GRADUATE COURSE ON
POLYNOMIAL METHODS FOR
ROBUST CONTROL
PART II.1

ROBUST STABILITY ANALYSIS: POLYTOPIC UNCERTAINTY

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Value set

Given a family of polynomials $p(s, q)$ with uncertainty $q \in Q$ the subset of the complex plane

$$\{p(z, q) : q \in Q\}$$

obtained when complex number z sweeps the stability boundary and q ranges over the whole uncertainty set Q is called the value set

Already encountered with continuous-time interval polynomials

- hyper-rectangle (box) Q
- stability boundary $z = j\omega$ for real $\omega \geq 0$
- value set = Kharitonov's rectangle

Useful when proving Kharitonov's theorem

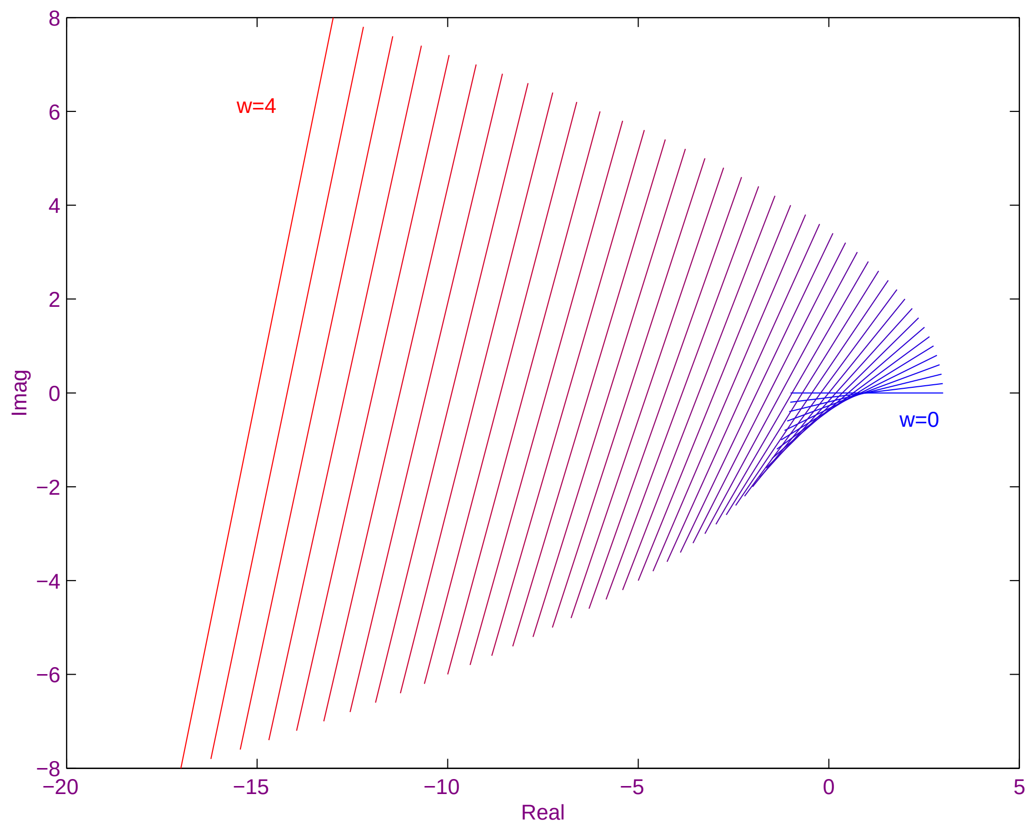
Can be generalized to any stability region and any pathwise connected uncertainty set Q

Value set as a segment

Example

Consider the uncertain polynomial
$p(s, q) = (3 - q) + (2 - q)s + s^3$, $q \in Q = [0, 4]$

For fixed real ω it holds Re $p(j\omega, q) = 3 - \omega^2 - q$ and Im $p(j\omega, q) = (2 - q)\omega$ so the value set $p(j\omega, Q)$ is a straight line segment joining $p(j\omega, 0)$ and $p(j\omega, 4)$



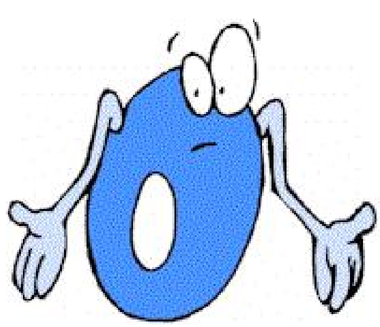
Zero exclusion condition

Given a family of polynomials $p(s, q)$ with
- open stability region (e.g LHP, unit disk)
- invariant degree (for unbounded stability region)
- coefficients continuous in parameter $q \in Q$
- pathwise connected uncertainty set Q
- at least one stable member $p(s, q_0)$

then

$p(s, q)$ is robustly stable iff
$0 \notin p(z, Q)$
for all complex z sweeping
the stability boundary

Very general graphical robust stability condition, requires one-dimensional sweeping of stability boundary



Robust Schur stability

A discrete-time polynomial is Schur when all its roots belong to the unit disk

Example

Is the interval polynomial

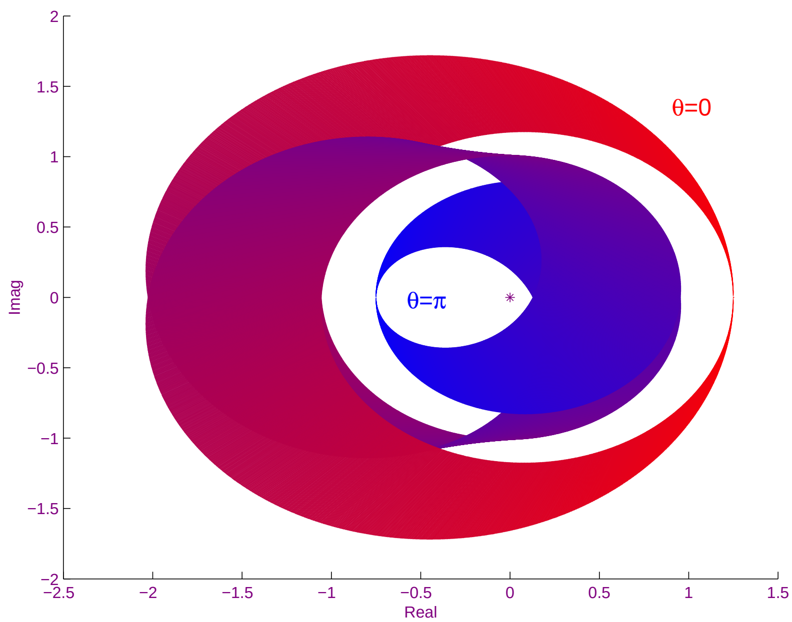
$p(z, q) = -(0.25 + q) + (0.5 + q)z^2 + z^3, \ |q| \leq 0.3$

robustly Schur ? We apply the zero exclusion condition:

