An Algorithm for Static Output Feedback Simultaneous Stabilization of Scalar Plants*

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Abstract

Static output feedback design and simultaneous stabilization are difficult control tasks for which no general efficient algorithm has been designed so far. In this note we show that, in the special case of scalar plants, the problem of simultaneous stabilization by static output feedback can however be solved in polynomial time using standard tools of numerical algebra.

Keywords

Linear systems, Static output feedback, Simultaneous stabilization.

1 Introduction

Static output feedback (SOF) design and simultaneous stabilization (SS) are amongst the most famous basic control problems for which no general, systematic and efficient resolution tool is available. Even though there is no computational complexity result for general SOF design, the problem was shown to be NP-hard when the gain matrix satisfies interval

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constraints [Blondel and Tsitsiklis, 2000]. Similarly, the problem of SS by output feedback was shown to be NP-hard, the most negative result in this direction being that the SS problem for more than two plants is rationally undecidable [Blondel and Tsitsiklis, 2000].

A lot of work has been devoted to SOF design and SS, but we are not aware of any result on the combined SOF/SS problem. The purpose of this note is to show that, in the special case of single-input single-output plants, both SOF design (traditionally approached with graphical root locus techniques) and the most intricate problem of SS of a set of plants via SOF can be solved in polynomial time using standard routines of numerical linear algebra.

2 Problem Statement

Let

$$\frac{p_i(s)}{q_i(s)}, \quad i = 1, \ldots, N$$

denote a set of N single-input single-output plants, where $p_i(s)$ and $q_i(s)$ are scalar polynomials of degree n. Our objective is to find a scalar static feedback gain k that simultaneous stabilizes the N plants, i.e. such that the roots of all the characteristic polynomials

$$r_i(s) = q_i(s) + kp_i(s), \quad i = 1, \ldots, N$$

belong to some given region of the complex plane (typically a half plane, a disk or a sector).

3 Solution

The Hermite stability criterion states that a scalar polynomial $r(s) = r_0 + r_1 s + \cdots + r_n s^n$ is stable if and only if the symmetric matrix

$$H = \sum_{j=0}^{n} \sum_{l=0}^{n} r_j r_l H_{jl}$$

is positive definite (henceforth denoted by $H \succ 0$), where matrices H_{jl} of size n depend on the stability region only. For example, when $r(s)$ is a polynomial of degree 3 and the stability region is the open left half-plane, matrix H is given by

$$H = \begin{bmatrix} 2r_0 r_1 & 0 & 2r_0 r_3 \\ 0 & 2r_1 r_2 - 2r_0 r_3 & 0 \\ 2r_0 r_3 & 0 & 2r_2 r_3 \end{bmatrix}.$$

When $r(s)$ is a polynomial of degree 3 and the stability region is the open unit disk, matrix H is given by

$$H = \begin{bmatrix} r_3^2 - r_0^2 & r_2 r_3 - r_0 r_1 & r_1 r_3 - r_0 r_2 \\ r_2 r_3 - r_0 r_1 & r_2^2 + r_3^2 - r_0^2 - r_1^2 & r_2 r_3 - r_0 r_1 \\ r_1 r_3 - r_0 r_2 & r_2 r_3 - r_0 r_1 & r_3^2 - r_0^2 \end{bmatrix}.$$

Construction of the matrix H and other forms of the Hermite stability criterion are reviewed with deep detail and historical perspective in [Lev-Ari et al., 1991].

The SOF/SS problem is then solved if and only if scalar k satisfies the quadratic matrix inequalities

$$H_i(k) = H_{i0} + kH_{i1} + k^2 H_{i2} \succ 0, \quad i = 1, \ldots, N$$

where matrices

$$\begin{array}{rcl} H_{i0} & = & \sum_{j=0}^{n} \sum_{l=0}^{n} q_{ij} q_{il} H_{jl} \\ H_{i1} & = & \sum_{j=0}^{n} \sum_{l=0}^{n} (q_{ij} p_{il} + q_{il} p_{ij}) H_{jl} \\ H_{i2} & = & \sum_{j=0}^{n} \sum_{l=0}^{n} p_{ij} p_{il} H_{jl} \end{array}$$

depend only on coefficients of polynomials $p_i(s)$, $q_i(s)$ and the stability region. SS of the N plants is then ensured if and only if scalar k is such that the eigenvalues of the block diagonal symmetric matrix $H(k) = \mathrm{diag}\,\{H_1(k), \ldots, H_N(k)\}$ are all positive.

Now consider $H(k)$ as a polynomial matrix of degree two in the indeterminate k and denote by $k_1, \ldots, k_m$ all the distinct real zeros[1] of $H(k)$, where $m \leq 2nN$. Then arrange these numbers in increasing order, such that

$$k_0 = -\infty \leq k_1 \leq \cdots \leq k_m \leq k_{m+1} = +\infty.$$

Let π_i denote the number of positive eigenvalues of $H(k)$ in the open interval $I_i =]k_i, k_{i+1}[$ for $i = 0, 1, \ldots, m$.

Theorem 1 *The scalar SOF/SS problem is solved if and only if feedback gain k belongs to the union of intervals I_i for which $\pi_i = nN$.*

Proof From the definition of the zeros of a polynomial matrix, in a given open interval I_i the number of positive eigenvalues of $H(k)$, denoted by π_i, remains constant. When k is such that $\pi_i = nN$, it means that all the matrices $H_i(k)$ are positive definite, hence that all the characteristic polynomials $r_i(s)$ are stable. □

Let us emphasize the fact that the condition in Theorem 1 is necessary and sufficient for scalar SOF/SS, and not only sufficient.

