C^*-algebras and noncommutative mathematics

Jan Hamhalter

Czech Technical University in Prague,
Department of Mathematics, Czech Republic

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Quantization program in mathematics

N.Weaver: “The fundamental idea of mathematical quantization is that sets are represented by Hilbert spaces".

Structure of sets	$\longrightarrow$	structure of subspaces in a Hilbert space.
functions on sets	$\longrightarrow$	operators acting on a Hilbert space.

classical theory	**quantum theory**
measure theory	quantum measure theory
probability theory	quantum probability theory
topological spaces	C^*-algebras
differential geometry	noncommutative geometry
topological groups	quantum groups
information theory	quantum information
computing	quantum computing
Banach spaces	operator spaces

The structure called $*$-normed algebra has the following ingredients:

linear space, multiplication, conjugation, norm:

$$(A, +, \cdot, *, \|\cdot\|)$$

More precisely,

1. $(A, +, \|\cdot\|)$ is a normed linear space
2. Produt, $\cdot$, is associative.
3. $a(b+c) = ab + bc\,,\ \ (b+c)a = ba + ca.$
4. $a^{**} = a$, $(ab)^* = b^* a^*$, $(\lambda a)^* = \overline{\lambda} a^*$, $(a+b)^* = a^* + b^*$.

Function algebra

In commutative mathematics we work with algebras of functions

(continuous functions, measurable functions, random variables, holomorphic functions,...)

K	...	compact Hausdorff topological space.
$A = C(K)$	...	continuous complex functions on K.

• usual arithmetic operations,
complex conjugation:

$$f(z) \to \overline{f(z)}$$

• norm

$$\|f(z)\| = \max_{z \in K} |f(z)| \,.$$

Axiomatization of function algebras

$$(A, +, \cdot, *, \|\cdot\|)$$

Observe that the following conditions are satisfied for every function algebra (for all $a, b \in A$):

1. $ab = ba$
2. A has a unit with respect to multiplication.
3. $\|ab\| \leq \|a\| \cdot \|b\|$.
4. $\|a\|^2 = \|a^* a\|$.
5. $(A, \|\cdot\|)$ is complete.

Gelfand representation

Gelfand Theorem

For each algebra A satisfying axioms above there is a unique compact Hausdorff space K such that A is isometrically $*$-isomorphic to $C(K)$.

Gelfand representation:

$K =$ set of all $\omega : A \to \mathbb{C}$ such that

1. ω is a linear functional.
2. $\omega(a^*) = \overline{\omega(a)}$.
3. $\omega(ab) = \omega(a)\omega(b)$.
4. $\omega(\mathbf{1}) = 1$.

Topology = pointwise convergence on elements of A.
Gelfand transformation is given by an evaluation
$a \in A \to \hat{a} \in C(K)$

$$\hat{a}(\omega) = \omega(a) \qquad \text{for all } \omega \in K\,.$$

Theorem

Let X and Y be compact spaces. Then $C(X)$ is isometrically $*$-isomorphic to $C(Y)$ if, and only if, X and Y are homeomorphic.

C^*-algebras are generalizations (noncommutative) of compact spaces. It is a basic strategy for Connes noncommutative geometry.

Matrix algebra

M_n .... *nxn* complex matrices.

a^* adjoint matrix of a.

$\|a\|^2 =$ spectral radius (a^*a).

Observe that the following conditions are satisfied for every matrix algebra (for all $a, b \in A$):

1. A is finite dimensional and only multiples of the unit commute with all elements of A.
2. A has a unit with respect to multiplication.
3. $\|ab\| \leq \|a\| \cdot \|b\|$.
4. $\|a\|^2 = \|a^*a\|$.
5. $(A, \|\cdot\|)$ is complete.

E.g.

$$\|a^*a\|^2 = \varrho(a^*aa^*a) = \varrho((a^*a)^2) =$$
$$= \varrho^2(a^*a) = \|a\|^4 .$$

Theorem

Every algebra A satisfying the axioms above is isometrically $*$-isomorphic to the algebra M_n for some n.

We drop conditions (i) in above two cases - so we abandon commutativity and finite dimensionality. This way we get a simultaneous generalization of function and matrix algebras.

Definition

Unital C^*-algebra is normed $*$-algebra A satisfying the following conditions:

1. A has a unit with respect to multiplication.
2. $\|ab\| \leq \|a\| \cdot \|b\|$.
3. $\|a\|^2 = \|a^* a\|$.
4. $(A, \|\cdot\|)$ is complete.