- member $p(z, 0)$ is stable
- stability boundary $z = e^{i\theta}, \ \theta \in [0, \ 2\pi[$

Issai Schur

(Mogilyov, Belarus 1875 - Tel Aviv, Palestine 1941)



Value set

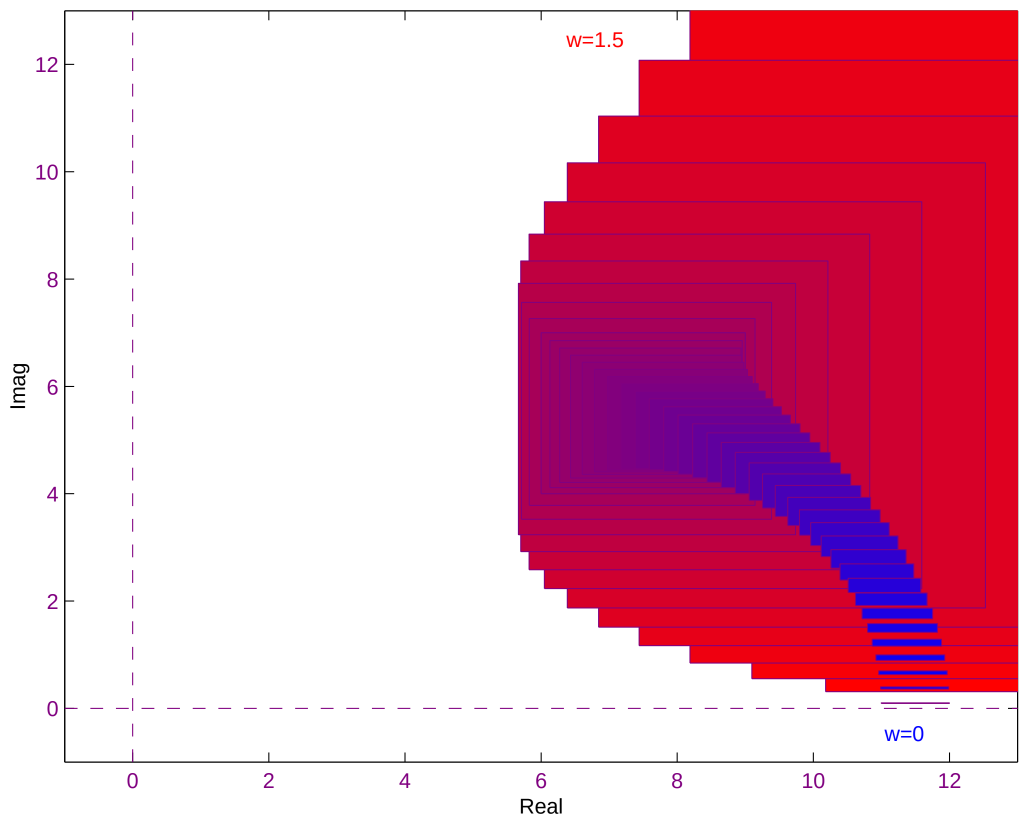
Graphically, zero is excluded from the value set so the polynomial is robustly Schur

Zero exclusion: warning

Example
Consider now the interval polynomial

$$[11, 12] + [9, 10]s + [7, 8]s^2 + [5, 6]s^3 + [3, 4]s^4 + [1, 2]s^5$$

We build its value set = Kharitonov's rectangles



Zero is excluded from the value set, but all 4 Kharitonov polynomials are unstable ! What is wrong ?

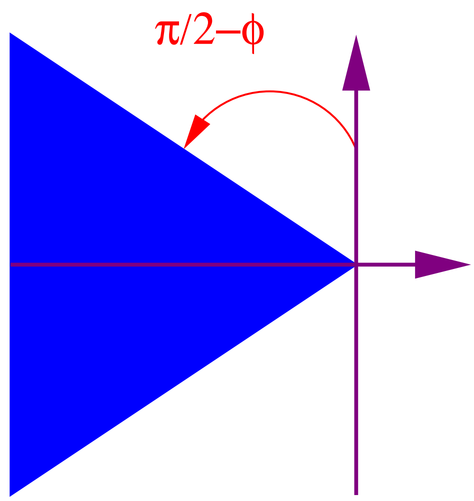
Boundary sweeping function

Testing the zero exclusion condition on the value set requires sweeping along the boundary of the stability region

Examples

- (left) half-plane $z = j\omega$
- shifted half-plane $z = -\alpha + j\omega$
- unit disk $z = e^{j\omega}$
- damping cone ϕ

$$z = \begin{cases} \omega \cos\phi + j\omega \sin\phi & \text{if } \omega \leq 0 \\ -\omega \cos\phi + j\omega \sin\phi & \text{if } \omega > 0 \end{cases}$$



