GRADUATE COURSE ON
POLYNOMIAL METHODS FOR
ROBUST CONTROL
PART I.1

ROBUST STABILITY ANALYSIS: SINGLE PARAMETER UNCERTAINTY

Didier HENRION

www.laas.fr/~henrion

henrion@laas.fr

Pont Neuf over river Garonne in Toulouse

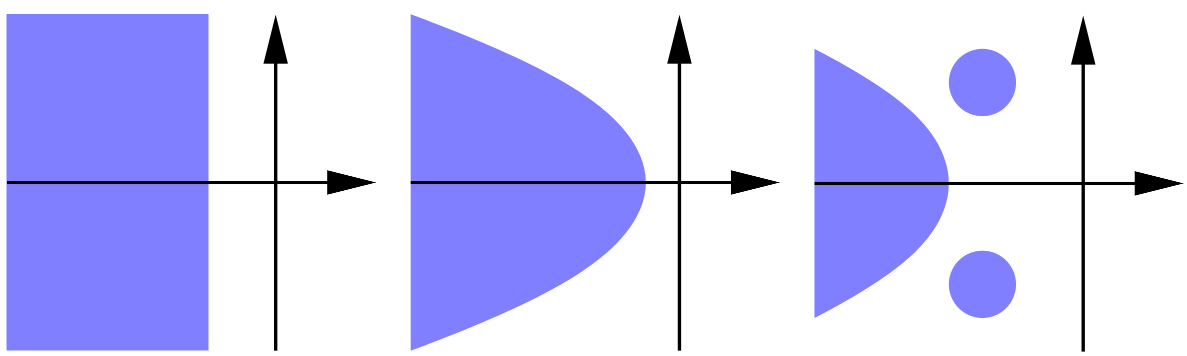
October-November 2001

Stability of polynomials

A polynomial $p(s)$ is stable if all its roots lie in some given region of the complex plane

The stability region depends on the nature of the system, most frequently

- the left half-plane (continuous-time), or
- the unit disk (discrete-time)

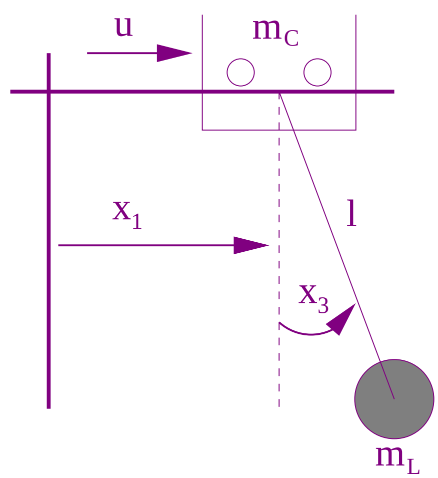


More sophisticated stability regions are however frequently required to ensure

- damping (parabola, sector)
- dominant behavior (shifted half-plane)
- bandwidth (shifted half-plane or disk)
- low-gain feedback preserving frequency behavior (disconnected region)

Example

Consider the linearized model of an anti-sway system for a crane



Closed-loop characteristic polynomial

$$\frac{600g}{lm_C} + \frac{2000g}{lm_C}s + \frac{10000 + 6000l + gm_C + gm_L}{lm_C}s^2 + \frac{2000}{m_C}s^3 + s^4$$

with gravity $g = 10$, crab mass $m_C = 1000$, rope length $l = 10$

Assume that the load mass m_L is not known exactly and belongs to the interval [50, 2395]

