Factorizable matrices

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1 Introduction

More than 10 years ago, I was lucky to show in $[1]$ that the well known companion matrix of a polynomial can be factorized into a product. In more detail, the $n \times n$ companion matrix

$$C = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ . & . & \cdots & . & . \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \qquad (1)$$

of the polynomial

$$x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$$

is a product

$$C = G_1 G_2 \cdots G_{n-1} G_n,$$

where for $k = 1, \ldots, n-1$,

$$G_k = \begin{bmatrix} I_{k-1} & & \\ & A_k & \\ & & I_{n-k-1} \end{bmatrix}$$

with

$$A_k = \begin{bmatrix} -a_k & 1 \\ 1 & 0 \end{bmatrix},$$

and $G_n = \mathrm{diag}(1, \ldots, 1, -a_n)$.

In addition, a remarkable property of this factorization is that for any permutation $(i_1, i_2, \ldots, i_n)$ of $(1, 2, \ldots, n)$, the product $G_{i_1} G_{i_2} \cdots G_{i_n}$ has the same eigenvalues as C, i.e. the same roots of the equation.

This factorization was during the last decade generalized in several ways and found applications, in particular in control theory.

We want now to present, partly with the help of Frank J. Hall, the most general case of such factorization.

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Let us start with the following observation.

Theorem 2.1. *Let p, q and n be positive integers, where $p + q < n$. Let A, B be $n \times n$ partitioned matrices (we could call them overlapping)*

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & I_p \end{bmatrix}, B = \begin{bmatrix} I_q & 0 \\ 0 & A_2 \end{bmatrix}. \tag{2}$$

Then the product $C = AB$ is a matrix the lower-left corner $p \times q$ submatrix of which is a zero matrix and the complementary $(n-p) \times (n-q)$ submatrix of which has rank at most $n - p - q$.

Conversely, let an $n \times n$ matrix C have the property that its lower-left corner $p \times q$ submatrix is a zero matrix and the complementary $(n-p) \times (n-q)$ submatrix has rank at most $n - p - q$. Then C can be factorized as $C = AB$, where A and B have the forms in (2).

Proof. For the first part, write A_1 as a block matrix $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, where A_{22} is $(n-p-q)\times(n-p-q)$, and A_2 as $\begin{bmatrix} B_{22} & B_{23} \\ B_{32} & B_{33} \end{bmatrix}$, where B_{22} is also $(n-p-q)\times(n-p-q)$. Then the product is

$$AB = \begin{bmatrix} A_{11} & A_{12}B_{22} & A_{12}B_{23} \\ A_{21} & A_{22}B_{22} & A_{22}B_{23} \\ 0 & B_{32} & B_{33} \end{bmatrix}. \tag{3}$$

The zero matrix has clearly dimension $p \times q$ and the complementary matrix rank at most $n - p - q$.

To prove the converse, write C in the partitioned form

$$C = \begin{bmatrix} C_{11} & C_{12} \\ 0 & C_{22} \end{bmatrix},$$

where the zero block is $p \times q$. By our assumption, C_{12} can be factorized as PQ, where P is $(n-p) \times (n-p-q)$ and Q is $(n-p-q) \times (n-q)$. It follows that $C = AB$, where A and B have the form (2) with $A_0 = [C_{11} \;\; P]$ and $B_0 = \begin{bmatrix} Q \\ C_{22} \end{bmatrix}$.

□

Suppose now that we have more than two, say t such generators $A_1, A_2, \ldots, A_t$ the corresponding intervals $J_1, J_2, \ldots, J_t$ of which are chained as follows: J_1 contains the number 1, J_2 (if $t > 1$) has with J_1 a non-void intersection, J_3 has with J_2 a non-void intersection but is disjoint with J_1, in general J_{k+1} has a non-void intersection with J_k but is disjoint with J_{k-1} for $k = 2, \ldots, t-1$; in addition, J_t contains the number n.

For $k = 1, \ldots, t$, define $n \times n$ matrices G_k by

$$G_k = \begin{bmatrix} I_{a_k} & & \\ & A_k & \\ & & I_{b_k} \end{bmatrix}. \qquad (4)$$

Then for permutations $(i_1, \ldots, i_t)$ of $(1, \ldots, t)$, products of the form

$$G_{i_1} G_{i_2} \cdots G_{i_t} \tag{5}$$

are called F-matrices.

The main result is the following theorem, in which the term essential indices means the indices in common of two consequent intervals.

Theorem 2.2. *Let $\mathcal{S} = \{G_1, G_2, \ldots, G_t\}$ be generators of an $n \times n$ F-matrix. Then, for every permutation $P = (i_1, i_2, \ldots, i_t)$ of $(1, 2, \ldots, t)$, the products $\Pi_P = G_{i_1} G_{i_2} \cdots G_{i_t}$ have the same spectrum.*

Proof. It suffices to show that every matrix $\Pi_P = G_{i_1} G_{i_2} \cdots G_{i_t}$ has the same spectrum as the matrix $G_1 G_2 \cdots G_t$. We use induction with respect to t. The assertion holds trivially for $t = 1$ and $t = 2$. Let $\mathcal{S} = \{G_1, G_2, \ldots, G_t\}$ be such system with $t > 2$ and suppose the assertion holds for all smaller t's. Since transposition changes the precedence properties, we can assume that $t-1$ precedes t in the given permutation. Also, cyclic permutations of the factors do not change the spectrum so that we can assume that G_t is the last factor and G_{t-1} is the next to the last factor. It follows that the given product is then cospectral with $\Pi = G_{l_1} G_{l_2} \cdots G_{l_{t-2}} (G_{t-1} G_t)$ and since the product $G_{t-1} G_t$ can be considered as a generator whose essential indices are formed by the union of essential indices of G_{t-1} and G_t, the product Π has $t-1$ factors. By the induction hypothesis, Π is cospectral with $G_1 G_2 \cdots G_{t-2}(G_{t-1}G_t)$. $\square$

Let me conclude with the following maybe interesting observation.

Observation 2.3. *If all diagonal blocks A_k in (4) are complete cycle permutations, then every product $G_{i_1}G_{i_2}\cdots G_{i_t}$ is also a complete cycle permutation.*

References

[1] M. Fiedler, A note on companion matrices, *Linear Algebra Appl.* 372(2003), 325–331.

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[3] M. Fiedler, Intrinsic products and factorizations of matrices, *Linear Algebra Appl.* 428(2008), 5–13.

[4] M. Fiedler and F. J. Hall, Some inheritance properties for complementary basic matrices, *Linear Algebra Appl.* 433(2010), 2060–2069.