4 Algorithm

Based on Theorem 1, a polynomial time algorithm for SOF/SS can be derived:

1. The first step consists in computing (possibly in parallel) the zeros of quadratic polynomial matrices $H_i(k)$ for $i = 1, \ldots, N$. There exist several methods to compute the zeros of a polynomial matrix. Specialized algorithm have been developed recently to compute the zeros of quadratic symmetric polynomial matrices [Tisseur and Meerbergen, 2001]. This step can be performed in $O(n^3 N)$.

[1] The zeros of a polynomial matrix are the roots of its determinant.

2. The second step then consists in ordering all the computed zeros k_i and finding the intervals $I_i =]k_i, k_{i+1}[$ for which $\pi_i = nN$. For this purpose, it is sufficient to evaluate the inertia of matrix $H(k)$ at any value of k within the interval I_i. For obvious numerical reasons, it is recommended to choose a value of k sufficiently far from the lower and upper bounds. Due to the special rank-displacement structure of each $H_i(k)$, this step can be performed in $O(n^2 N \log n)$ [Lev-Ari et al., 1991].

5 Numerical Examples

5.1 Aircraft

Consider the problem of simultaneously stabilizing four operating points of the longitudinal short period mode of the F4E fighter aircraft . The system is described by Ackermann's state-space model

$$\dot{x} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & -30 \end{bmatrix} x + \begin{bmatrix} b_1 \\ 0 \\ 30 \end{bmatrix} u$$

where the state-space vector components represent the normal acceleration, the pitch rate and the elevator angle [Howitt and Luus, 1991]. We assume that the only output available for feedback is the second state component, i.e.

$$y = x_2.$$

The values of the parameters $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}$ and b_1 at each operation point are given in [Howitt and Luus, 1991]. The $N = 4$ corresponding scalar transfer functions of order $n = 3$ are given by

$$\begin{aligned} p_1(s)/q_1(s) &= (-351.1 - 367.6s)/(-113.0 + 51.46s + 31.84s^2 + s^3) \\ p_2(s)/q_2(s) &= (-676.5 - 346.6s)/(-31.50 + 38.53s + 31.32s^2 + s^3) \\ p_3(s)/q_3(s) &= (-455.4 - 978.4s)/(-262.5 + 84.85s + 33.12s^2 + s^3) \\ p_4(s)/q_4(s) &= (-538.7 - 790.3s)/(576.7 + 71.46s + 31.74s^2 + s^3). \end{aligned}$$

Applying the algorithm described in the previous section, we obtain the values of k_i and π_i given in Table 1. The only interval for which the number of positive eigenvalues of

i	0	1	2	3	4	5	6	7	8	9
k_i	$-\infty$	-0.5764	-0.3219	-0.0466	0.0689	0.0962	0.1216	0.1543	1.0705	$+\infty$
π_i	**12**	11	10	9	7	7	7	7	8	

Table 1: Real zeros k_i and inertias π_i for the aircraft.

$H(k)$ is equal to $nN = 12$ is the interval I_0 therefore the four plants are simultaneously stabilizable by a static output feedback $u = ky$ for any finite value of k such that

$$k < -0.5764$$

5.2 Reactor

Consider the continuous stirred tank reactor model studied in [Howitt and Luus, 1991]. The non-linear model is

$$
\begin{aligned}
x_1 &= (x_2 + 0.5)\exp(Ex_1/(x_1 + 2)) - (2 + u)(x_1 + 0.25) \\
x_2 &= 0.5 - x_2 - (x_2 + 0.5)\exp(Ex_1/(x_1 + 2))
\end{aligned}
$$

where E is a parameter related to the activation energy. During the life of the reactor, some representative values of E are 20, 25 and 30. Assuming that only

$$y = x_1$$

is available for feedback, the $N = 3$ linearized systems of order $n = 2$ to be simultaneously stabilized are given by

$$
\begin{aligned}
p_1(s)/q_1(s) &= (0.5 - 0.25s)/(11 - 5s + s^2) \\
p_2(s)/q_2(s) &= (-0.5 - 0.25s)/(-2.25 - 2.25s + s^2) \\
p_3(s)/q_3(s) &= (-0.5 - 0.25s)/(-3.5 - 3.5s + s^2).
\end{aligned}
$$

Applying the algorithm described in the previous section, we obtain the values of k_i and π_i given in Table 2. The only interval for which the number of positive eigenvalues of

i	0	1	2	3	4	5	6	7
k_i	$-\infty$	-22	-20	-14	-9	-7	-4.5	$+\infty$
π_i	5	**6**	4	2	0	1	2	

Table 2: Real zeros k_i and inertias π_i for the reactor.

$H(k)$ is equal to $nN = 6$ is $I_1 =]-22, -20[$, therefore the three plants are simultaneously stabilizable by a static output feedback $u = ky$ for any value of k such that

$$-22 < k < -20.$$

6 Conclusion

We have shown that the problem of simultaneously stabilizing a set of scalar plants by a static output feedback can be solved very easily with standard tools of numerical linear algebra. The algorithm described in the paper will be implemented in the next release of the Polynomial Toolbox for Matlab [Polyx Ltd., 2001].

Due to the bilinearity of the Hermite matrix in the design parameters, there is unfortunately no direct extension of these results to dynamic output feedback controllers design. The lack of a polynomial matrix version of the Hermite criterion also prevents us from generalizing our results to multi-input and/or multi-output systems. The only methods that are available so far for addressing these difficult and open control problems are heuristics.

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