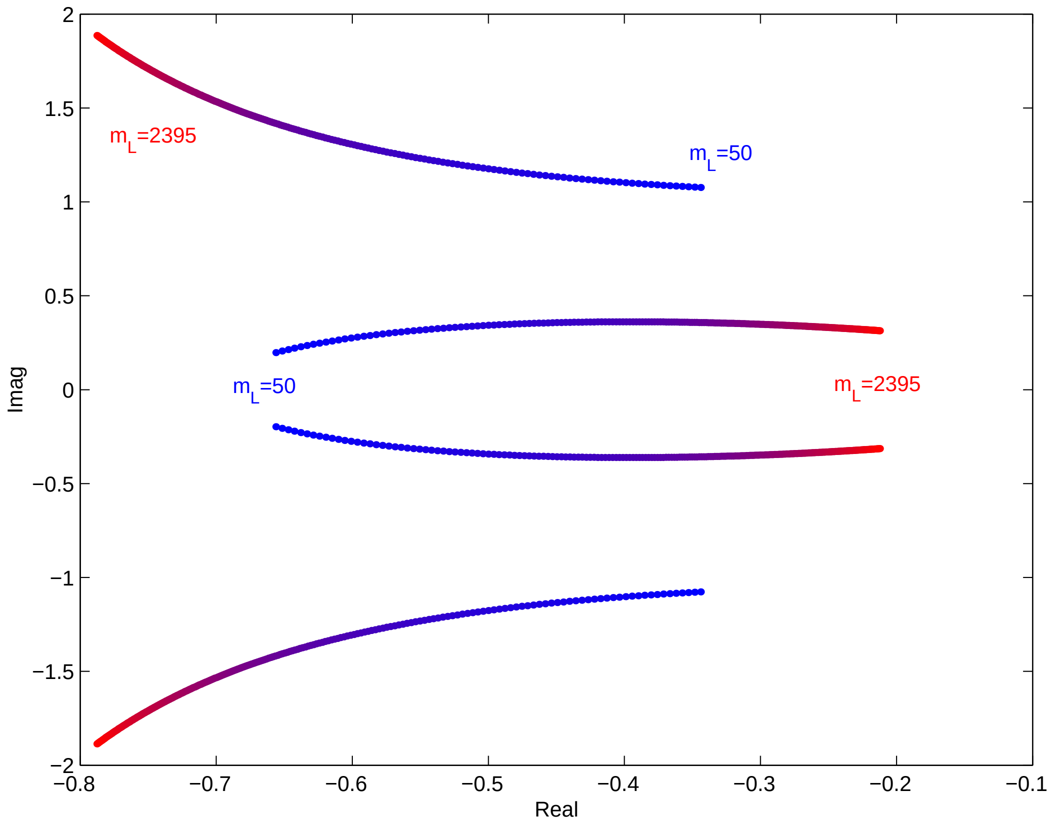
The robustness analysis question is:
does polynomial $d(s)$ remains stable
for all admissible values of m_L ?

Graphical robustness analysis

We draw the root locus of the characteristic polynomial

$$0.6 + 2s + (2.6 + 0.001m_L)s^2 + 2s^3 + s^4$$

for all admissible values of the parameter $m_L \in [50, 2395]$



The robust root locus remains in the left half-plane so the closed-loop system is robustly stable

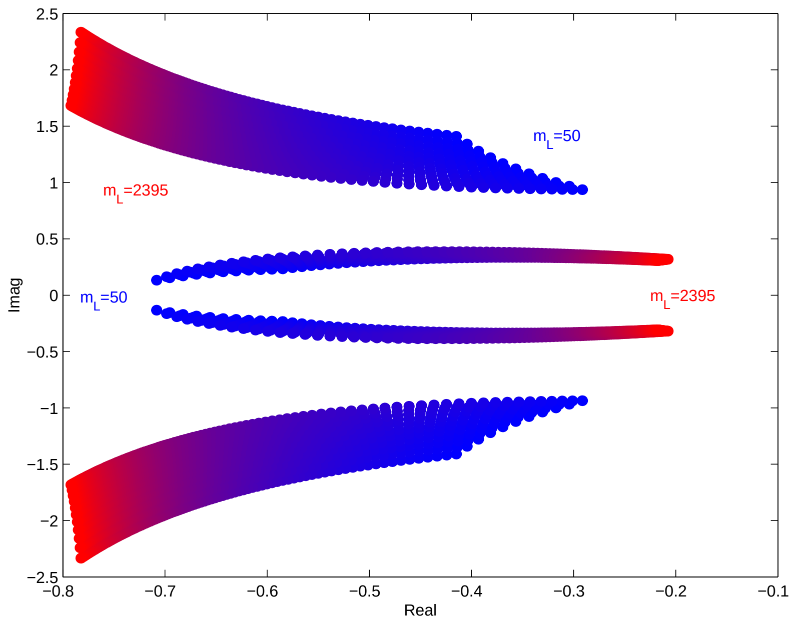
The robust root locus is obtained by a straightforward one-dimensional gridding