Example: $B(H)$ — bounded operators on a Hilbert space H endowed with operator norm, composition as multiplication, and conjugation associating to an operator T its adjoint T^* characterised by

$$< T\xi, \eta >=< \xi, T^*\eta > \text{ for all } \xi, \eta \in H\,.$$

Gelfand-Naimark-Segal Theorem, 1941

For each C^*-algebra A there is a Hilbert space H and isometric $*$-isomorphism $\pi : A \to B(H)$ such that $\pi(A)$ is a closed $*$-subalgebra of $B(H)$.

How to get an inner product and a representation?

Simple case when A admits a linear functional ϱ such that

1. $\varrho(\mathbf{1}) = 1$.
2. $\varrho(x^*x) \geq 0$ for all $x \in A$
3. $\varrho(x^*x) = 0$ implies $x = 0$.

It can be proved:

$$\varrho(b^*a^*ab) \leq \|a^*a\|\varrho(b^*b)\,.$$

Inner product:

$$< x, y >= \varrho(y^*x)\,.$$

H — completion of $(A, < \cdot, \cdot >)$.

$\pi : A \to B(H)$ determined by

$$\pi(a)b = ab \qquad b \in A\,.$$

$$\|ab\|_H = \varrho(b^*a^*ab) \leq \|a^*a\| \cdot \varrho(b^*b) = \|a\|^2 \varrho(b^*b) = \|a\|^2 \|b\|_H\,.$$

It implies

$$\|\pi(a)\| \leq \|a\|\,.$$

As $\pi(a)(\mathbf{1}) = a$, and $\|\mathbf{1}\|_H = \varrho(\mathbf{1}) = 1$ we obtain that

$$\|\pi(a)\| = \|a\|\,.$$

Physical Interlude

Spectroscopic data: (i, j) - pairs consisting of frequency and its respond.
Heisenberg: two dimensional data are forming grupoid Δ with multiplication.
Jordan and Bohr - Heisenberg's multiplication is a matrix multiplication.

p ... momentum
q ... position
Commutation relations:

$$pq - qp = -i\frac{h}{2\pi}I\,.$$

p and q do not commute. It is not possible to consider finite dimensional matrices:

$$trace(pq - qp) = 0 \text{ while } trace\left(-i\frac{h}{2\pi}I\right) \neq 0\,.$$

Concrete realisation:

- $H = L^2(\mathbb{R})$.
- $qf(x) = xf(x)$
- $p = \frac{h}{2\pi}FqF^{-1}$, where F is the Fourier-Plancherel transform.

Classical versus noncommutative - original results

A ... C^*-algebra with unit $\mathbf{1}$

$(Abel(A), \subset)$ ... set of all abelian C^*-subalgebras of A containing the unit $\mathbf{1}$, ordered by set theoretic inclusion.

- $Abel(A)$ is a semilattice : $C_1 \wedge C_2 = C_1 \cap C_2$; $C_1 \vee C_2$ exists if, and only if, C_1 and C_2 mutually commute; with a least element $\{\lambda \mathbf{1} \mid \lambda \in \mathbb{C}\}$.
- $\cup_{C \in Abel(A)} C =$ normal elements of A.

Let A and B be C^*-algebras. A map $\psi : Abel(A) \to Abel(B)$ is an order isomorphism if

- ψ is a bijection
- $C_1 \subset C_2 \iff \psi(C_1) \subset \psi(C_2)$ for all $C_1, C_2 \in Abel(A)$.

- If A, B are $*$-isomorphic unital C^*-algebras, then $Abel(A)$ and $Abel(B)$ are isomorphic semilattices. So

$$A \to Abel(A)$$

is an invariant. However, this invariant is not complete:

A.Connes (1962)

There is a C^*-algebra A such that A is not $*$-isomorphic to its opposite algebra A^o.

So in this case $Abel(A) = Abel(A^o)$, but there is no $*$-isomorphism between A and A^0!

We shall be interested in a detailed relationship between order isomorphisms of the structure of subalgebras and relevant isomorphisms between operator algebras.

Definition

Let A and B be C^*-algebras. An order isomorphism $\varphi : Abel(A) \to Abel(B)$ is said to be implemented by a map $\psi : A \to B$ if

$$\varphi(C) = \psi(C) \qquad \text{for all } C \in Abel(A)\,.$$

The question of encoding the C^*-structure in $Abel(A)$ is nontrivial even if A is abelian.

Mendivill (1999)

Let X and Y be compact Hausdorff spaces. If $Abel(C(X))$ is isomorphic to $Abel(C(Y))$, then X and Y are homeomorphic.