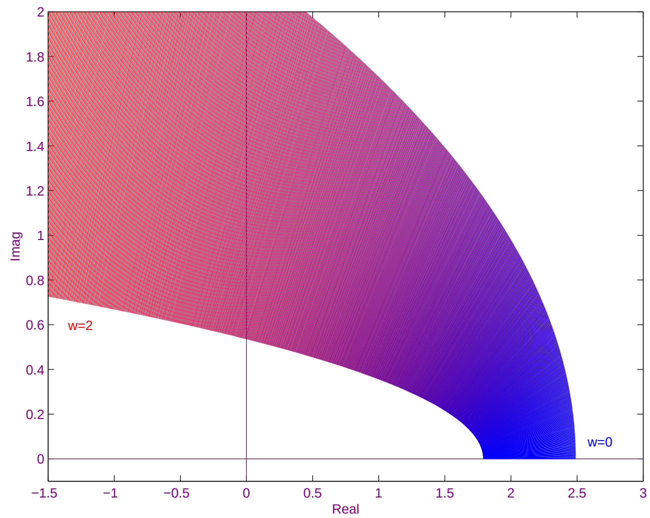
with ω real parameter

Robust stability margin and damping

Example

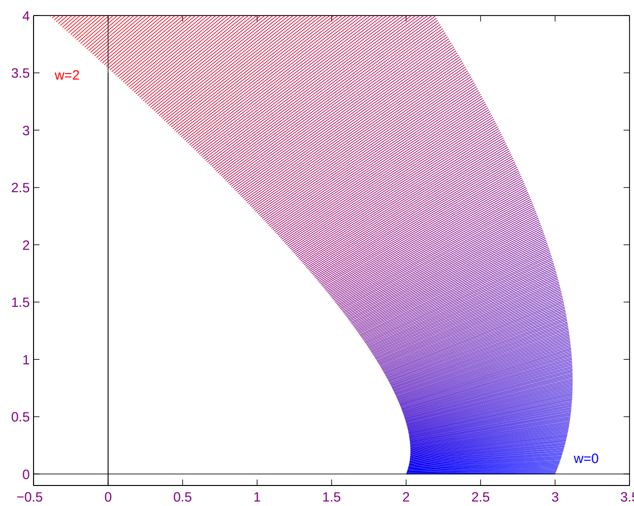
Consider the uncertain polynomial

$$p(s,q) = (3-q) + (2-q)s + s^2,\ q \in Q = [0,\ 1]$$



Robust Hurwitz stability with margin 3/10

Boundary sweeping function $z = -0.3 + j\omega,\ \omega \in [0,\ 2]$



Robust Hurwitz stability with damping $\phi = 2\pi/5$

Boundary sweeping function $z = \omega(\cos\phi + j\sin\phi)$

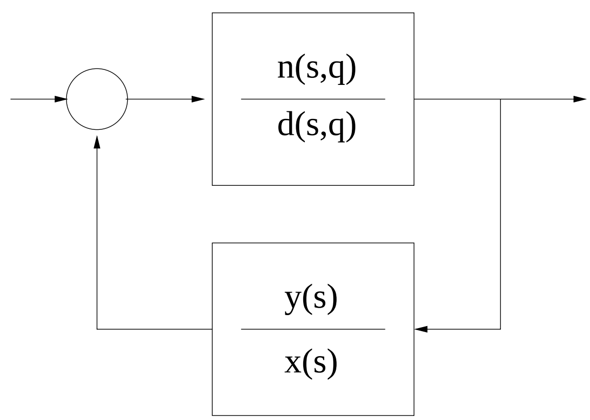
Affine uncertainty

Because interval polynomials have too conservative independent uncertainty structure we will describe a more general uncertainty model

When coefficients of an uncertain polynomial $p(s, q)$ or rational function $n(s, q)/d(s, q)$ depend affinely on parameter q, such as

$$a^T q + b$$

we speak about affine uncertainty



The above feedback interconnection

$$\frac{n(s, q)x(s)}{d(s, q)x(s) + n(s, q)y(s)}$$

preserves the affine uncertainty structure of the plant

Nonlinear uncertainty overbounding

Example

Consider the uncertain polynomial

$$\begin{aligned} p(s,q) = \; & (6q_1 + q_2 + 4) + (q_1^2 + 2q_2^2 + 2)s \\ & +(4q_1^2 + 2q_1 + 3q_2 + 3)s^2 + s^3, \; |q_i| \leq 1 \end{aligned}$$

with nonlinear uncertainty structure

It can be overbounded by the affine uncertainty structure

$$\begin{aligned} \tilde{p}(s,\tilde{q}) = \; & (6\tilde{q}_1 + \tilde{q}_2 + 4) + (\tilde{q}_3 + 2\tilde{q}_4 + 2)s \\ & +(4\tilde{q}_3 + 2\tilde{q}_1 + 3\tilde{q}_2 + 3)s^2 + s^3, \; |\tilde{q}_i| \leq 1 \end{aligned}$$

by defining new variables

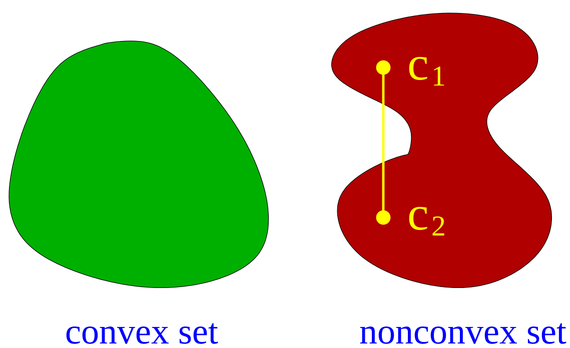
$$\begin{aligned} \tilde{q}_1 &= q_1 \qquad & \tilde{q}_3 &= q_1^2 \\ \tilde{q}_2 &= q_2 \qquad & \tilde{q}_4 &= q_2^2 \end{aligned}$$

In general it is less conservative than interval uncertainty overbounding

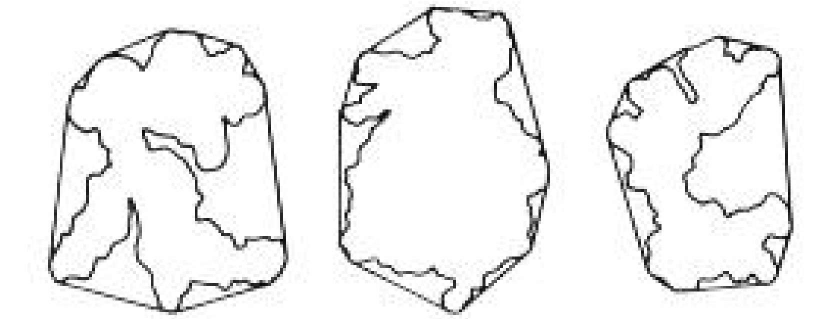
Convex set

A set C is convex if the line joining any two points c_1 and c_2 in C remains entirely in C

Given c_1 and c_2 in C and $\lambda \in [0, 1]$ we call $\lambda c_1 + (1 - \lambda)c_2$ in C a convex combination of c_1 and c_2