Two uncertain parameters

Now we assume that the rope length l is also uncertain, and we draw by brute-force 2-D gridding the robust root locus of the polynomial

$$\frac{6}{l} + \frac{20}{l}s + \frac{0.6l + 20 + 0.01m_L}{l}s^2 + 2s^3 + s^4$$

for all possible values of $m_L \in [50, 2395]$ and $l \in [7, 12]$



Here too, the closed-loop system remains robustly stable

Even for this small example, it took about one minute to draw the above root locus on a SunBlade 100 work-station

Computational time is an exponential function of the number of uncertain parameters, so the approach becomes unpractical for even middle-size problems

Robust analysis tools

As shown in the previous slide, we need more suitable tools to assess robustness

In the sequel we will describe such tools and we will mainly show how computational complexity depends on the uncertainty model

In increasing order of complexity, we will distinguish between

- single parameter uncertainty $q \in [q_{\text{min}}, q_{\text{max}}]$
- interval uncertainty $q_i \in [q_{i\text{min}}, q_{i\text{max}}]$
- polytopic uncertainty $\lambda_1 q_1 + \cdots + \lambda_N q_N$
- multilinear uncertainty $q_0 + q_1 \cdot q_2 \cdot q_3$

For each uncertainty class, we will describe a specific robustness analysis tool

We start with single parameter uncertainty and the eigenvalue criterion

Single parameter uncertainty

We consider the continuous-time polynomial

$$p(s,q) = p_0(s) + q p_1(s)$$

where

- $p_0(s)$ is a stable nominal polynomial
- $p_1(s)$ is an arbitrary polynomial
- q is an uncertain real parameter lying within a given interval $[q_{\text{min}}, q_{\text{max}}]$

Example

First order plant with one uncertain pole

$$P(s,q) = \frac{1}{s-q}, \quad |q| \leq 2$$

compensated by unity feedback $C(s) = 1$

Uncertain closed-loop polynomial

$$p(s,q) = s + 1 - q$$

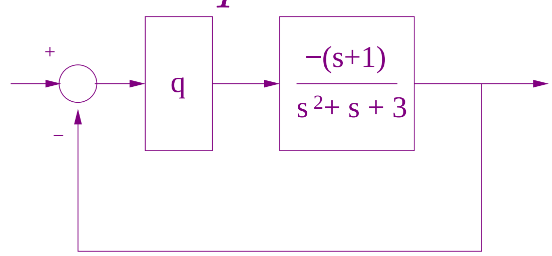
is not robustly stable since the root leaves the left half-plane when $q \geq 1$

Example

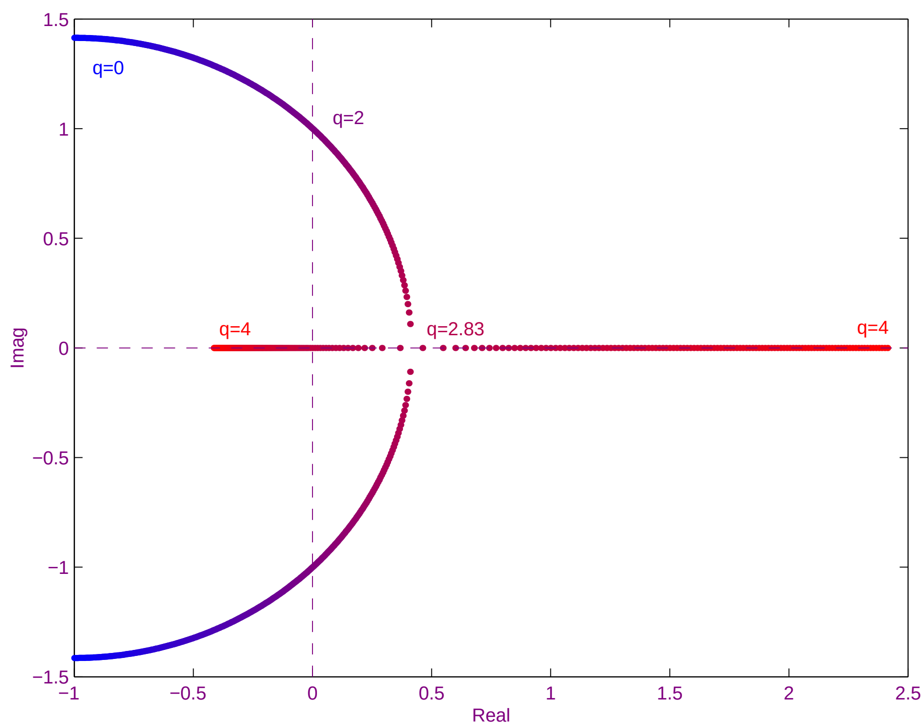
Uncertain polynomial

$$\begin{aligned} p(s,q) &= s^2 + (2-q)s + (3-q) \\ &= p_0(s) + q p_1(s) \end{aligned}$$

