 \end{array} \right]$$

Positive definite matrix Q characterizing the uncertainty ellipsoid $\{a : (a - \bar{a})' Q (a - \bar{a}) \leq 1\}$ is readily available as a by-product of the identification technique

Application in robust design

Second example (end)

We perform robust pole placement in the inner ellipsoidal approximation

$$\{p \,:\, (p-\bar{p})'P(p-\bar{p}) \leq 1\}$$

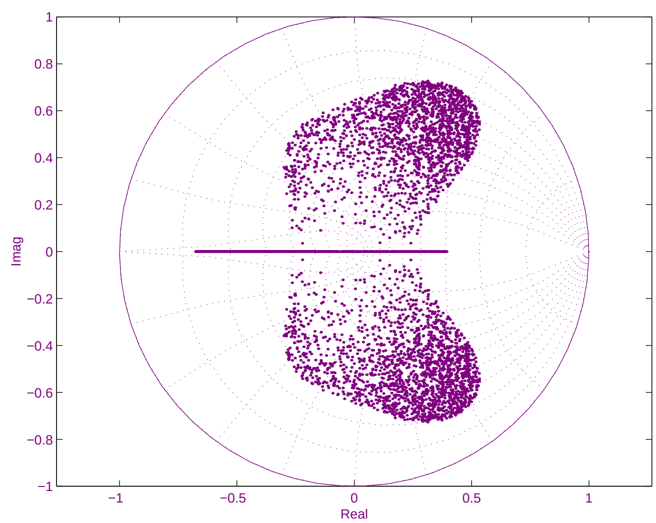
of the 3rd degree non-convex stability region, with

$$P = \begin{bmatrix} 2.3378 & 0 & 0.5397 \\ 0 & 2.1368 & 0 \\ 0.5397 & 0 & 1.7552 \end{bmatrix} \quad \bar{p} = \begin{bmatrix} 0 \\ 0.1235 \\ 0 \end{bmatrix}$$

We solve a convex quadratic optimization problem to obtain a first-order controller

$$\frac{y(z)}{x(z)} = \frac{0.3377 + 166.0z}{0.6212 + z}$$

robustly stabilizing the plant



Robust closed-loop root-locus for random admissible ellipsoidal uncertainty