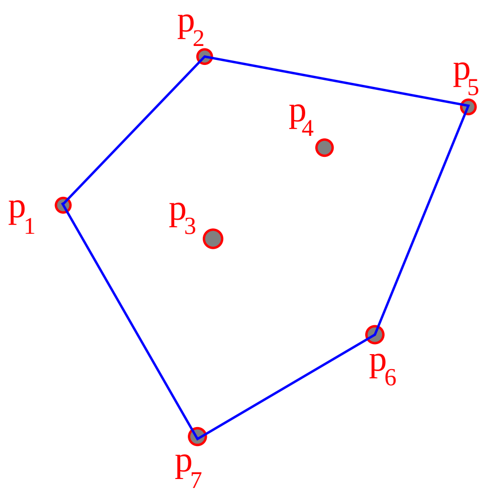
Given a set C (not necessarily convex) its convex hull is the smallest convex set which contains C



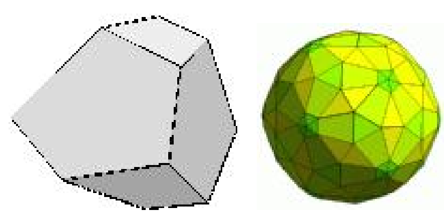
Polytope

A polytope is the convex hull of a finite set of points $P = \text{convex hull}\,\{p_1, p_2, \ldots, p_N\}$

A vertex is a point that cannot be generated as the convex combination of two distinct points



Polytope generated by vertices
p_1, p_2, p_5, p_6, p_7



Two 3-D polytopes

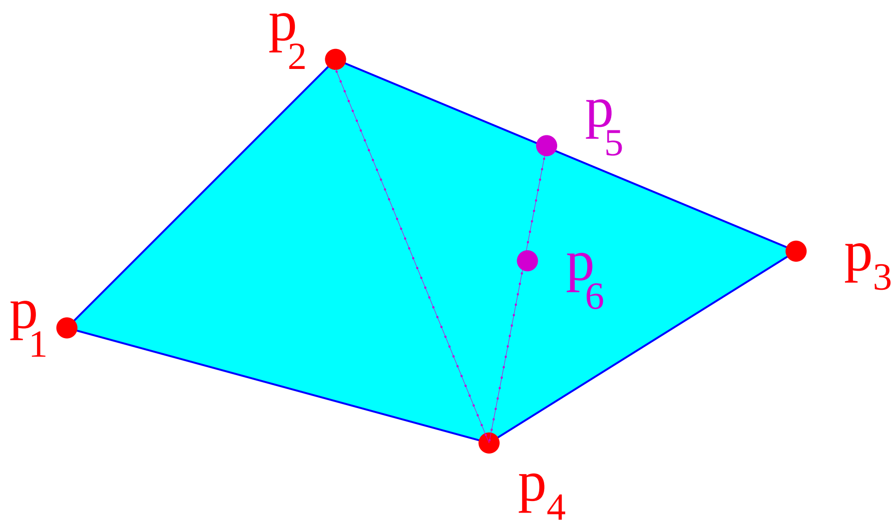
Every point p in a polytope P can be generated as the convex combination of the vertices of P

$$p = \sum_{i=1}^{N} \lambda_i p_i, \quad \sum_{i=1}^{n} \lambda_i = 1, \quad \lambda_i \geq 0$$

The constraint set for λ is called the unit simplex

Convex combinations in a polytope

Example



The polytope is generated by 4 vertices

$$P = \text{convex hull}\,\{p_1, p_2, p_3, p_4\}$$

and every point in P can be written as

$$p = \sum_{i=1}^{4} \lambda_i p_i, \quad \lambda_i \geq 0, \quad \sum_{i=1}^{4} \lambda_i = 1$$

For example

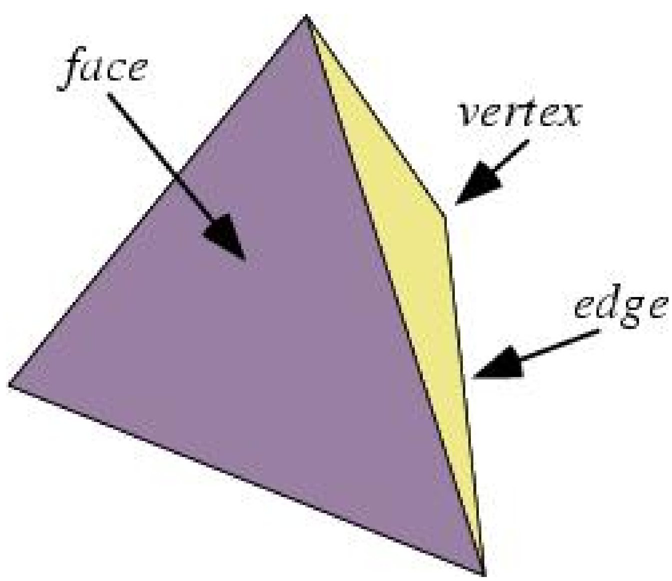
$$\begin{aligned} p_5 &= 0p_1 + \tfrac{1}{2}p_2 + \tfrac{1}{2}p_3 + 0p_4 \\ p_6 &= 0p_1 + \tfrac{1}{3}p_2 + \tfrac{1}{3}p_3 + \tfrac{1}{3}p_4 \\ &= \tfrac{1}{3}p_4 + \tfrac{2}{3}p_5 \end{aligned}$$

Faces, edges and vertices

A face is the intersection of a polytope with a tangent hyperplane

An edge (or side) is a 1-D face (a line segment) where two 2-D faces of a polytope meet

A vertex (or extreme point) is a 0-D face (a point) at which three or more edges meet



3-D polytope

Polytopes of polynomials

A family of polynomials $p(s, q)$, $q \in Q$ is said to be a polytope of polynomials if

- $p(s, q)$ has an affine uncertainty structure
- Q is a polytope

There is a natural isomorphism between a polytope of polynomials and its set of coefficients

Example

$p(s, q) = (2q_1 - q_2 + 5) + (4q_1 + 3q_2 + 2)s + s^2$, $|q_i| \leq 1$

Uncertainty polytope has 4 generating vertices

$$\begin{array}{lll} q^1 = [-1,\ -1] & & q^2 = [-1,\ 1] \\ q^3 = [1,\ -1] & & q^4 = [1,\ 1] \end{array}$$

Uncertain polynomial family has 4 generating vertices

$$\begin{array}{ll} p(s, q^1) = 4 - 5s + s^2 & p(s, q^2) = 2 + s + s^2 \\ p(s, q^3) = 8 + 3s + s^2 & p(s, q^4) = 6 + 9s + s^2 \end{array}$$

Any polynomial in the family can be written as

$$p(s, q) = \sum_{i=1}^{4} \lambda_i p(s, q^i),\ \sum_{i=1}^{4} \lambda_i = 1,\ \lambda_i \geq 0$$