Stability interval on q ?



Draw root-locus for fictitious plant $P(s) = \frac{p_1(s)}{p_0(s)}$



Stability interval $q \in]-\infty, 2[$

Degree dropping and infinite roots

Polynomial roots depend continuously on coefficients so stability is lost when a root crosses the stability boundary (the imaginary axis in continuous-time)

However, stability loss can also occur at infinity by degree dropping

Example

Consider the uncertain polynomial

$$p(s,q) = qs^2 - s - 1,\ q \in [0,\ 1]$$

- $p(s,0) = -(s+1)$ stable
- $p(s,1) = (s+0.6180)(s-1.6180)$ unstable

No stability boundary crossing occurred since $p(j\omega,q) = -1 - q\omega^2 - j\omega \neq 0$ for all $q \in [0,\ 1]$

Necessary assumption on invariant degree of $p(s,q)$ over the uncertainty range

Valid for unbounded stability regions such as left half-plane, exterior of unit disk

Hurwitz matrix

Given a continuous-time polynomial

$$p(s) = p0 + p1s + \cdots + p_{n-1}s^{n-1} + p_n s^n$$

with $p_n > 0$ we define its $n \times n$ Hurwitz matrix

$$H(p) = \begin{bmatrix} p_{n-1} & p_{n-3} & & 0 & 0 \\ p_n & p_{n-2} & & \vdots & \vdots \\ 0 & p_{n-1} & \ddots & 0 & 0 \\ 0 & p_n & & p_0 & 0 \\ \vdots & \vdots & & p_1 & 0 \\ 0 & 0 & & p_2 & p_0 \end{bmatrix}$$

Hurwitz stability criterion: Polynomial $p(s)$ is stable iff all principal minors of $H(p)$ are > 0

Adolf Hurwitz
(Hanover 1859 - Zürich 1919)

Orlando's formula: $H(p)$ is singular iff $p(s)$ has a root s_i on the imaginary axis, because

$$\det H(p) = \alpha \prod_{1 \le i < j \le n} (s_i + s_j)$$

Largest stability interval and eigenvalue criterion

Consider the uncertain polynomial

$$p(s,q) = p_0(s) + qp_1(s)$$

where

- $p_0(s)$ nominally stable with positive coefs
- $p_1(s)$ such that $\deg p_1(s) < \deg p_0(s)$

What is the largest stability interval

$$q \in]q_{\text{min}},\, q_{\text{max}}[$$

such that $p(s,q)$ is robustly stable ?

Hurwitz matrix

$$\begin{aligned} H(p) &= H(p_0(s) + qp_1(s)) \\ &= H(p_0(s)) + qH(p_1(s)) \\ &= H_0 + qH_1 \end{aligned}$$

with $\det H_0 > 0$

From Orlando's formula $H(p)$ becomes singular when q is such that $p(s,q)$ has a root on the imaginary axis

Since $\det H(p) = \det[H_0 + qH_1] = \det[q^{-1} + H_0^{-1}H_1]$ and there is no root crossing at infinity we obtain Białas' eigenvalue criterion:

The largest stability interval is given by

$$\begin{aligned} q_{\text{max}} &= 1/\lambda^+_{\text{max}}(-H_0^{-1}H_1) \\ q_{\text{min}} &= 1/\lambda^-_{\text{min}}(-H_0^{-1}H_1) \end{aligned}$$

where λ^+_{max} is the maximum positive real eigenvalue and λ^-_{min} is the minimum negative real eigenvalue