Theorem, 2011

Let A and B be commutative unital C^*-algebras. For any order isomorphism

$$\varphi : Abel(A) \to Abel(B)$$

there is a $*$-isomorphism

$$\psi : A \to B$$

such that ψ implements φ.
Moreover, if A has not dimension two, then there is only one $*$-isomorphism ψ implementing φ.

Remark: If $A = \mathbb{C} \oplus \mathbb{C}$, then identity map and the $*$-isomorphism exchanging the direct summands both implement the identical automorphism of $Abel(A)$.

Conclusion: For abelian algebras there is a one-to-one correspondence between the order isomorphisms of the structure of abelian subalgebras and $*$-isomorphisms of global algebras in the sense of implementation.

Le A and B be C^*-algebras. A linear map $\psi : A \to B$ is a Jordan $*$-homomorphism if

1. $\psi(x^2) = \psi(x)^2$ for all self-adjoint elements $x \in A$.
2. $\psi(x^*) = \psi(x)^*$ for all $x \in A$.

Jordan $*$-isomorphism is a bijective Jordan $*$-homomorphism.

For showing that a continuous map preserves abelian subalgebras it suffices to assume linearity only on abelian subalgebras. It motivates:

Definition

Let A be C^*-algebra and X a linear space. A map $\psi : A \to X$ is called quasi-linear if the following conditions are satisfied:

1. $\psi(\lambda x) = \lambda \psi(x)$ for all $\lambda \in \mathbb{C}$, $x \in A$.
2. $\psi(x + y) = \psi(x) + \psi(y)$ for all $x, y \in A$ such that $xy = yx$.

A map $\psi : A \to B$ between C^*-algebras A and B is called a quasi linear Jordan $*$-isomorphism if the following holds:

1. ψ is a quasi-linear bijection,
2. $\psi(x^2) = \psi(x)^2$ for all self-adjoint $x \in A$,
3. $\psi(x^*) = \psi(x)^*$ for all $x \in A$.

Example of order isomorphism

Let $\psi : A \to B$ be a quasi linear $*$-isomorphism. Then the map

$$\varphi : Abel(A) \to Abel(B)$$

defined by

$$\varphi(C) = \psi(C) \qquad C \in Abel(A)$$

is an order isomorphism.

Question

Is every order isomorphism implemented by a unique quasi-Jordan $*$-isomorphism?

Another obstacle for unicity: $A = B = M_2(\mathbb{C})$:
Identity order isomorphism can be implemented by many quasi Jordan isomorphisms.

Theorem, 2011

Let A and B be unital C^*-algebras. Suppose that A is not isomorphic either to $\mathbb{C}^2$ or to $M_2(\mathbb{C})$. Let

$$\varphi : Abel(A) \to Abel(B)$$

be an order isomorphism. Then there is exactly one quasi Jordan $*$-isomorphism $\psi : A \to B$ implementing φ.

Quasi-linearity problem: When is a quasi-linear map between C^*-algebras linear?

Definition

A C^*-subalgebra of $B(H)$ is called a von Neumann algebra if $A'' = A$, where $A' = \{x \in B(H) \mid xy = yx \text{ for all } y \in A\}$

Generalized Gleason Theorem: The answer is yes for von Neumann algebras without Type I_2 direct summnad and bounded maps.

Credits: Aaarnes, Gunson, Christensen, Paszckiewicz, Yeadon, Bunce, Wright (see J.Hamhalter: Quantum Measure Theory, (2003)).

Corollary

Any order isomorphism with domain $Abel(M)$, where M is a von Neumann algebra without Type I_2 direct summand, $\dim M \geq 3$, is implemented by a unique Jordan $*$-isomorphism.

Some consequences for quantum theory

In the spirit of Bohr, quantum system can be viewed only through classical system. Formalization of this idea:

- Quantum system: Self-adjoint operators in a C^*-algebra A
- Structure of classical subsystems: $Abel(A)$

Intuitionistic quantum logic is based on the domain $Abel(A)$
There has been a very rapid development in this area recently.

J.Hamhalter: Quantum Measure Theory, Kluwer Academic Publishers, Dordrecht, Boston, London (2003). Book Series: Fundamental Theories of Physics, Vol. 134. pp 418, ISBN 1-4020-1714-6.

J.Hamhalter: Quantum Structures and Operator Algebras, Handbook of Quantum Logic and Quantum Structures: Quantum Structures, 285-333, Elsevier B.V., Amsterdam, 2007, Edited by K.Engesser, Dov M. Gabbay, D.Lehmann. ISBN: 978-0-444-52870-4.

J.Hamhalter: Isomorphisms of ordered structures of abelian C^*- subalgebras of C^*-algebras, Journal of Mathematical Analysis and Applications, to appear.