Interval polynomials

Interval polynomials are a special case of polytopic polynomials

$$p(s,q) = \sum_{i=0}^{n} [q_i^-,\, q_i^+] s^i$$

with at most 2^{n+1} generating vertices

$$p(s,q^k) = \sum_{i=0}^{n} q_i^k s^i, \qquad q_i^k = \begin{cases} q_i^- \\ \text{or} \\ q_i^+ \end{cases} \qquad 1 \le k \le 2^{n+1}$$

Example

The interval polynomial

$$p(s,q) = [5,6] + [3,4]s + 5s^2 + [7,8]s^3 + s^4$$

can be generated by the $2^3 = 8$ vertex polynomials

$$\begin{aligned}
p(s,q^1) &= 5 + 3s + 5s^2 + 7s^3 + s^4 \\
p(s,q^2) &= 6 + 3s + 5s^2 + 7s^3 + s^4 \\
p(s,q^3) &= 5 + 4s + 5s^2 + 7s^3 + s^4 \\
p(s,q^4) &= 6 + 4s + 5s^2 + 7s^3 + s^4 \\
p(s,q^5) &= 5 + 3s + 5s^2 + 8s^3 + s^4 \\
p(s,q^6) &= 6 + 3s + 5s^2 + 8s^3 + s^4 \\
p(s,q^7) &= 5 + 4s + 5s^2 + 8s^3 + s^4 \\
p(s,q^8) &= 6 + 4s + 5s^2 + 8s^3 + s^4
\end{aligned}$$

Polygonal value set

The value set of a polytope of polynomials is a polygon, i.e. a 2-D polytope

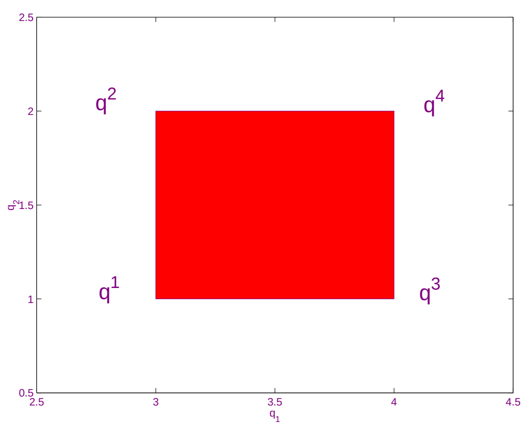
Example

Consider the interval polynomial

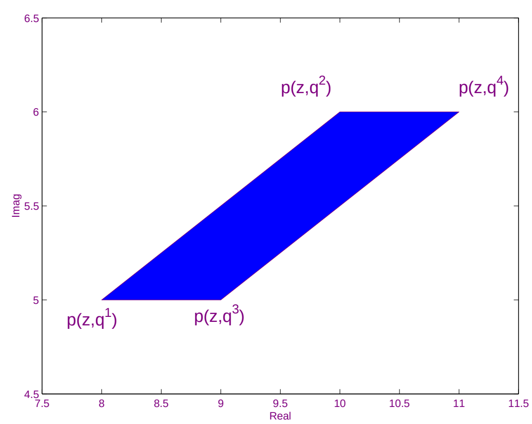
$$p(s,q) = [3,\ 4] + [1,\ 2]s + s^2$$

and its value set at $z = 2 + j$, i.e.

$$\begin{aligned} p(2+j, q^1 = [3\ 1]) &= 8 + j5 \\ p(2+j, q^2 = [3\ 2]) &= 10 + j6 \\ p(2+j, q^3 = [4\ 1]) &= 9 + j5 \\ p(2+j, q^3 = [4\ 2]) &= 11 + j6 \end{aligned}$$



Parameter box



Value set

Polygonal value set

More generally, if $p(s,q)$ denotes a polynomial with uncertainty polytope Q and generating vertices $p(s,q^i)$, then the value set $p(z,Q)$ is a polygon with generating vertices $p(z,q^i)$, i.e. $p(z,Q) = \text{convex hull}\,\{p(z,q^i)\}$

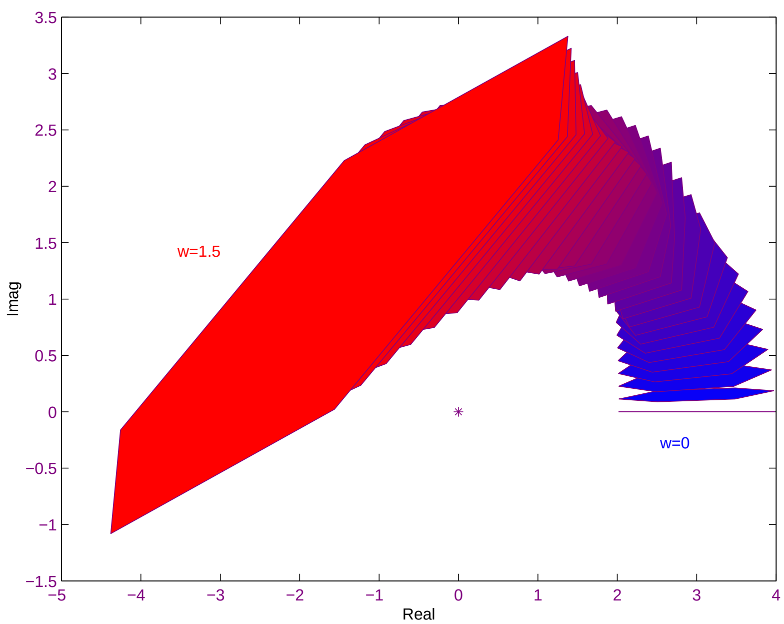
Moreover, all the edges of polygon $p(z,Q)$ are obtained from the edges of polytope Q, but some vertices of Q can be mapped into the interior of $p(z,Q)$

Example

Given the polytopic polynomial

$$\begin{aligned} p(s,q) \;&=\; (q_1 - q_2 + 2q_3 + 3) + (3q_1 + q_2 + q_3 + 3)s \\ &\quad + (3q_1 - 3q_2 + q_3 + 3)s^2 + (2q_1 - q_2 + q_3 + 1)s^3 \end{aligned}$$

with $|q_i| \leq 0.245$ we obtain for $w \in [0,\ 1.5]$ the following value set $p(jw,\ Q)$



Note that even though there are eight generators of the polytope, the value set has only six vertices