Eigenvalue criterion

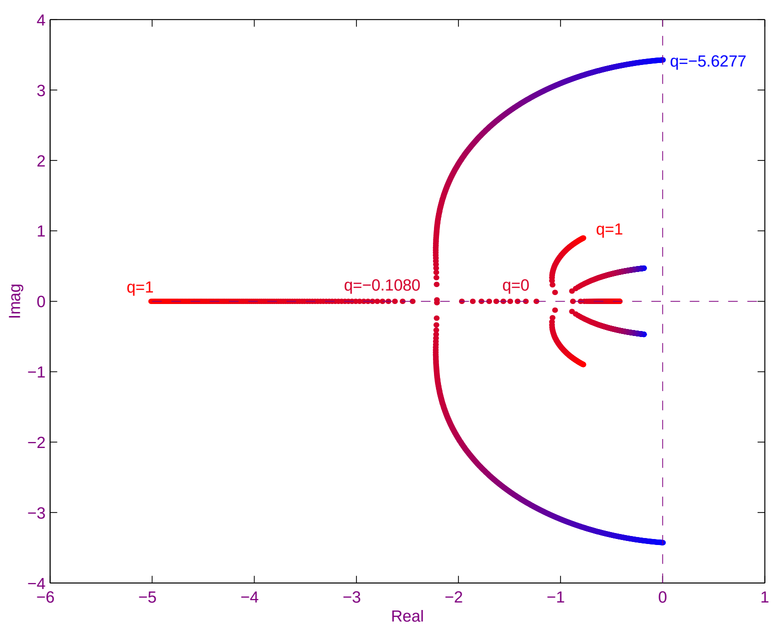
Example
Take the polynomial

$$\begin{aligned} p(s,q) &= s^4+(6+q)s^3+12s^2+(10+q)s+3 \\ &= s^4+6s^3+12s^2+10s+3+q(s^3+s) \end{aligned}$$

Hurwitz matrix $H(p,q) = H_0 + qH_1$ with

$$H_0 = \begin{bmatrix} 6 & 10 & 0 & 0 \\ 1 & 12 & 3 & 0 \\ 0 & 6 & 10 & 0 \\ 0 & 1 & 12 & 3 \end{bmatrix}, \quad H_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Eigenvalues of $-H_0^{-1}H_1 = \{0, 0, -0.0879, -0.1777\}$
hence $q_{\min} = -5.6277$, no positive eigenvalues
hence $q_{\max} = +\infty$



Higher powers of a single parameter

Now consider the continuous-time polynomial

$$p(s,q) = p_0(s) + qp_1(s) + q^2 p_2(s) + \cdots + q^m p_m(s)$$

with $p_0(s)$ stable and $\deg p_0(s) > \deg p_i(s)$

Using the zeros (roots of determinant) of the polynomial Hurwitz matrix

$$H(p) = H(p_0) + qH(p_1) + q^2 H(p_2) + \cdots + q^m H(p_m)$$

we can show that

$$\begin{aligned} q_{\min} &= 1/\lambda^-_{\min}(M) \\ q_{\max} &= 1/\lambda^+_{\max}(M) \end{aligned}$$

where

$$M = \begin{bmatrix} 0 & & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & & I & 0 \\ 0 & & 0 & I \\ -H_0^{-1}H_m & \cdots & -H_0^{-1}H_2 & -H_0^{-1}H_1 \end{bmatrix}$$

is a block companion matrix

Higher powers of a single parameter

Example

Consider the linearized model of an A4D jet fighter

$$P(s,q) = \frac{b(s,q)}{a(s,q)}$$
$$= \frac{-1486+15.37s+3.444s^2-3.426q}{0.3707+2.335s+39.09s^2+4.329s^3+s^4-q(-0.004508+0.06050s+s^2)}$$

where

- input is elevator angle
- output is forward velocity
- uncertain real parameter q represents changes in longitudinal static stability derivative

An observer-based controller

$$C(s,q) = \frac{y(s,q)}{x(s,q)}$$
$$= \frac{1674+1543s+199.9s^2+35.33s^3-q(41.88+34.52*s+0.1919s^2)}{21670+5235s+688.9s^2+49.62s^3+s^4-q(-14.83+6.006s+0.8815s^2)}$$

put in a negative feedback configuration gives rise to the following closed-loop characteristic polynomial ...

