Tractable characterizations of polynomials nonnegative on a set

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In many problems in control, optimal and robust control, one has to solve global optimization problems of the form: $\mathbf{P}:\ f^* = \min_{\mathbf{x}}\{f(\mathbf{x}) : \mathbf{x} \in \mathbf{K}\}$, or, equivalently, $f^* = \max\{\lambda : f - \lambda \geq 0 \text{ on } \mathbf{K}\}$, where f is a polynomial (or even a semi-algebraic function) and $\mathbf{K}$ is a basic semi-algebraic set. One may even need solve the "robust" version $\min\{f(\mathbf{x}) : \mathbf{x} \in \mathbf{K};\ h(\mathbf{x}, \mathbf{u}) \geq 0,\ \forall \mathbf{u} \in \mathbf{U}\}$ where $\mathbf{U}$ is a set of parameters. For instance, some static output feedback problems can be cast as polynomial optimization problems whose feasible set $\mathbf{K}$ is defined by a polynomial matrix inequality (PMI). And robust stability regions of linear systems can be modeled as parametrized polynomial matrix inequalities (PMIs) where parameters $\mathbf{u}$ account for uncertainties and (decision) variables $\mathbf{x}$ are the controller coefficients.

Therefore, to solve such problems one needs *tractable characterizations* of polynomials (and even semi-algebraic functions) which are nonnegative on a set, a topic of independent interest and of primary importance because it also has implications in many other areas.

We will review two kinds of *tractable* characterizations of polynomials which are non-negative on a basic closed semi-algebraic set $\mathbf{K} \subset \mathbb{R}^n$. The first type of characterization is when knowledge on $\mathbf{K}$ is through its defining polynomials, i.e., $\mathbf{K} = \{\mathbf{x} : g_j(\mathbf{x}) \geq 0,\ j = 1, \ldots, m\}$, in which case some powerful certificates of positivity can be stated in terms of some sums of squares (SOS)-weighted representation. For instance, this allows to define a hierarchy fo semidefinite relaxations which yields a monotone sequence of *lower bounds* converging to f^* (and in fact, finite convergence is generic). There is also another way of looking at nonnegativity where now knowledge on $\mathbf{K}$ is through *moments* of a measure whose support is $\mathbf{K}$. In this case, checking whether a polynomial is nonnegative on $\mathbf{K}$ reduces to solving a sequence of *generalized eigenvalue* problems associated with a countable (nested) family of real symmetric matrices of increasing size. When applied to $\mathbf{P}$, this results in a monotone sequence of *upper bounds* converging to the global minimum, which complements the previous sequence of upper bounds. These two (dual) characterizations provide convex *inner* (resp. *outer*) approximations (by spectrahedra) of the convex cone of polynomials nonnegative on $\mathbf{K}$.