Improvement over rectangular overbounding

Example

Consider the polytope of polynomials

$$
\begin{aligned}
p(s,q) &= (q_1 - 2q_2 + 2) + (q_2 + 1)s \\
&\quad + (2q_1 - q_2 + 4)s^2 + (2q_2 + 1)s^3 + s^4
\end{aligned}
$$

where $q_1 \in [-0.5,\ 2]$ and $q_2 \in [-0.3,\ 0.3]$

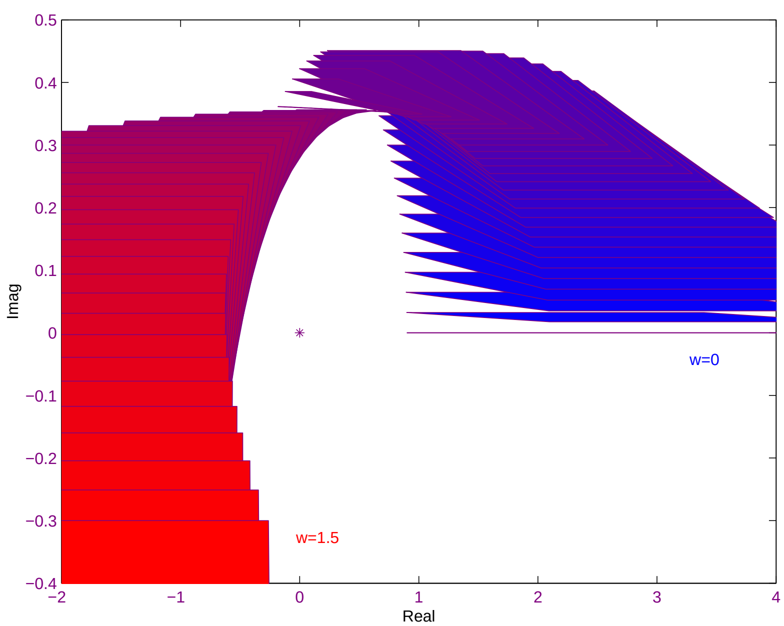
Using the overbounding interval polynomial

$\tilde{p}(s,\tilde{p}) = [0.9, 4.6] + [0.7, 1.3]s + [2.7, 8.3]s^2 + [0.4, 1.6]s^3 + s^4$

we find that the Kharitonov polynomial

$$p^{+-}(s) = 4.6 + 0.7s + 2.7s^2 + 1.6s^3 + s^4$$

is unstable, so we cannot conclude about robust stability



From the polygonal value set we check graphically that the zero exclusion condition holds, and hence that $p(s,q)$ is robustly stable

Gridding polytopes

In order to check whether the zero exclusion condition holds or not, so far we have swept the whole uncertainty polytope to build the value set: this may be really computationally demanding

Recall that there is no vertex result for assessing stability of polytopes of polynomials

Example

First vertex: $0.57+6s+s^2+10s^3$ stable

Second vertex: $1.57+8s+2s^2+10s^3$ stable

But middle of segment:

$1.07+7s+1.50s^2+10s^3$ unstable

In the sequel we will see that it is however not necessary to grid the interior of the uncertainty polytope

The edge theorem

Let $p(s, q)$, $q \in Q$ be a polynomial with invariant degree over polytopic set Q

> Polynomial $p(s, q)$ is robustly stable over the whole uncertainty polytope Q iff $p(s, q)$ is stable along the edges of Q

In other words, it is enough to check robust stability of the single parameter polynomial

$$\lambda p(s, q^{i_1}) + (1 - \lambda) p(s, q^{i_2}), \quad \lambda \in [0,\ 1]$$

for each pair of vertices q^{i_1} and q^{i_2} of Q

This can be done with the eigenvalue criterion

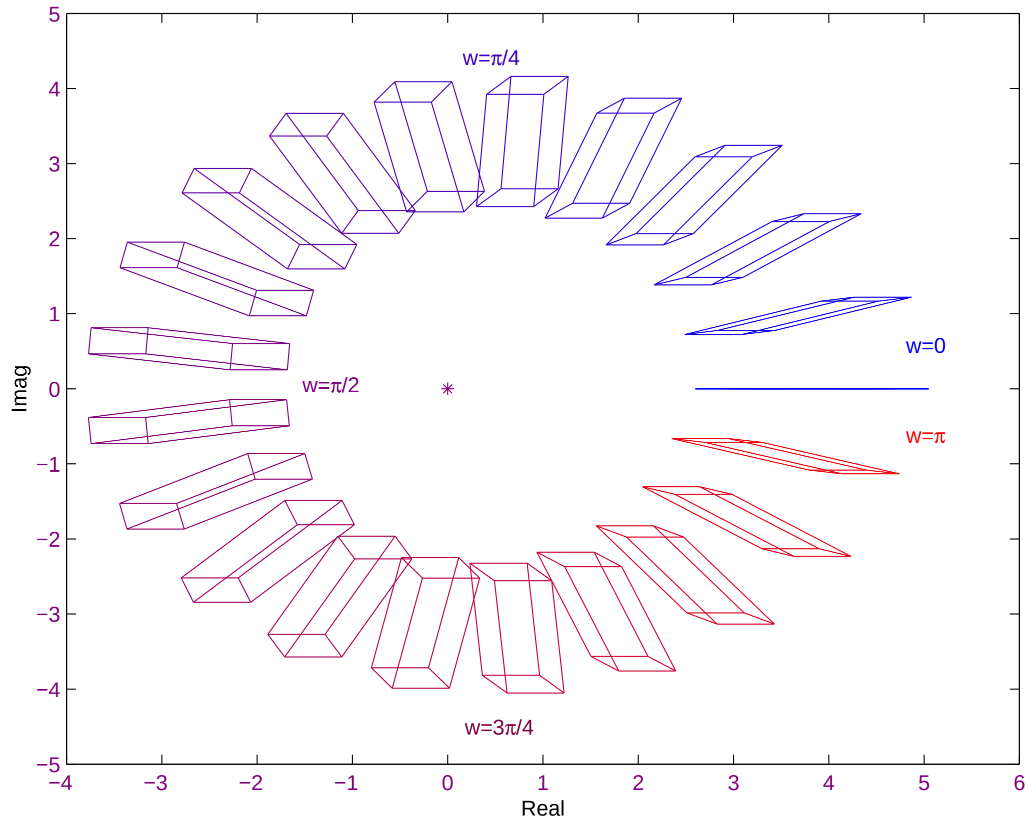
Necessity is obvious, and to prove sufficiency we use the zero exclusion condition: an edge of the polygonal value set corresponds to an edge of Q, so stability is lost along the stability boundary along an edge of Q