Higher powers of a single parameter

Closed-loop characteristic polynomial

$$\begin{aligned} p(s,q) &= a(s,q)x(s,q) + b(s,q)y(s,q) \\ &= p_0(s) + qp_1(s) + q^2p_2(s) \end{aligned}$$

where

$$\begin{aligned} p_0(s) &= 2496000 + 2320000s + 1127000s^2 \\ &\quad +344200s^3 + 70150s^4 + 1004s^5 \\ &\quad +942.8s^6 + 53.95s^7 + s^8 \\ p_1(s) &= -5643 - 4670s - 2150s^2 \\ &\quad -5335s^3 - 766.5s^4 - 59.50s^5 \\ &\quad -1.882s^6 \\ p_2(s) &= -143.6 - 117.4s + 14.53s^2 \\ &\quad +6.059s^3 + 0.8815s^4 \end{aligned}$$

Using the eigenvalue criterion on the 8×8 quadratic Hurwitz matrix of p, we obtain the robust stability bounds

$$q \in]-433.2,\ 40.14[$$

F4D Skyhawk

Stability of extreme points

Ensuring robust stability of the parametrized polynomial

$$p(s,q) = p_0(s) + qp_1(s)$$
$$q \in [q_{\mathrm{min}},\ q_{\mathrm{max}}]$$

amounts to ensuring robust stability of the whole segment of polynomials

$$\lambda p(s,q_{\mathrm{min}}) + (1-\lambda)p(s,q_{\mathrm{max}})$$
$$\lambda = \frac{q_{\mathrm{max}}-q}{q_{\mathrm{max}}-q_{\mathrm{min}}} \in [0,\ 1]$$

A natural question arises: does stability of two vertices imply stability of the segment ?

Unfortunately, the answer is no

Example

First vertex: $0.57 + 6s + s^2 + 10s^3$ stable

Second vertex: $1.57 + 8s + 2s^2 + 10s^3$ stable

But middle of segment:

$1.07 + 7s + 1.50s^2 + 10s^3$ unstable

Other stability regions

The eigenvalue criterion can be also used in the discrete-time case, when the stability region is the unit disk

- no degree dropping at infinity
  because stability region is bounded

- Jury matrix replaces Hurwitz matrix
  $\det J(p) = \alpha \prod_{1 \leq i < j \leq n} (1 - z_i z_j)$

- stability is also lost at -1 and $+1$

More generally for any other stability region we can define a guardian map such as the Hurwitz or Jury matrix whose rank drops when stability is lost

MIMO systems

Uncertain multivariable systems are modeled by uncertain polynomial matrices

$$P(s,q) = P_0(s) + qP_1(s) + q^2P_2(s) + \cdots + q^mP_m(s)$$

where $p_0(s) = \det P_0(s)$ is a stable polynomial

We can apply the scalar procedure to the determinant polynomial

$$\det P(s,q) = p_0(s) + qp_1(s) + q^2p_2(s) + \cdots + q^rp_r(s)$$

Example

MIMO design on the plant with left MFD

$$A^{-1}(s,q)B(s,q) = \begin{bmatrix} s^2 & q \\ q^2+1 & s \end{bmatrix}^{-1} \begin{bmatrix} s+1 & 0 \\ q & 1 \end{bmatrix}$$
$$= \frac{\begin{bmatrix} s^2+s-q^2 & -q \\ qs^2-(q^2+1)s-(q^2+1) & s^2 \end{bmatrix}}{s^3-q^2-q}$$

with uncertain parameter $q \in [0,1]$

MIMO systems

Using some design method, we obtain a controller with right MFD

$$Y(s)X^{-1}(s) = \begin{bmatrix} 94-51s & -18+17s \\ -55 & 100 \end{bmatrix} \begin{bmatrix} 55+s & -17 \\ -1 & 18+s \end{bmatrix}$$

Closed-loop system with characteristic denominator polynomial matrix

$$\begin{aligned} D(s,q) &= A(s,q)X(s) + B(s,q)Y(s) \\ &= D_0(s) + qD_1(s) + q^2D_2(s) \end{aligned}$$

Nominal system poles: roots of $\det D_0(s)$

Applying the eigenvalue criterion on $\det D(s,q)$ yields the stability interval

$$q \in]-0.93,\ 1.17[\ \supset\ [0,\ 1]$$

so the closed-loop system is robustly stable