Discrete-time interval polynomial

Example
We want to assess robust stability of the discrete-time interval polynomial

$$[0.2,\ 0.8] + [-0.1,\ 0.25]z + [2.5,\ 4]$$

We draw the $3 \cdot 2^2 = 12$ edges of his polygonal value set



Zero is excluded so we conclude that the polynomial is robustly stable

Root set

There exists another version of the edge theorem that gives more information about the actual root locations for polytopic polynomials

Given an polynomial $p(s,q)$ with polytopic uncertainty $q \in Q$ we define its root set

$$\{z : p(z,q) = 0 \text{ for some } q \in Q\}$$

Obviously, $p(s,q)$ is robustly stable iff its root set lies within the stability region

Using the same arguments as in the proof of the edge theorem, we can show that

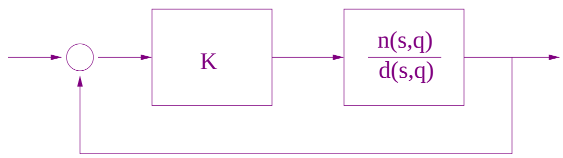
> Boundary of root set of polynomial $p(s,q)$ is contained in root set of edges of Q

So once more it suffices to check the edges of polytope Q to conclude about robust stability

Interval feedback system

Example

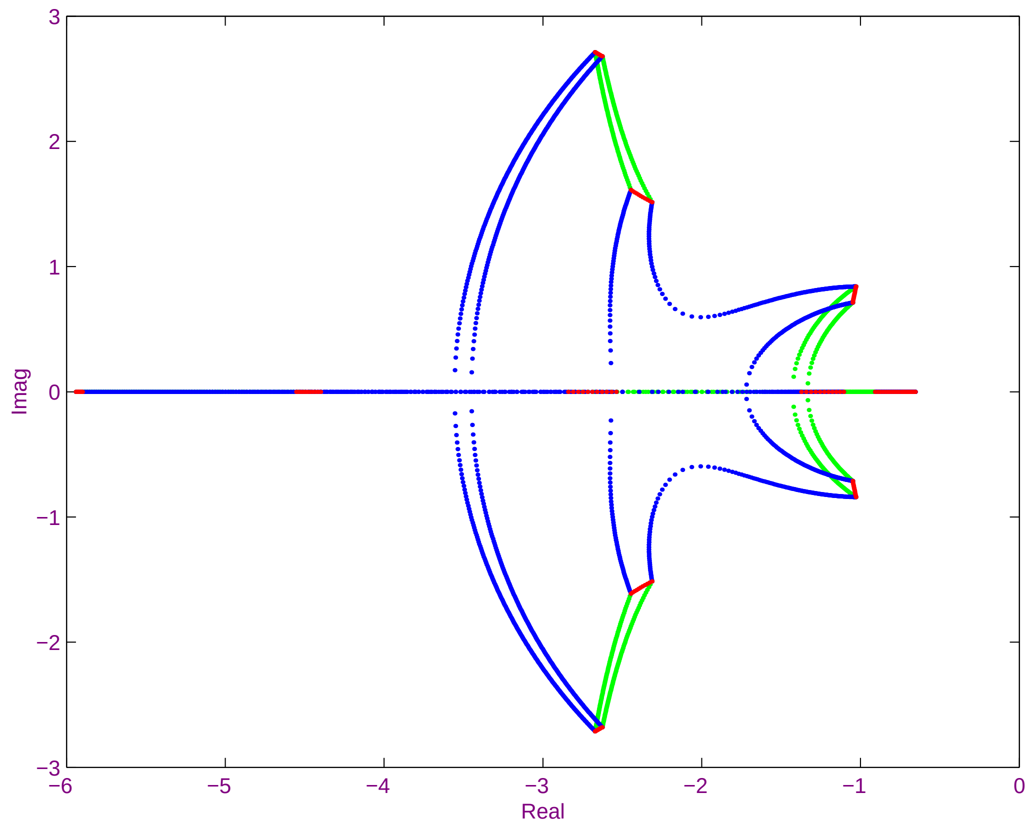
We consider the interval control system



with $n(s,q) = [6,\ 8]s^2 + [9.5,\ 10.5]$, $d(s,q) = s(s^2 + [14,\ 18])$ and characteristic polynomial

$$K[9.5,\ 10.5] + [14,\ 18]s + K[6,\ 8]s^2 + s^3$$

For $K = 1$ we draw the 12 edges of its root set



The closed-loop system is robustly stable

Number of edges

The edge theorem drastically reduces the number of value sets (or root locii) to be computed in order to assess robust stability

In the case of interval uncertainty with n uncertain parameters the number of vertices 2^n and edges $n2^{n-1}$ are given in the table below

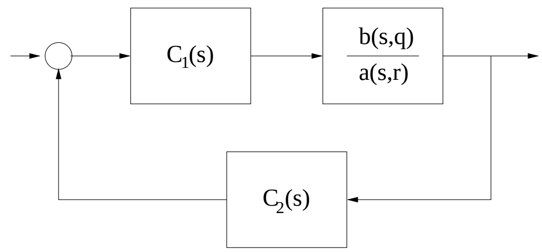
n	vertices	edges
1	2	1
2	4	4
3	8	12
4	16	32
5	32	80
10	1024	5120
20	1048576	10485760

so that for a large number of parameters the number of edges to be tested becomes prohibitively large

We will see that in some cases the number of edges to be tested can be reduced

32 plant theorem

We consider an interval plant with fixed compensator



where

$$\frac{b(s,q)}{a(s,r)} = \frac{\sum_{i=0}^{m} [q_i^-,\ q_i^+] s^i}{\sum_{i=0}^{n} [r_i^-,\ r_i^+] s^i}$$

and

$$C(s) = C_1(s) C_2(s) = \frac{y(s)}{x(s)}$$

The closed-loop characteristic polynomials is given by

$$p(s,q,r) = a(s,r)x(s) + b(s,q)y(s)$$

Let $a_1(s), \ldots, a_4(s)$ and $b_1(s), \ldots, b_4(s)$ denote Kharitonov's polynomials of the plant num. and den. resp.

Then robust stability is guaranteed iff all 32 edge polynomials

$$\begin{array}{l} [\lambda a_{i_1}(s) + (1-\lambda) a_{i_2}(s)] x(s) + b_{i_3}(s) y(s) \\ a_{i_3}(s) x(s) + [\lambda b_{i_2}(s) + (1-\lambda) b_{i_1}(s)] y(s) \end{array}$$

are stable for all $\{i_1,\ i_2\} \in \{(1,\ 3),\ (1,\ 4),\ (2,\ 3),\ (2,\ 4)\}$, all $i_3 \in \{1, \ldots, 4\}$ and all $\lambda \in [0,\ 1]$

The proof is based on the fact that the value set is octagonal (only 8 edges are relevant)

32 plant theorem

Example

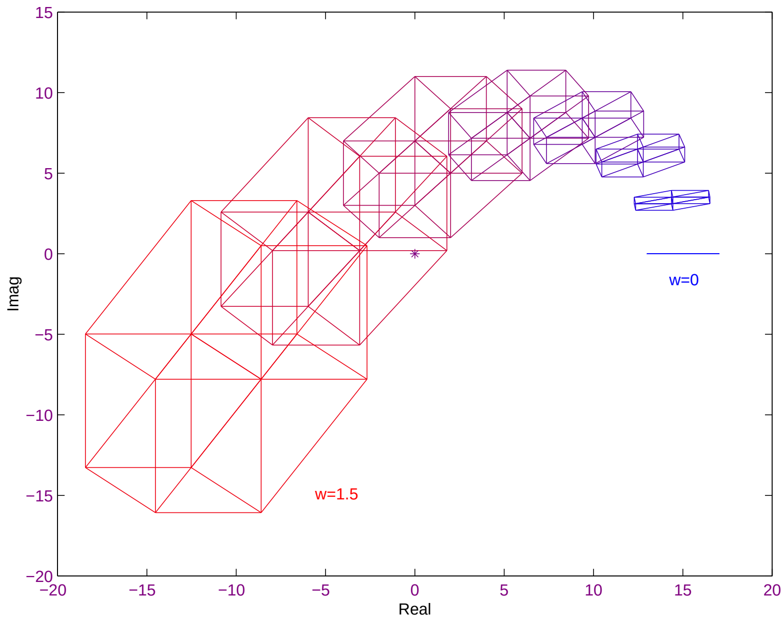
We consider the interval plant

$$\frac{[6,\ 8] + [2,\ 4]s + [3,\ 5]s^2 + [4,\ 6]s^3}{[7,\ 9] + [5,\ 7]s + [4,\ 6]s^2 + s^3}$$

connected with compensators

$$C_1(s) = 1, \quad C_2(s) = \frac{1}{s+1}$$

The nominal closed-loop polynomial is stable so we can apply the zero exclusion condition: we draw the value set octagons by sweeping the 32 edge polynomials

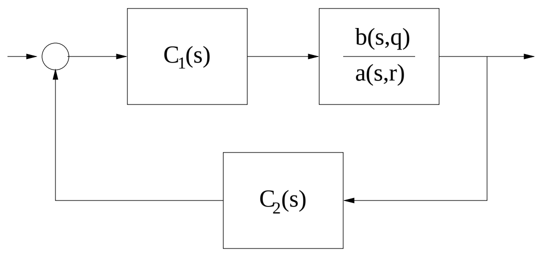


The origin is included in the value set so the closed-loop system is not robustly stable

16 plant theorem

In the special case that the compensator is first order testing the edges is not necessary

Consider an interval plant



$$\frac{b(s,q)}{a(s,r)} = \frac{\sum_{i=0}^{m}[q_i^-,\ q_i^+]s^i}{\sum_{i=0}^{n}[r_i^-,\ r_i^+]s^i}$$

strictly proper ($m < n$) and monic ($r_n^- = r_n^+ = 1$)
controlled by a first order compensator

$$C(s) = C_1(s)C_2(s) = \frac{K(s-z)}{s-p}$$

Let $a_1(s), \ldots, a_4(s)$ and $b_1(s), \ldots, b_4(s)$ denote Kharitonov's polynomials of the plant num. and den. resp.

Then $C(s)$ robustly stabilizes the plant iff
it stabilizes each of the 16 plants

$$\frac{b_{i_1}(s)}{a_{i_2}(s)},\ 1 \leq i_1,\ i_2 \leq 4$$

Application in robust synthesis...

Synthesis for interval plant

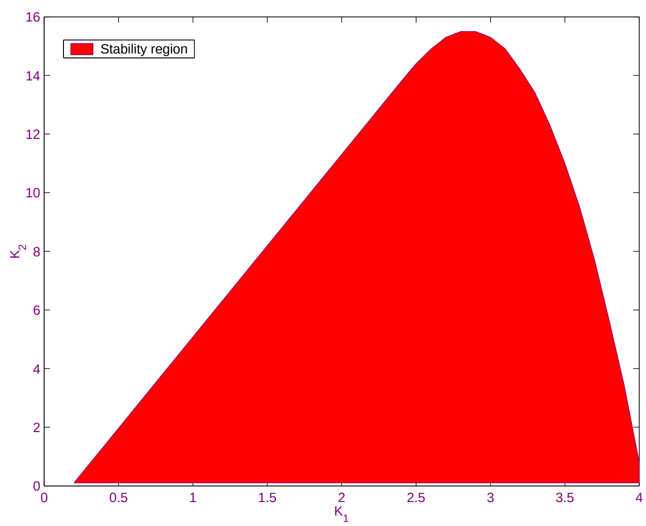
Example

Consider the model of an experimental oblique wing aircraft with uncertain transfer function

$$\frac{[90,\ 166] + [54,\ 74]s}{[-0.1,\ 0.1] + [30.1,\ 33.9]s + [50.4,\ 80.8]s^2 + [2.8,\ 4.6]s^3 + s^4}$$

that we want to stabilize with a PI compensator $K_1 + \frac{K_2}{s}$

For Kharitonov's 1st num. and 2nd den. polynomials we obtain the closed-loop char. polynomial $166K_2 + (-0.1 + 166K_1 + 74K_2)s + (30.1 + 74K_1)s^2 + 80.8s^3 + 4.6s^4 + s^5$ for which stability is guaranteed in the red region below (obtained with Hurwitz condition)



Proceeding this way for all 16 plants, we obtain graphically the robustly stabilizing controller

$$C(s) = 0.9 + \frac{0.2}{s}$$